Asymptotic performance of port-based teleportation

Felix Leditzky

(JILA & CTQM, University of Colorado Boulder)

Joint work with

M. Christandl, C. Majenz, G. Smith, F. Speelman, M. Walter

(to appear soon on arXiv)

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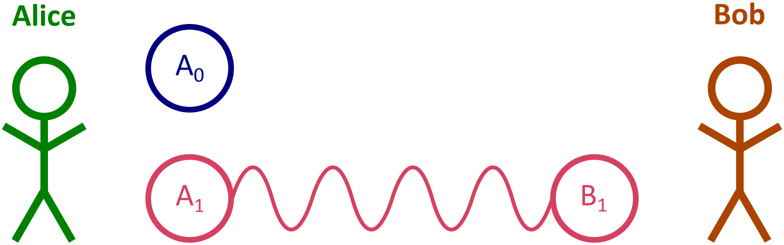
JILA CU Boulder and NIST

Center for Theory of Quantum Matter CTQM

Standard teleportation protocol



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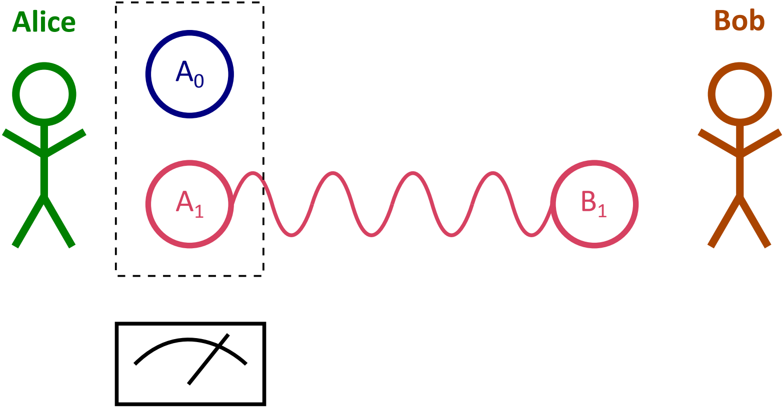

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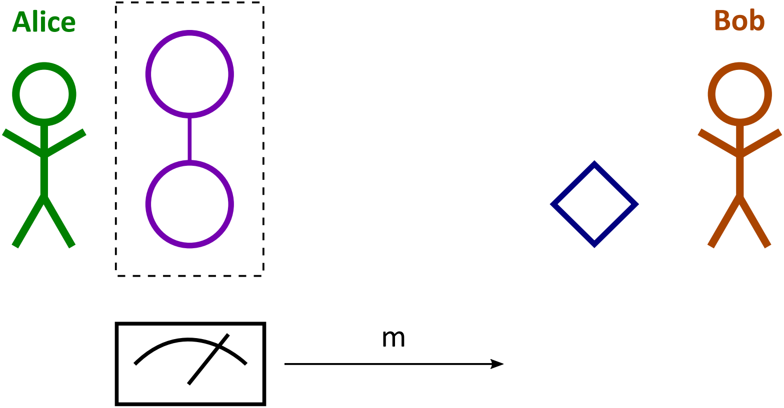

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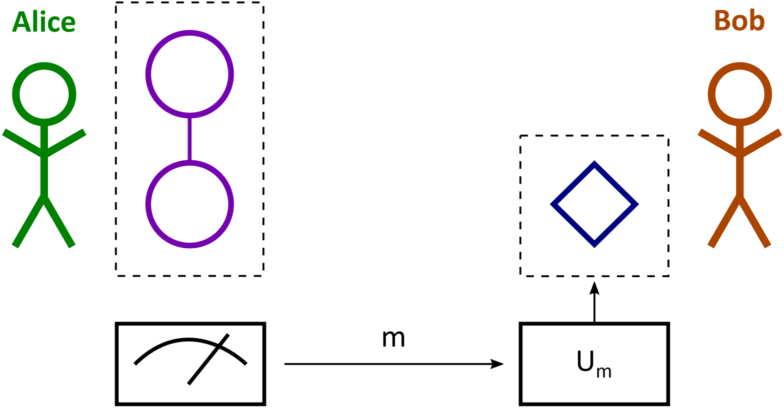

