Strong converse theorem for state redistribution using Rényi entropies

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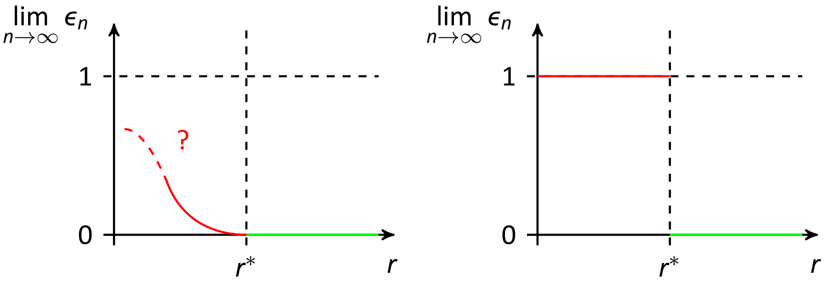
Introduction: Weak vs. strong converse

- Consider: information-theoretic task with optimal rate r^*.
- Code with rate r, blocklength n, and error ϵ_n.



Weak converse **Strong converse**

- This talk: Strong converse for **state redistribution**.

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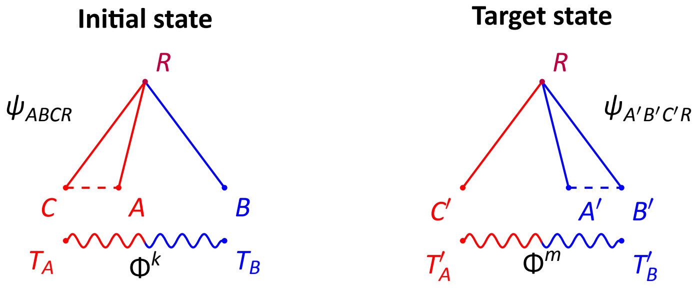
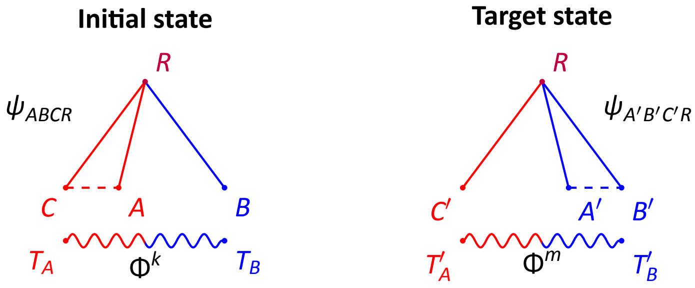
State redistribution



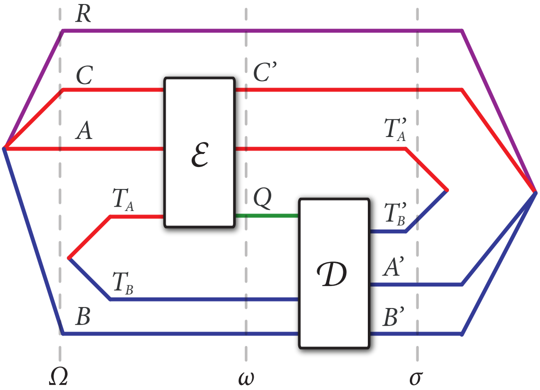
- Consider a tripartite state ρ_{ABC} shared between Alice (AC) and Bob (B), with purification ψ_{ABCR} where R is a reference system.
- Furthermore, Alice and Bob share a maximally entangled state $\Phi^k_{T_A T_B}$ of Schmidt rank k.

State redistribution



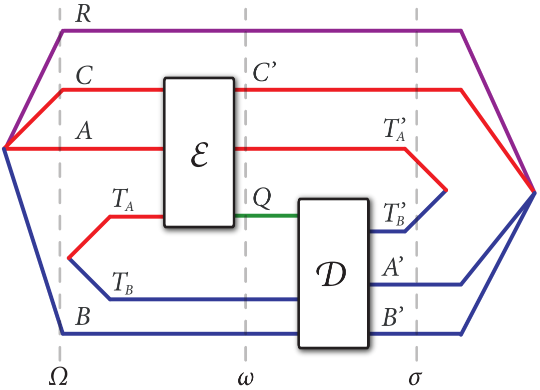
- **State redistribution:** Transfer A-part of ρ from Alice to Bob.
- Use shared entanglement, local encoding $\mathcal{E}$ (Alice) and decoding $\mathcal{D}$ (Bob), and quantum communication Q from Alice to Bob.

State redistribution



- Use shared entanglement, local encoding $\mathcal{E}$ (Alice) and decoding $\mathcal{D}$ (Bob), and quantum communication Q from Alice to Bob.

State redistribution



- State redistribution **generalizes many quantum information-processing tasks** like Schumacher compression, coherent state merging, and quantum state splitting.

