Second order asymptotics of mixed quantum source coding

arXiv:1407.6616

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17 September 2015 QIPC 2015, Leeds

Contents

Optimal rates in information theory

- **Coding theorems:** optimal rate of an information-processing tasks in terms of an entropic quantity
- Well-known example:
  Quantum Source Coding Theorem
  Minimum compression length $m(\rho)$ of quantum source ρ given by von Neumann entropy $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$
  [B. Schumacher (1995). *Physical Review A* 51.4, p. 2738]
- Coding theorems are proved in **asymptotic, memoryless setting**:
  1. Asymptotic limit: **Infinite number** of input copies to the protocol.
  2. Memoryless: Input copies are **independent and identically distributed**.
- Asymptotic limit ($n \to \infty$): unrealistic assumption in applications.
- What about finite n? $\longrightarrow$ **Second order asymptotics**

Second order asymptotics of memoryless tasks

Example: **(One-shot) quantum source coding**

- Quantum source emits pure states $|\psi_i\rangle \in \mathcal{H}$ with probability p_i.
- **(Average) source state:** $\rho = \sum_i p_i \psi_i$
- **Task:** Encode signals as $|\psi_i^c\rangle \in \mathcal{H}_c$ where $M := \dim \mathcal{H}_c < \dim \mathcal{H}$.
- *Good* ϵ-code $:\Leftrightarrow$ Decoded state is ϵ-close to initial state, where $\epsilon \in (0,1)$ is the error incurred in the protocol.
- ϵ-**error minimum compression length** $m^\epsilon(\rho)$:

  Infimum of $\log M$, optimized over all good codes.

- Second order asymptotics: Evaluate $m^\epsilon(\rho^{\otimes n})$

Second order asymptotics of quantum source coding

Theorem (Datta and Leditzky 2015)

$$m^{\epsilon}(\rho^{\otimes n}) = nS(\rho) - \sqrt{n}\sigma(\rho)\Phi^{-1}(\epsilon) + O(\log n)$$

[N. Datta and F. Leditzky (2015). *IEEE Trans. Inf. Th.* 61.1, pp. 582–608]

- **First order** (n): von Neumann entropy $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$
- **Second order** ($\sqrt{n}$):
  - Quantum information variance

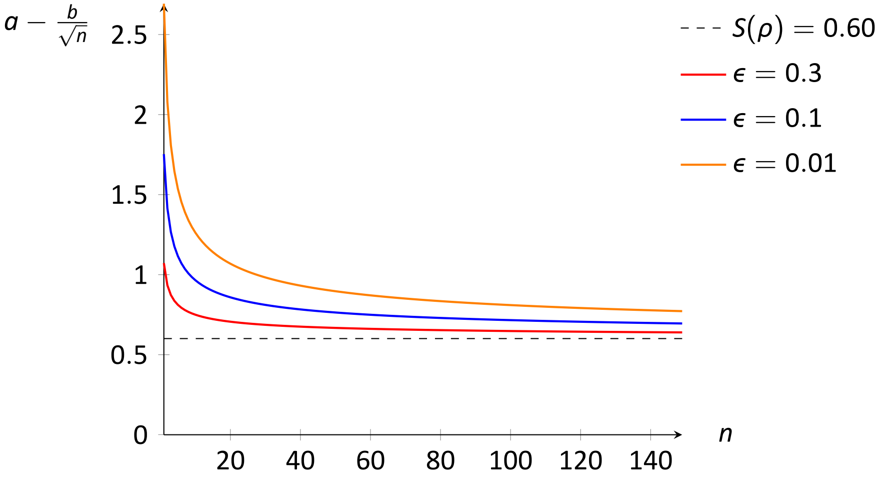
$$\sigma(\rho) := \operatorname{Tr}\left[\rho(\log \rho)^2\right] - S(\rho)^2$$

  - inverse of c.d.f. of normal distribution:

$$\Phi^{-1}(\epsilon) := \sup\{x \in \mathbb{R} \mid \Phi(x) \leq \epsilon\}$$

Second order asymptotics of quantum source coding

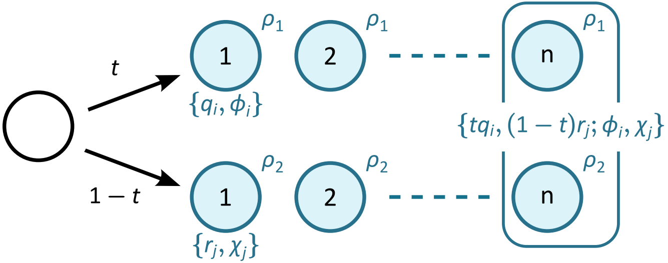
- **Source:** $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$
- **Parameters:** $a = S(\rho)$, $b = -\sigma(\rho)\Phi^{-1}(\epsilon)$