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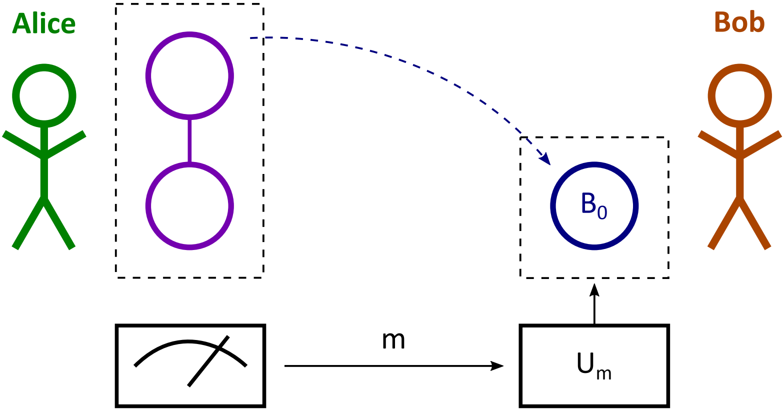
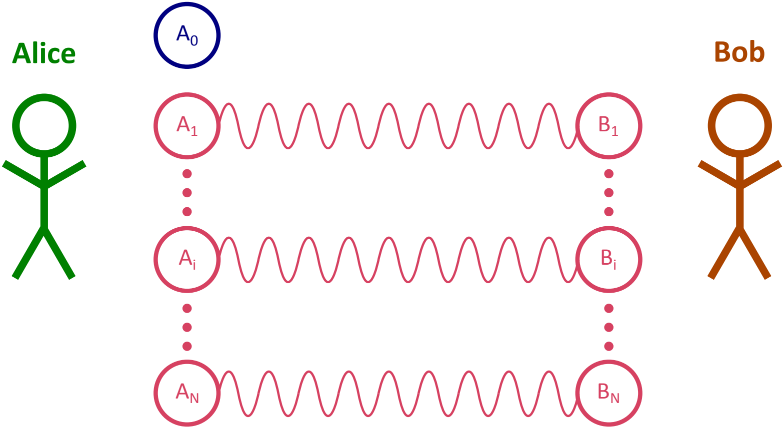


Port-based teleportation



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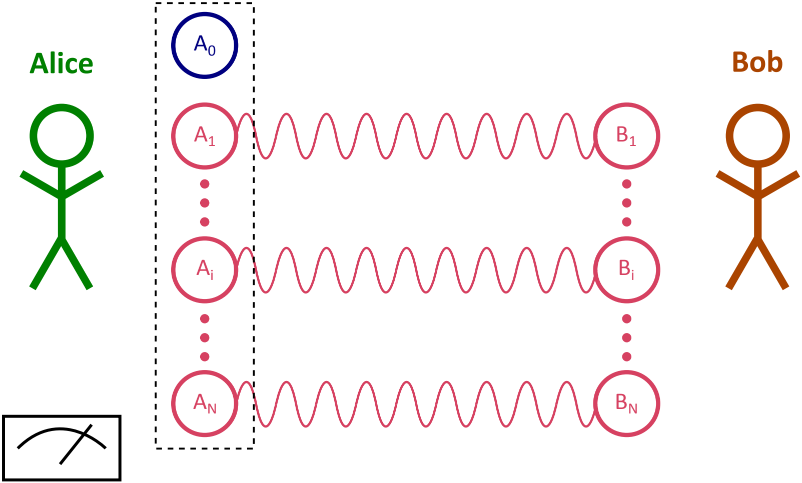

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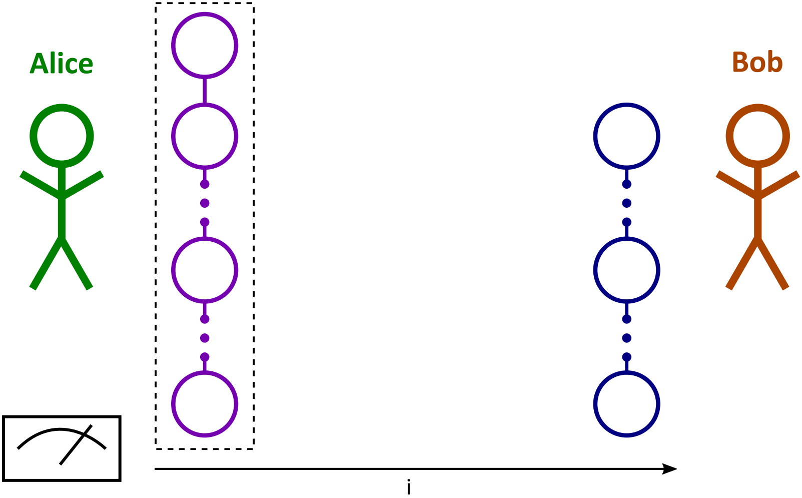