State redistribution

- Consider now **n i.i.d. copies** of initial state ρ, and a shared maximally entangled state Φ^{k_n} of Schmidt rank k_n.
- **Target state:** n i.i.d. copies of ρ, and some maximally entangled state Φ^{m_n} of Schmidt rank m_n.
- State redistribution protocol outputs state σ_n.
- **Figure of merit:** Fidelity

$$F_n := F\left(\sigma_n, \Phi^{m_n} \otimes \psi^{\otimes n}\right),$$

where $F(\omega, \tau) := \|\omega^{1/2}\tau^{1/2}\|_1$.

State redistribution

- **Operational quantities:**
  - Quantum communication cost $q_n := \frac{1}{n} \log |Q^n|$
  - Entanglement cost

    $e_n := \frac{1}{n}(\log k_n - \log m_n) = \frac{1}{n}(\log |T_A^n| - \log |T_A'^n|)$
- Note that e_n can be negative, in which case entanglement is gained in the protocol rather than consumed.

Definition (Achievable rates)

A rate pair (e, q) is achievable if there exists a sequence of state redistribution protocols such that

$$\limsup_{n\to\infty} e_n = e, \qquad \limsup_{n\to\infty} q_n = q, \qquad \lim_{n\to\infty} F_n = 1.$$

State redistribution: Optimal rates

Theorem (Optimal rates)

Luo and Devetak (2009) and Yard and Devetak (2009)

The rate pair (e, q) *is achievable if and only if*

$$q + e \geq S(A|B)_\rho \qquad \textit{and} \qquad q \geq \frac{1}{2} I(A;C|B)_\rho,$$

where $S(A|B)_\rho = S(AB)_\rho - S(B)_\rho$ *is the conditional entropy and* $I(A;C|B)_\rho = S(A|B)_\rho - S(A|BC)_\rho$ *is the conditional mutual information.*

State redistribution: Optimal rates

- Preceding theorem proves a **weak converse**:

$$\limsup(q_n + e_n) < S(A|B)_\rho \quad \vee \quad \limsup q_n < \frac{1}{2} I(A;C|B)_\rho$$

$$\Longrightarrow \lim F_n < 1$$

- Our goal is to prove **strong converse**, that is,

$$F_n \leq \exp(-Kn)$$

  for some suitable constant $K > 0$, yielding $\lim F_n = 0$.

- Note: A strong converse theorem has also been obtained by Berta et al. (2016) using smooth entropies.
- We use the so-called Rényi entropy method (cf. Arimoto (1973) and Ogawa and Nagaoka (1999)).

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Rényi entropies

Definition (Sandwiched Rényi divergence)

Müller-Lennert et al. (2013) and Wilde et al. (2014)

Let ρ be a quantum state and σ be positive semidefinite with $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$. For $\alpha \in (0,1) \cup (1,\infty)$, we set $\gamma = (1-\alpha)/2\alpha$ and define the **α-sandwiched Rényi divergence** as

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr}\left(\sigma^\gamma \rho \sigma^\gamma\right)^\alpha .$$

Quantum generalization of the classical Rényi divergence

$$D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \log \sum\nolimits_x P(x)^\alpha Q(x)^{1-\alpha}.$$

That is, $\widetilde{D}_\alpha(\rho_X\|\sigma_X) = D_\alpha(P\|Q)$ for $\rho_X = \bigoplus_x P(x)$, $\sigma_X = \bigoplus_x Q(x)$.

Rényi entropies

Properties of the sandwiched Rényi divergence:

- Recovers the quantum relative entropy:

$$\lim_{\alpha \to 1} \widetilde{D}_\alpha(\rho \| \sigma) = D(\rho \| \sigma),$$

where $D(\rho \| \sigma) = \mathrm{Tr}(\rho(\log \rho - \log \sigma))$.

- **Additivity:**

$$\widetilde{D}_\alpha(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \widetilde{D}_\alpha(\rho_1 \| \sigma_1) + \widetilde{D}_\alpha(\rho_2 \| \sigma_2)$$

- **Data processing inequality:** Frank and Lieb (2013)

For every quantum operation Λ and $\alpha \geq 1/2$,

$$\widetilde{D}_\alpha(\rho \| \sigma) \geq \widetilde{D}_\alpha(\Lambda(\rho) \| \Lambda(\sigma)).$$

Rényi entropies

Definition (Derived entropies)

Let ρ_{AB} be a bipartite quantum state with marginal ρ_A. Then we define the following quantities:

- **Rényi entropy:** $S_\alpha(A)_\rho := -\widetilde{D}_\alpha(\rho_A \| \mathbb{1}_A)$;
- **Rényi conditional entropy:** $\widetilde{S}_\alpha(A|B)_\rho := -\min_{\sigma_B} \widetilde{D}_\alpha(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B)$;
- **Reyni mutual information:** $\widetilde{I}_\alpha(A;B)_\rho := \min_{\sigma_B} \widetilde{D}_\alpha(\rho_{AB} \| \rho_A \otimes \sigma_B)$.

These recover the von Neumann quantities in the limit $\alpha \to 1$, and satisfy additivity and data processing inequality as well.

