ASTR 3730: Problem Set 1
(due Tuesday September 6^{th})

(1) The *FERMI* space telescope has recently detected flares from the Crab Nebula in gamma-rays with an energy of around 100 MeV. Suppose that the quiescent flux from the Crab in photons of this energy is 10^{-10} erg cm^{-2} s^{-1} at Earth.

(a) What is the frequency and wavelength of 100 MeV photons?
(b) If the effective collecting area of *FERMI* for photons of 100 MeV energy is 10^3 cm^2, how many photons from the Crab does the telescope collect per day, on average?

(2) Estimate the *theoretical* resolution of the human eye, assuming a pupil diameter of 0.5 cm and a wavelength corresponding to that of green light ($\lambda = 0.5$ μm). Express the answer in arcminutes.

(3) A globular cluster has 10^5 stars distributed within a sphere of characteristic radius of 1 parsec (1 pc). *Estimate* the distance, in kpc, out to which the *Hubble Space Telescope* ought to be able to resolve individual stars within the cluster. Assume that the *HST* has a mirror of diameter 2.4m, and works at a wavelength of $\lambda = 0.5$ μm.

State clearly any assumptions you make about the distribution of stars within the cluster!

(4) The supermassive black hole at the Galactic Center (distance 8 kpc) has a characteristic size given by the Schwarzschild radius:

$$R_s = \frac{2GM}{c^2}.$$

The mass of the black hole is about 4×10^6 M_{Sun}. Estimate the diameter, D, of a radio telescope that would be needed to resolve structure on the scale of the Schwarzschild radius, if the telescope works at a wavelength of 0.1 mm.

ASTR 3730 : Problem Set 1 Solutions

(1) (a) Use $E = h\nu$

$E = 100$ MeV $\Rightarrow$ $\nu = 2.42 \times 10^{22}$ Hz (1)

Use $c = \lambda\nu$ $\Rightarrow$ $\lambda = 1.24 \times 10^{-12}$ cm (1)

(b) Energy received by collecting area dA in time dt

$$E = F\, dA\, dt$$

For photon energy ΔE

$$N = \frac{E}{\Delta E} = \frac{F.\, dA.\, dt}{\Delta E}$$

$\Rightarrow$ $N \simeq 54$ (3)

TOTAL FOR QUESTION : 5

(2) Angular resolution:

$$\theta = 1.22 \frac{\lambda}{D}$$

$$\Rightarrow \quad \theta \simeq \cancel{10^{-4}} \text{ radians} \quad 1.22 \times 10^{-4} \text{ rad}$$

$$\Rightarrow \quad \theta \approx \cancel{0.34} \ \cancel{\text{radian}} = 0.42 \ \overset{\text{arcmin}}{\cancel{\text{rad}}}$$

Ok to assume $\theta \sim \frac{\lambda}{D} \Rightarrow \cancel{0.28}\ \overset{0.34}{\text{arcmin}}$ also.

TOTAL FOR QUESTION : $\underline{\cancel{\ }\ 2}$

(3) Angular resolution of HST @ this wavelength:

$$\Theta_{HST} = 1.22 \frac{\lambda}{D} = \overset{2.54 \times 10^{-7} \text{ rad}}{2.08 \times 10^{-7}} \text{ rad.} \quad (2)$$

Suppose cluster has N stars, within radius R, and is observed at distance d.

To order of magnitude, density of stars in the plane of the sky

$$\simeq \frac{N}{\pi R^2} \quad (2) \quad \text{(units stars per square pc.)}$$

Characteristic projected physical separation:

$$\Delta l = \left(\frac{N}{\pi R^2}\right)^{-1/2} \quad (2)$$

5.6×10^{-3} pc

Projected angular separation:

$$\Delta\Theta = \frac{\Delta l}{d} \quad \text{which we set equal to } \Theta_{HST}.$$

Solving:

$$\frac{1}{d}\left(\frac{N}{\pi R^2}\right)^{-1/2} = \Theta_{HST}$$

$$\Rightarrow \quad d = \frac{1}{\Theta_{HST}}\left(\frac{N}{\pi R^2}\right)^{-1/2}$$

$$\Rightarrow \quad \underline{\underline{d = 27 \text{ kpc}}} \quad (2)$$

TOTAL FOR QUESTION: 8 22 kpc

Ok of course to do this properly and consider the density projected through the cluster core ⇒ smaller final distance...

(4) $R_s = \frac{2GM}{c^2}$

$R_s = 1.18 \times 10^{12}$ cm (1)

$\Rightarrow$ require angular resolution $\theta \simeq \frac{R_s}{d} = 4.775 \times 10^{-11}$ rad. (1)

$\theta \simeq 1.22 \frac{\lambda}{D}$ (1)

$D = \frac{1.22 \lambda}{\theta}$

$\Rightarrow$ $\underline{\underline{D \simeq 2500 \text{ km}}}$ (2)

TOTAL FOR QUESTION : $\underline{\underline{5}}$

TOTAL SCORE : 20

ASTR 3730: Problem Set 2

(due Tuesday September 20th)

(1) The collapse of the core of a massive star in a Type 2 supernova results in the formation of a neutron star and the release of large numbers of neutrinos. Neutrinos have a very small cross-section for interaction with matter, but the high densities encountered during stellar collapse mean that there can be circumstances where the neutrinos become temporarily trapped before they scatter and escape.

(i) Consider a static star of mass M, radius r, made up of particles with mass m_n. If the cross-section for interaction between neutrinos and matter is σ, derive an expression for the radius of a star that is just optically thick to neutrino emission from the core.