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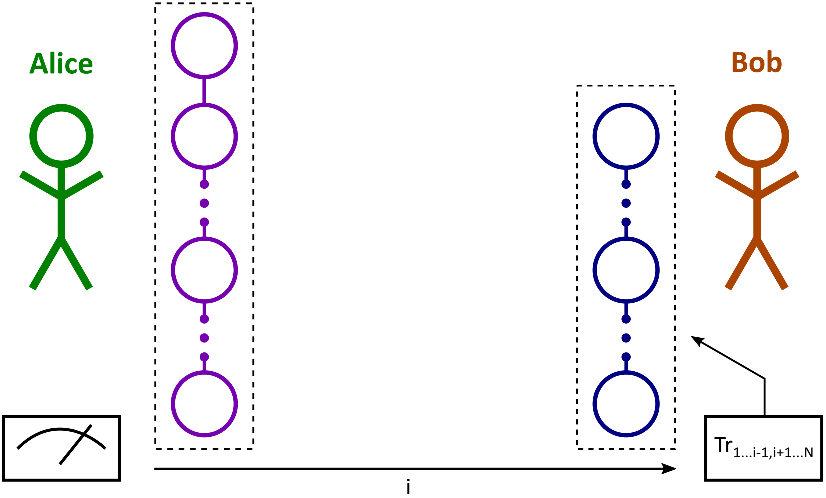

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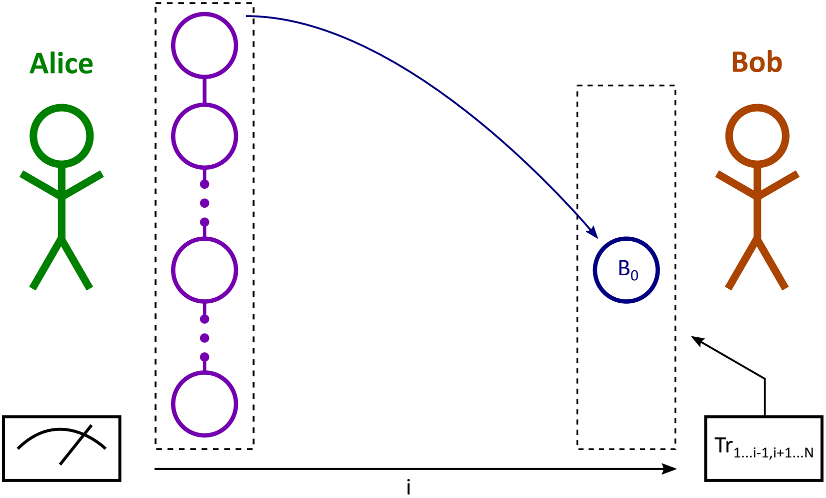

Why is PBT interesting?

- Since partial trace commutes with $U^{\otimes N}$, PBT is **unitarily covariant** (details later).
- PBT enables instantaneous non-local quantum computation (INQC). [Beigi and König 2011]
- INQC can be used to break position-based cryptography. [Buhrman et al. 2014]

Caveat

Unitary covariance leads to the fact that
perfect PBT is impossible with finite resources.

[Ishizaka and Hiroshima 2008]

Variants of PBT

- Hence, need to allow for error in protocol, leading to deterministic and probabilistic PBT.
- **Deterministic PBT**: Protocol always yields final state that approximates target state.
- **Probabilistic PBT:** Protocol yields exact target state with certain success probability.
- **Goal of this talk:** Understand symmetries of PBT and determine asymptotic performance of standard protocols.