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Main result: Strong converse theorem

Theorem

Let $\alpha \in (1/2, 1)$ *and* β *be such that* $1/\alpha + 1/\beta = 2$. *Then for all* $n \in \mathbb{N}$ *we have the following bounds on* $F_n = F\left(\sigma_n, \Phi^{m_n} \otimes \psi^{\otimes n}\right)$*:*

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[S_\beta(AB)_\rho - S_\alpha(B)_\rho - (q_n + e_n)\right]\right\}$$
$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[\widetilde{S}_\beta(R|B)_\rho - \widetilde{S}_\alpha(R|AB)_\rho - 2q_n\right]\right\}$$
$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[\widetilde{I}_\alpha(R;AB)_\rho - \widetilde{I}_\beta(R;B)_\rho - 2q_n\right]\right\}$$

where $\kappa(\alpha) = (1-\alpha)/2\alpha$*, and* q_n *and* e_n *are the quantum communication cost and entanglement cost, respectively.*

Main result: Strong converse theorem

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[S_\beta(AB)_\rho - S_\alpha(B)_\rho - (q_n + e_n)\right]\right\}$$

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[\widetilde{S}_\beta(R|B)_\rho - \widetilde{S}_\alpha(R|AB)_\rho - 2q_n\right]\right\}$$

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[\widetilde{I}_\alpha(R;AB)_\rho - \widetilde{I}_\beta(R;B)_\rho - 2q_n\right]\right\}$$

- Recall that state redistribution is possible iff $q + e \geq S(A|B)_\rho$.
- We have $S_\beta(AB)_\rho - S_\alpha(B)_\rho \xrightarrow{\alpha\to 1} S(A|B)_\rho$.
- Hence, if $q_n + e_n < S(A|B)_\rho$, there is $\alpha_0 < 1$ such that

$$K(\alpha_0) := \kappa(\alpha_0)[S_{\beta(\alpha_0)}(AB)_\rho - S_{\alpha_0}(B)_\rho - (q_n + e_n)] > 0,$$

and $F_n \leq \exp(-nK(\alpha_0))$.

Main result: Strong converse theorem

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[S_\beta(AB)_\rho - S_\alpha(B)_\rho - (q_n + e_n)\right]\right\}$$

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[\widetilde{S}_\beta(R|B)_\rho - \widetilde{S}_\alpha(R|AB)_\rho - 2q_n\right]\right\}$$

$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[\widetilde{I}_\alpha(R;AB)_\rho - \widetilde{I}_\beta(R;B)_\rho - 2q_n\right]\right\}$$

- Recall that state redistribution is possible iff $2q \geq I(A;C|B)_\rho$.
- $\widetilde{S}_\beta(R|B)_\rho - \widetilde{S}_\alpha(R|AB)_\rho \xrightarrow{\alpha\to 1} I(A;C|B)_\rho$ and $\widetilde{I}_\alpha(R;AB)_\rho - \widetilde{I}_\beta(R;B)_\rho \xrightarrow{\alpha\to 1} I(A;C|B)_\rho$.
- Hence, if $2q < I(A;C|B)_\rho$, there is $\alpha_0 < 1$ and $K(\alpha_0) > 0$ such that $F_n \leq \exp(-nK(\alpha_0))$.

Key ingredient in the proof of the main result

Theorem (Fidelity bound)

Let ρ_{AB} and σ_{AB} be bipartite quantum states, and for $\alpha \in (1/2, 1)$ let β be such that $1/\alpha + 1/\beta = 2$, then

(i) $S_\alpha(A)_\rho - S_\beta(A)_\sigma \geq \kappa(\alpha)^{-1} \log F(\rho_A, \sigma_A)$

(ii) $\widetilde{S}_\alpha(A|B)_\rho - \widetilde{S}_\beta(A|B)_\sigma \geq \kappa(\alpha)^{-1} \log F(\rho_{AB}, \sigma_{AB})$.

If $\rho_A = \sigma_A$, then also:

(iii) $\widetilde{I}_\beta(A;B)_\rho - \widetilde{I}_\alpha(A;B)_\sigma \geq \kappa(\alpha)^{-1} \log F(\rho_{AB}, \sigma_{AB})$.

Here, $\kappa(\alpha) = (1-\alpha)/2\alpha$ as before.

- Generalizes van Dam and Hayden (2002), who proved (i).
- Our proof method: Hölder inequality for Schatten p-norms.

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Summary

- We established a full strong converse theorem for state redistribution.
- Employed "Rényi entropy method" based on entropic quantities derived from sandwiched Rényi divergence, together with fidelity bounds.
- Same method yields strong converse for:
  - measurement compression with quantum side information
  - randomness extraction
  - data compression with quantum side information

(see full paper at arXiv:1506.02635)

Open questions

- Are these bounds optimal?

  $\longrightarrow$ Rényi quantities as strong converse exponents?

- Can we use the fidelity bounds for proving strong converse theorems for other tasks?

Thank you very much!

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