Second order asymptotics of memoryless tasks

- More examples of second order expansions in QIT (**not** exhaustive!):
  - Quantum hypothesis testing
    [K. Li (2014). *Annals of Statistics* 42.1, pp. 171–189]
    [M. Tomamichel and M. Hayashi (2013). *IEEE Trans. Inf. Th.* 59.11, pp. 7693–7710]
  - Capacity of classical-quantum channels
    [M. Tomamichel and V. Y. Tan (2015). *Comm. Math. Phys.* 338.1, pp. 103–137]
  - Noisy dense-coding
    [N. Datta and F. Leditzky (2015). *IEEE Trans. Inf. Th.* 61.1, pp. 582–608]
  - Entanglement conversion
    [W. Kumagai and M. Hayashi (2013). *Physical Review Letters* 111.13, p. 130407]
    [N. Datta and F. Leditzky (2015). *IEEE Trans. Inf. Th.* 61.1, pp. 582–608]
- Common feature: **i.i.d. resource** (e.g. source state, channel, entanglement resource)
- Second order asymptotics of information-processing **tasks with memory**?

Mixed quantum source



- Simple example of a task with memory.
- Take two memoryless sources $\rho_1 \leftrightarrow \{q_i, \phi_i\}$ and $\rho_2 \leftrightarrow \{r_j, \chi_j\}$.
- Construct mixed source that emits signals from either ρ_1 with probability t or ρ_2 with probability $1 - t$
- **Ensemble:** $\mathfrak{E}_{\text{mix}} = \{tq_i, (1-t)r_j; \phi_i, \chi_j\}$
- **Source state**: $\rho = t\rho_1 + (1-t)\rho_2$

Mixed source coding: protocol

- n uses of the mixed source:

$$\textbf{Source state: } \rho^{(n)} = t\rho_1^{\otimes n} + (1-t)\rho_2^{\otimes n}$$

$$\textbf{Ensemble: } \mathfrak{E}_{\text{mix}}^{(n)} = \{tq_{\underline{i}}, (1-t)r_{\underline{j}}; \phi_{\underline{i}}, \chi_{\underline{j}}\} \equiv \{p_{\underline{k}}, \psi_{\underline{k}}\}$$

where $\underline{i} = i_1 \dots i_n$, $q_{\underline{i}} := q_{i_1} \dots q_{i_n}$, $|\phi_{\underline{i}}\rangle := |\phi_{i_1}\rangle \otimes \dots \otimes |\phi_{i_n}\rangle$

- For $n \in \mathbb{N}$ define a **code** $(\mathcal{V}_n, \mathfrak{D}_n, M_n)$:
  - the *visible* encoding map $\mathcal{V}_n : \{\underline{i}\} \to \mathcal{D}(\mathcal{H}_c^n)$
  - a decoding CPTP map $\mathfrak{D}_n : \mathcal{D}(\mathcal{H}_c^n) \to \mathcal{D}(\mathcal{H}^{\otimes n})$
  - $M_n := \dim \mathcal{H}_c^n$
- Figure of merit: **ensemble average fidelity**

$$\bar{F}\left(\mathfrak{E}_{\text{mix}}^{(n)}, \mathcal{C}_n\right) := \sum_{\underline{k}} p_{\underline{k}} \operatorname{Tr}((\mathfrak{D}_n \circ \mathcal{V}_n)(\underline{k})\psi_{\underline{k}})$$

Mixed source coding: achievable rates

- Fix $a \in \mathbb{R}$, $\epsilon \in (0, 1)$.
- $b \in \mathbb{R}$ is called an (a, ϵ)-**achievable rate** if there is a sequence $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of codes $\mathcal{C}_n = (\mathcal{V}_n, \mathfrak{D}_n, M_n)$ such that

$$\liminf_{n\to\infty} \bar{F}\left(\mathfrak{E}^{(n)}_{\text{mix}}, \mathcal{C}_n\right) \geq 1 - \epsilon$$

$$\limsup_{n\to\infty} \frac{\log M_n - na}{\sqrt{n}} \leq b.$$

Compare: $\log M_n = na + \sqrt{n}b + O(\log n)$

- **Second order asymptotic rate** $b(a, \epsilon|\rho)$ is defined as the infimum over all (a, ϵ)-achievable rates b.
- **Remark:** $b(a, \epsilon|\rho)$ is finite $\iff$ a equals the first order rate.