(ii) Evaluate this radius for a star of mass $M = 1.4\ M_{Sun}$, assuming a neutrino-matter cross-section $\sigma = 10^{-44}\ \text{cm}^2$. 5

(2) Suppose some impulsive process releases a burst of radiation (photons or neutrinos) at the center of a star of radius r. The optical depth between the surface and the center is $\tau >> 1$. Estimate the characteristic time scale of the burst of radiation that would be seen by an observer, after the radiation has diffusively propagated to the surface. 5

(3) Consider a small, optically thin cloud of gas that lies close to a nearby, luminous point source of radiation. The gas has opacity κ, which we will take to be independent of frequency, while the luminous source has mass M and luminosity L. Suppose that the luminosity *exceeds* the Eddington limit, so that the action of radiation will eject the cloud. If the cloud starts at radius r, and is initially at rest, find an expression for the terminal velocity v that the cloud will have when it is very far from the luminous source. 5

(4) We will be interested later in radiation transport within stars. Show that in a spherical co-ordinate system, with the center of the star at the origin, the transfer equation can be written in the form,

$$\frac{\cos\theta}{\kappa_\nu \rho}\frac{dI_\nu}{dr} = S_\nu - I_\nu$$

5

where θ is the angle made between the direction of a ray and the outward radial direction.

PROBLEM SET # 2: SOLUTIONS

(1) Star mass M, radius r. Assume composed of neutrons mass M_n, cross-section σ.

(i) Density $\rho = \frac{M}{\frac{4}{3}\pi r^3}$

Number density $n = \frac{\rho}{M_n} = \frac{3}{4\pi} \frac{M}{M_n r^3}$

Column density to center of star $\Sigma = nr$

Just optically thick: $\sigma \Sigma = 1$

$$\sigma \frac{3}{4\pi} \frac{M}{M_n r^2} = 1$$

$$\Rightarrow \quad r = \left(\frac{3\sigma M}{4\pi M_n} \right)^{1/2}$$

(ii) Evaluating: $M = 1.4 M_\odot$

$\sigma = 10^{-44}\ cm^{-2}$

$M_n = 1.67 \times 10^{-24}\ g$

Mass of neutron.
OK to take mass proton.
Mass H, etc.

$$\Rightarrow \quad r \simeq 2.0 \times 10^6\ cm$$

$$\simeq 20\ km$$

(2) Time scale of radiation will be $\simeq$ "diffusion time" from center of star to surface, or equivalently the time scale for a random walk of a photon from center.

Ok to just quote formula, but doing this from scratch.

Let mean-free-path be l

After N scatterings, distance travelled:

$$d = \sqrt{N}\, l$$

Time taken $\Delta t = \frac{N l}{c}$

Set d = radius: $N = r^2/l^2$

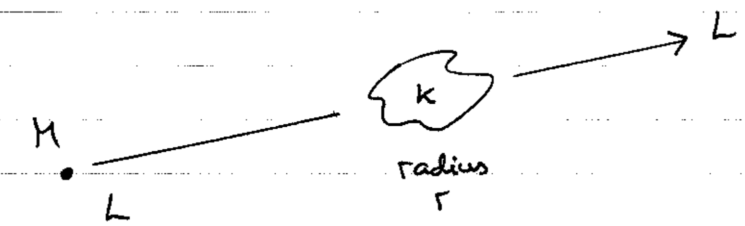
$$\Rightarrow \quad \Delta t = \frac{r^2}{l c}$$

In terms of optical depth $\tau \simeq \frac{r}{l}$

$\Rightarrow$ Characteristic time scale $\Delta t \simeq \frac{r}{c}\,\tau$

i.e. factor τ longer than free-streaming.

(3)



Gravitational force per unit mass $f_g = \frac{GM}{r^2}$

Radiative force per unit mass $f_{rad} = \frac{kL}{4\pi c r^2}$

Easiest way to solve problem is to note that since both f_g and f_{rad} $\propto 1/r^2$, can define effective potential and use conservation of energy:

$$\Phi_{eff} = -\left(GM - \frac{kL}{4\pi c} \right) \frac{1}{r}$$

Initially $v=0$, finally $\Phi_{eff}=0$. Equating energy:

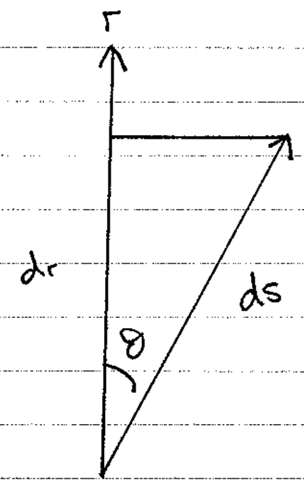
Terminal velocity:

$$\frac{1}{2} v^2 = \left(\frac{kL}{4\pi c} - GM \right) \frac{1}{r}$$

$$v = \left[\frac{2}{r} \left(\frac{kL}{4\pi c} - GM \right) \right]^{1/2}$$

Expect that most people will try to solve this by integrating an equation for $\ddot{r}$. Full credit if that works, partial if it fails along the way...

(4)



Along a ray, eq^n of radiative transfer:

$$\frac{1}{\rho \kappa_\nu} \frac{dI_\nu}{ds} = S_\nu - I_\nu$$

In spherical symmetry:

$$dr = ds \cos\theta$$

$$ds = \frac{dr}{\cos\theta}$$

$$\Rightarrow \quad \frac{\cos\theta}{\rho \kappa_\nu} \frac{dI_\nu}{dr} = S_\nu - I_\nu$$