Relaxation time for a stellar cluster

The factor $\ln[b_{max} / b_{min}]$ depends upon the limits of integration. Usually take:

- **b_{min}** to be the strong encounter radius r_s (~1 au for the Sun). Approximations made in deriving the relaxation time are definitely invalid for $r < r_s$.

- **b_{max}** to be the characteristic size of the whole stellar system - for the Sun would be reasonable to adopt either the thickness of the disk (300 pc) or the size of the galaxy (30 kpc).

$$\ln\left[b_{\max} / b_{\min}\right] = 18 - 23$$

Because the dependence is only logarithmic, getting the limits exactly right isn't critical.

$$t_{\rm relax} \approx \frac{2\times 10^{12}}{\ln[b_{\max}/b_{\min}]}\left(\frac{V}{10\ {\rm km\ s^{-1}}}\right)^{3}\left(\frac{m}{M_{sun}}\right)^{-2}\left(\frac{n}{1\ {\rm pc^{-3}}}\right)^{-1}\ {\rm yr}$$

Evaluate the relaxation time for different conditions:

	Sun	**Globular cluster**	**Open cluster**
V / km s^{-1}	30	10	1
n / pc^{-3}	0.1	10^4	10
size / pc	1000	5	5
	3×10^{13} yr	~100 Myr	~100 Myr

Predict that clusters ought to evolve due to star-star interactions during the lifetime of the Galaxy.

Can use the virial theorem (ASTR 3730 #14) to write this result in an alternate form:

$$2\langle \mathrm{KE} \rangle + \langle \mathrm{PE} \rangle = 0$$

Average value of the kinetic energy

Average value of gravitational potential energy

For a cluster of N stars, each of mass m, moving at average velocity V in a system of size R:

- total mass M = Nm
- total kinetic energy: $\frac{1}{2} NmV^2$
- gravitational potential energy: $\sim \frac{GM^2}{R} = \frac{G(Nm)^2}{R}$

Applying the virial theorem: $V = \sqrt{\frac{GNm}{R}}$

Number density of stars = number of stars / volume:

$$n = \frac{N}{\frac{4}{3}\pi R^3}$$

Range of radii over which weak interactions can occur is:

$$\frac{b_{\max}}{b_{\min}} = \frac{R}{r_s} = \frac{RV^2}{2Gm} = \frac{R}{2Gm} \times \frac{GNm}{R} = \frac{N}{2}$$

Finally define the **crossing time** for a star in the cluster:

$$t_{\text{cross}} = \frac{R}{V}$$

Ratio of the relaxation time to the crossing time is:

$$\frac{t_{\text{relax}}}{t_{\text{cross}}} = \frac{\dfrac{V^3}{8\pi G^2 m^2 n \ln\left[b_{\max}/b_{\min}\right]}}{\dfrac{R}{V}} = \frac{V^4}{8\pi G^2 m^2 n R \ln\left[b_{\max}/b_{\min}\right]}$$

Substitute for V, n and [b_{max} / b_{min}] and this simplifies to:

$$\frac{t_{\text{relax}}}{t_{\text{cross}}} = \frac{N}{6\ln\left[N/2\right]}$$

In a cluster, number of orbits a star makes before it is significantly perturbed by other stars depends only on the number of stars in the system.

Interactions are negligible for galaxy size systems, but very important for small clusters.

Consequences of relaxation

Evaporation: two-body relaxation allows stars to exchange energy amongst themselves. If at some moment a star becomes unbound (kinetic + potential energy > 0) then it will escape the cluster entirely.

Evaporation time $t_{evap} \sim 100\ t_{relax}$ (see textbook 3.2.3), and although long limits the lifetime of open clusters:

Pleiades

~500 stars, plus population of brown dwarfs.

~100 Myr old.

Mass segregation: two-body relaxation tries to equalize the **kinetic energy** of different mass stars, rather than their velocity. Since:

$$\mathrm{KE} = \frac{1}{2} m V^2$$

…more massive stars tend to have smaller velocities and sink to the center of the cluster.

Core collapse: stars in the cluster core tend to have higher velocities. If they attempt to equalize kinetic energy with stars outside the core, they lose energy, and sink even further toward the center.

Limit of this process is called *core collapse* - eventually contraction is probably halted by injection of energy from binary stars.

Analysis suggests that large-N, roughly spherical systems are stable, long lived structures (elliptical galaxies, the bulges of spiral galaxies).

Does **not** mean that anything goes as far as galaxy shapes:

- most obviously, need to have consistency between the mean stellar density and the gravitational force:

Gravitational potential

Allowed orbits of stars in the galaxy

Average stellar density at any point in the galaxy

- also more subtle issues - e.g. if the potential of the galaxy admits *chaotic* orbits, then even small perturbations can shift stars into qualitatively different orbits.