Main result

Three cases (cf. Nomura and Han 2013); $S_i \equiv S(\rho_i), \sigma_i \equiv \sigma(\rho_i)$

1. $S_1 = S_2 \equiv S$
2. $S_1 > S_2, t > \epsilon$
3. $S_1 > S_2, t < \epsilon$

Theorem (Second order rates of mixed source coding)

1. $b(S, \epsilon|\rho) = L$, *where L is defined through*

$$t\Phi\left(\frac{L}{\sigma_1}\right) + (1-t)\Phi\left(\frac{L}{\sigma_2}\right) = 1 - \epsilon.$$

2. $b(S_1, \epsilon|\rho) = -\sigma_1 \Phi^{-1}\left(\frac{\epsilon}{t}\right)$
3. $b(S_2, \epsilon|\rho) = -\sigma_2 \Phi^{-1}\left(\frac{\epsilon - t}{1 - t}\right)$

Unequal von Neumann entropies (Cases 2, 3)

Case 2

$S_1 > S_2, t > \epsilon$

$$b(S_1, \epsilon|\rho) = -\sigma_1 \Phi^{-1}\left(\frac{\epsilon}{t}\right)$$

Case 3

$S_1 > S_2, t < \epsilon$

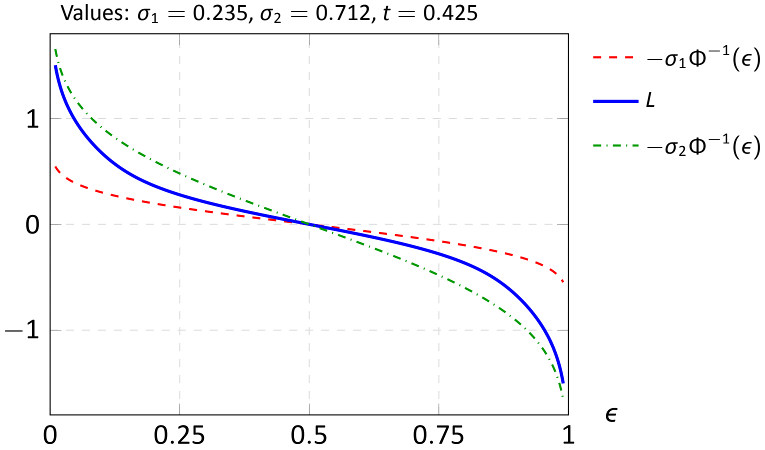
$$b(S_2, \epsilon|\rho) = -\sigma_2 \Phi^{-1}\left(\frac{\epsilon - t}{1 - t}\right)$$

- Memory of the source: For $S_1 \neq S_2$, first order rate is function of ϵ.
- Known optimal and strong converse rates:
  [G. Bowen and N. Datta (2006). *arXiv:quant-ph/0610003*]
  - **Optimal rate** ($\epsilon \to 0$): $\max\{S_1, S_2\}$
  - **Strong converse rate** ($\epsilon \to 1$): $\min\{S_1, S_2\}$
- Consequence: For $S_1 \neq S_2$ optimal rate of mixed source coding **does not satisfy** strong converse property!

Equal von Neumann entropies (Case 1)

If $S_1 = S_2 \equiv S$, then $b(S, \epsilon|\rho) = L$ where

$$t\Phi\left(L/\sigma_1\right) + (1-t)\Phi\left(L/\sigma_2\right) = 1 - \epsilon.$$



Proof idea

- Main ingredient: Second order asymptotics of **information spectrum**

Theorem (Strassen 1962)

Let $\rho \in \mathcal{D}(\mathcal{H})$ with $S = S(\rho)$ and $\sigma = \sigma(\rho)$. There is a $K > 0$ such that for any $L \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\left| \mathrm{Tr}\left(\rho^{\otimes n} \left\{ \rho^{\otimes n} \leq 2^{-nS+\sqrt{n}L} I_n \right\} \right) - \Phi\left(L/\sigma \right) \right| \leq \frac{K}{\sqrt{n}}.$$

[V. Strassen (1962). *Trans. 3rd Prague Conf. on Inf. Th.* Pp. 689–723]

Notation: For Hermitian operator A with spectral decomposition $A = \sum_i \lambda_i E_i$ and $\lambda_i \in \mathbb{R}$, define projector

$$\{A \geq 0\} := \sum_{i:\ \lambda_i \geq 0} E_i.$$

- Proof: essentially Berry-Esseen Theorem

Conclusion and open questions

Conclusion

- Second order asymptotics are not restricted to memoryless tasks!
- Mixed source coding is an easy but tractable example of a task with memory.
- Tasks with memory can violate the strong converse property.

Open questions

- Can we apply our methods to mixed classical-quantum channels?
- Can we derive second order asymptotics of more complex tasks with memory?

Thank you for your attention!

Bowen, G. and N. Datta (2006). *arXiv:quant-ph/0610003*.

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