Outline

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Operational setting

- **Goal:** teleport an unknown quantum state $|\psi\rangle_{A_0}$ from Alice to Bob.
- **Resources:** shared entanglement $|\varphi\rangle_{A_1,\ldots,A_N,B_1,\ldots,B_N}$ and (forward) classical communication.
- **Ports:** quantum systems B_i in Bob's possession.
- Each quantum system appearing above is assumed to have local dimension d.

Operational setting

- **Protocol:** Alice implements a collective measurement on $A_0 A_1 \dots A_N$ using a POVM $E = \{E^{(i)}\}_{i=1}^{N}$.
- Alice sends measurement outcome to Bob.
- Bob selects the right port that holds an (approximate) copy of target state and discards the rest.
- Selecting B_i corresponds to the quantum operation $\mathrm{Tr}_{B_i^c}$.
- The pair $(|\varphi\rangle, E)$ determines the PBT protocol.

Deterministic PBT

- In deterministic PBT the protocol always yields a final state as an approximation to the target state.
- Hence, PBT protocol should **simulate ideal *d*-dimensional channel**.
- **Figure of merit:** entanglement fidelity of PBT protocol Λ to identity channel $\mathrm{id} : \mathbb{C}^d \to \mathbb{C}^d$:

$$F_d = F(\Lambda, \mathrm{id}) = \langle \Phi^+_{A'A} | (\mathrm{id} \otimes \Lambda)(\Phi^+_{A'A}) | \Phi^+_{A'A} \rangle.$$

$|\Phi^+\rangle_{A'A} = \frac{1}{\sqrt{d}} \sum_x |x\rangle_{A'} |x\rangle_A$ is a max. entangled state.

- Alternative: diamond norm distance to identity channel.

Deterministic PBT and state discrimination

- Deterministic PBT $\iff$ state discrimination of the uniformly drawn states $\omega^{(i)}_{A^N B_i} = \mathrm{Tr}_{B_i^c}\, \varphi_{A^N B^N}$. [Ishizaka and Hiroshima 2009]
- Success probability q of discriminating between $\omega^{(i)}$:
$$q = \frac{d^2}{N} F_d.$$
- State discrimination: **pretty good measurement** as POVM.
- Further protocol simplification: N maximally entangled states $|\Phi^+\rangle_{A_i B_i} = \frac{1}{\sqrt{d}} \sum_x |x\rangle_{A_i} |x\rangle_{B_i}$ as ports.

Probabilistic PBT

- Probabilistic PBT yields the exact target state with **success probability** p_d and aborts otherwise.
- Extended POVM $E_{\text{prob}} = \{E^{(i)}\}_{i=0}^{N}$, where $E^{(0)}$ corresponds to abortion of the protocol.
- Probabilistic PBT is a **special case of deterministic PBT**.
- Again: consider special case where $|\varphi\rangle_{A^N B^N} = |\Phi^+\rangle_{AB}^{\otimes N}$.

Existing results: optimal performance of PBT

- Det. PBT with PGM as POVM and EPR pairs as ports:

$$F_d^{\text{PGM,EPR}} \geq 1 - \frac{d^2 - 1}{N}.$$

[Ishizaka and Hiroshima 2008; Beigi and König 2011]

- Converse bound for arbitrary POVM and resource state:

$$F_d^* \leq 1 - \frac{1}{4(d-1)N^2} + O(N^{-3}).$$

[Ishizaka 2015]

- Closed form for $d = 2$: [Ishizaka and Hiroshima 2009]

$$F_2^{\text{PGM}} = F_2^{\text{PGM,EPR}} = 1 - \frac{3}{4N} + o(N^{-1})$$

$$p_2^{\text{EPR}} \sim 1 - \left(\frac{8}{\pi N}\right)^{-1/2} + o(N^{-1/2}).$$

Existing results: optimal performance of PBT

- PBT has a lot of inherent **symmetries**

  $\longrightarrow$ use **representation theory** (RT)!

- [Studziński et al. 2017] and [Mozrzymas et al. 2017]: exact expressions for F_d and p_d in terms of RT quantities, hard to calculate.

- To state these: understand symmetries and RT of PBT.

- Our main results: Determine the asymptotics of these expressions for $F_d^{\mathrm{PGM,EPR}}$ and p_d^{EPR} to first order.

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Symmetries

- **Permutation symmetry:**

  Every port B_i is equally good for teleportation

  $\longrightarrow S_N$-symmetry of $\rho_{B^N} = \mathrm{Tr}_{A^N} \varphi$.

- Similar symmetry for POVM elements

  $\longrightarrow S_N$-symmetry of $E^{(i)}$.

- **Unitary invariance:**

  The protocol works equally well for all input states

  $\longrightarrow U_d$-symmetry of ρ_{B^N}.

Symmetries

Proposition: Symmetries of PBT

Every PBT protocol can be symmetrized to one with F_d no worse than the original one, and

- resource state is a purification of a symmetric Werner state, i.e., invariant under $U^{\otimes N} \otimes \overline{U}^{\otimes N}$ and S_N;
- POVM elements form an orbit under the S_N-action, and are each invariant under $\overline{U}_{A_0} \otimes U_A^{\otimes N}$;
- $\Lambda\colon \mathbb{C}^d \to \mathbb{C}^d$ is covariant w.r.t. the unitary group.

"Folklore" results, proofs in our paper and C. Majenz's PhD thesis.

Schur-Weyl duality

- Resource state invariant under both U_d- and S_N-action
  $\longrightarrow$ structure determined by **Schur-Weyl duality**.
- S_N-action: $|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \xrightarrow{\pi} |\psi_{\pi^{-1}(1)}\rangle \otimes \ldots \otimes |\psi_{\pi^{-1}(N)}\rangle$
- U_d-action: $|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \xrightarrow{U} U|\psi_1\rangle \otimes \ldots \otimes U|\psi_N\rangle$
- **Schur-Weyl decomposition:**

$$(\mathbb{C}^d)^{\otimes N} \cong \bigoplus_{\lambda \vdash_d N} [\lambda] \otimes V_\lambda$$

- **Irreducible representations:**
  - $[\lambda]$ is an irrep of S_N with $\dim[\lambda] = d_\lambda$.
  - V_λ is an irrep of U_d with $\dim V_\lambda = m_\lambda$.

Exact expressions for F_d and p_d using RT

Exact expressions for deterministic and probabilistic PBT

[Studziński et al. 2017; Mozrzymas et al. 2017]

- $$F_d^{\text{PGM,EPR}} = \frac{1}{d^{N-2}} \sum_{\alpha \vdash_d N-1} \left(\sum_{\mu = \alpha + \Box} \sqrt{d_\mu m_\mu} \right)^2,$$

  where $\mu = \alpha + \Box$ denotes a Young diagram $\mu \vdash_d N$ obtained from $\alpha \vdash_d N - 1$ by adding a single box (!).

- $$p_d^{\text{EPR}} = \frac{1}{d^N} \sum_{\alpha \vdash_d N-1} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}},$$

  where μ^* is the Young diagram obtained from $\alpha \vdash_d N - 1$ by adding a single box such that $N \frac{m_\mu d_\alpha}{m_\alpha d_\mu}$ is maximal.

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Main result 1: deterministic PBT

For deterministic PBT using PGM and EPR, we prove:

$$F_d^{\text{PGM,EPR}} = 1 - \frac{d^2 - 1}{4N} + o(N^{-1}).$$

- Recovers qubit result $F_2^{\text{PGM,EPR}} = 1 - \frac{3}{4N} + o(N^{-1})$.
- Shows that $F_d^{\text{PGM,EPR}} \geq 1 - \frac{d^2 - 1}{N}$ is not tight.
  (previously known from numerics)

Main result 2: probabilistic PBT

For probabilistic PBT using EPR, we prove:

$$p_d^{\mathrm{EPR}} = 1 - \sqrt{\frac{d}{N-1}}\mathbb{E}[\lambda_{\max}(\mathbf{G})] + o(N^{-1}),$$

where **G** is a Hermitian, traceless random $d \times d$ matrix with independent Gaussian RVs as entries.

- For qubits (i.e., **G** is a 2×2 matrix), one can prove

$$\mathbb{E}[\lambda_{\max}(\mathbf{G})] = 2\pi^{-1/2}.$$

- Hence, our result asymptotically recovers the qubit result

$$p_2^{\mathrm{EPR}} \sim 1 - \sqrt{\frac{8}{\pi N}} + o(N^{-1/2}).$$

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Spectrum estimation & random matrix theory

- Recall the decomposition of $(\mathbb{C}^d)^{\otimes N}$ obtained from Schur-Weyl duality,

$$(\mathbb{C}^d)^{\otimes N} \cong \bigoplus_{\lambda \vdash_d N} [\lambda] \otimes V_\lambda.$$

- Consider the projective measurement $\{P_\lambda\}_{\lambda \vdash_d N}$, where P_λ projects onto $[\lambda] \otimes V_\lambda$.

Spectrum estimation [Keyl and Werner 2005]

Let $\mathbf{Y}_N$ denote the outcome of the measurement $\{P_\lambda\}_{\lambda \vdash_d N}$ applied to $\rho^{\otimes N}$ where ρ is a state. Then, as $N \to \infty$,

$$\tfrac{1}{N}\mathbf{Y}_N \xrightarrow{D} \mathrm{spec}(\rho).$$

Spectrum estimation & random matrix theory

- For $\rho = \frac{1}{d}\mathbb{1}$ the corresponding outcome probability distribution $p_{d,N}$ of the measurement $\{P_\lambda\}_{\lambda \vdash_d N}$ is called **Schur-Weyl distribution**:

$$p_{d,N} = \frac{1}{d^N} d_\lambda m_\lambda.$$

- Spectrum estimation: $\frac{1}{N}\mathbf{Y}_N \xrightarrow{D} (1/d, \ldots, 1/d)$

- A "typical" Young diagram w.r.t. $p_{d,N}$ has

$$\lambda_1 \approx \frac{N}{d} + 2\sqrt{N} \qquad \lambda_d \approx \frac{N}{d} - 2\sqrt{N},$$

and the remaining λ_i's interpolate between these.

Spectrum estimation & random matrix theory

- To make this exact, define the centered and normalized RV

$$\mathbf{A}_N = \frac{\lambda_N - (N/d, \ldots, N/d)}{\sqrt{N/d}},$$

where $\lambda_N \sim p_{d,N}$ takes values in the set of Young diagrams.

- Let $\mathbf{M}$ be a Hermitian random matrix whose entries are independent Gaussian RVs

(df: $\exp(-\frac{1}{2} \operatorname{Tr} \mathbf{H}^2)$ where $\mathbf{H}$ is a Hermitian matrix-valued RV.)

- Define $\mathbf{M}_0 = \mathbf{M} - \frac{\operatorname{Tr}(\mathbf{M})}{d} \mathbb{1}$, called the **traceless Gaussian unitary ensemble** $\mathrm{GUE}_0(d)$.

Spectrum estimation & random matrix theory

Fluctuations of Schur-Weyl distribution [Johansson 2001]

For the centered and normalized RV $\mathbf{A}_N$,

$$\mathbf{A}_N \xrightarrow{D} \mathrm{spec}(\mathbf{G}),$$

where $\mathbf{G} \sim \mathrm{GUE}_0(d)$ is drawn from the traceless Gaussian unitary ensemble.

- A strengthening of Johansson's result allows us to determine asymptotics of the RT-formulae by [Studziński et al. 2017] and [Mozrzymas et al. 2017].

Proof idea: probabilistic PBT

- Recall formula: $p_d^{\mathrm{EPR}} = \frac{1}{d^N} \sum_{\alpha \vdash_d N-1} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}$,

  where μ^* is the Young diagram obtained from $\alpha \vdash_d N-1$ by adding a single box such that $N \frac{m_\mu d_\alpha}{m_\alpha d_\mu}$ is maximal.
- First step: It can be shown that

$$\frac{m_\mu d_\alpha}{m_\alpha d_\mu} = \frac{\alpha_i - i + d + 1}{N}$$

  $\longrightarrow$ maximized by choosing $i = 1$.
- Second step: Rewrite $p_d^{\mathrm{EPR}} = \frac{N}{d} \mathbb{E}_\alpha[(\alpha_1 + d)^{-1}]$.
- Show result by proving **uniform integrability** of A_i^k.

  (such that $\mathbb{E}_\alpha[A_1^k] \xrightarrow{N \to \infty} \mathbb{E}[\lambda_{\max}(\mathbf{G})^k]$)

Proof idea: deterministic PBT

- Recall: $F_d^{\text{PGM,EPR}} = \frac{1}{d^{N-2}} \sum_{\alpha \vdash_d N-1} \left(\sum_{\mu=\alpha+\square} \sqrt{d_\mu m_\mu} \right)^2,$

  where $\mu = \alpha + \square$ denotes a Young diagram $\mu \vdash_d N$ obtained from $\alpha \vdash_d N-1$ by adding a single box.

- Rewrite: $F_d^{\text{PGM,EPR}} \propto \mathbb{E}_\alpha \left[\left(\sum_{\mu=\alpha+\square} f(\alpha_i^{-1}, \mu_i^{-1}) \right)^2 \right].$

- Remember: Typical Young diagrams have echelon form, where you can add a box to every row.

- Contribution of atypical YD is negligible in the expectation value $\mathbb{E}_\alpha[\cdot]$, and hence $\mu = \alpha + \square \longrightarrow \mu = \alpha + e_i$.

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Summary

- We discussed port-based teleportation (PBT) and its two variants:
  - deterministic PBT with entanglement fidelity F_d;
  - probabilistic PBT with success probability p_d.
- PBT protocols have a number of inherent symmetries that give rise to closed representation-theoretic formulas for F_d and p_d.
- Using a connection between Young diagrams and Gaussian unitary ensembles, we were able to determine the asymptotics of some of these formulae.

Converse bound

- We also show the following general converse on PBT:

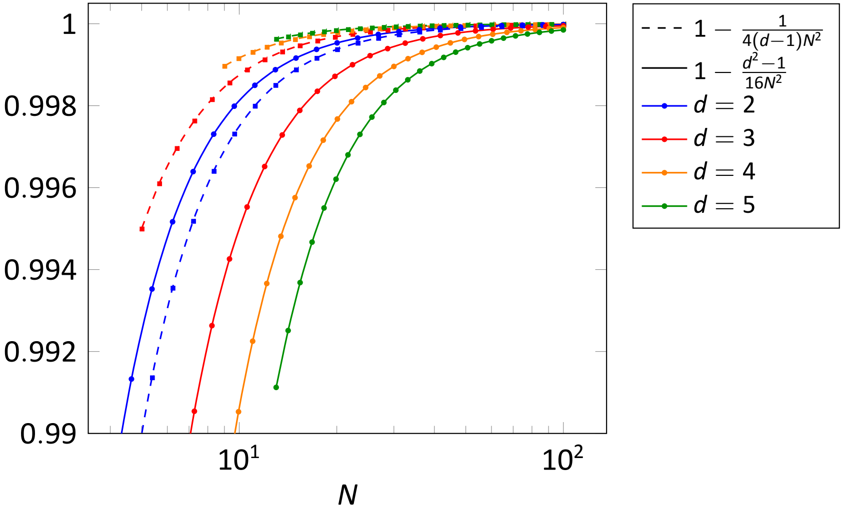
Main result 3: Converse bound

For an arbitrary PBT protocol,

$$F_d \leq \begin{cases} \dfrac{\sqrt{N}}{d} & \text{if } N \leq d^2/2, \\ 1 - \dfrac{d^2-1}{16N^2} & \text{else.} \end{cases}$$

- Tighter for $d > 2$ than converse bound $F_d \leq 1 - \frac{1}{4(d-1)N^2} + O(N^{-3})$ from [Ishizaka 2015].

Converse bounds: comparison



Open problems

- [Mozrzymas et al. 2017] derive the following closed expression of F_d^* in the fully optimized case:
$$F_d^* = d^{-N-2} \max_{c_\mu} \sum_{\alpha \vdash_d N-1} \left(\sum_{\mu = \alpha + \square} \sqrt{c_\mu d_\mu m_\mu} \right)^2 ,$$
where $\{c_\mu\}$ are such that
$$\sum_{\mu \vdash_d N} \frac{c_\mu d_\mu m_\mu}{d^N} = 1.$$
- Just figured out: There is a choice of $\{c_\mu\}$ giving the asymptotic lower bound
$$F_d^*(N) \geq 1 - \frac{d-1}{4N} + o(N^{-1})$$
(with possible improvement in N dependence to N^{-2}...)
- Open problem: Determine full asymptotics of F_d^*!

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Thank you very much for your attention!