Second order asymptotics in Quantum Information Theory: Quantum source coding

arXiv:1403.2543, 1407.6616

Felix Leditzky

(joint work with Nilanjana Datta)

UNIVERSITY OF CAMBRIDGE

18 August 2015

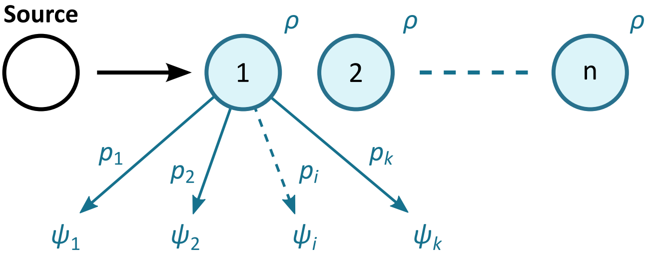
YRM 2015, Oxford

Contents

Quantum source coding

Quantum system: Modelled by Hilbert space $\mathcal{H}$, possible states of system correspond to vectors $|\phi\rangle \in \mathcal{H}$ ("pure states").

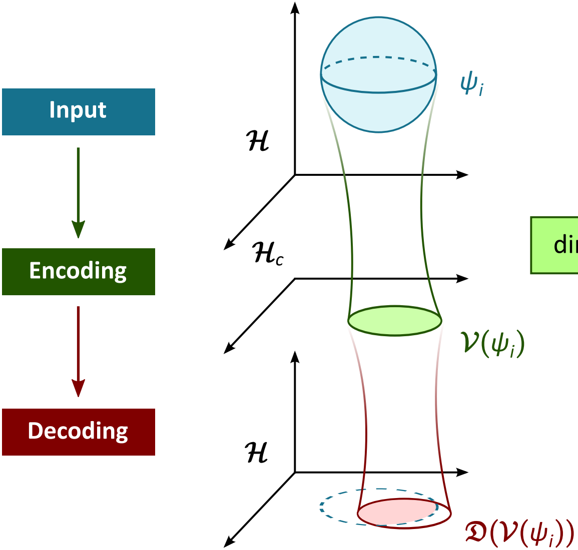
Quantum source: Atom emitting electrons, laser emitting photons, ...



Goal: Compress signals such that they can be recovered reliably.

Compression: Reduce dimension of the Hilbert space $\mathcal{H}$ supporting ψ_i.

Quantum source coding



$$\dim \mathcal{H}_c < \dim \mathcal{H}$$

Quantum source coding

In more detail:

- Quantum source emits pure states $|\psi_i\rangle \in \mathcal{H}$ with probability p_i.
- Average **source state**: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
  ("mixed state": $\rho \in \mathcal{L}(\mathcal{H})$ with $\rho \geq 0$, $\operatorname{Tr}\rho = 1$)
- n uses of the source correspond to n independent and identically distributed copies of $\rho \longrightarrow$ source state $\rho^{\otimes n}$ defined on $\mathcal{H}^{\otimes n}$.
- Use encoding $\mathcal{E}\colon \mathcal{L}(\mathcal{H}^{\otimes n}) \to \mathcal{L}(\mathcal{H}_c^n)$ where $M_n := \dim \mathcal{H}_c^n < \dim \mathcal{H}^{\otimes n}$.
- Goal: Use decoding operation $\mathfrak{D}\colon \mathcal{L}(\mathcal{H}_c^n) \to \mathcal{L}(\mathcal{H}^{\otimes n})$ such that $(\mathfrak{D} \circ \mathcal{E})(\rho^{\otimes n}) \approx \rho^{\otimes n}$.
- Triple $(\mathcal{E}, \mathfrak{D}, M_n)$ is called a **source code**.

Quantum source coding

- Measure of distance between initial and final state of the protocol: **ensemble average fidelity**

$$\bar{F}_n \equiv \bar{F}(\mathcal{E}, \mathfrak{D}, M_n) := \sum_{\underline{i}} p_{\underline{i}} \langle \psi_{\underline{i}} | (\mathfrak{D} \circ \mathcal{E})(\psi_{\underline{i}}) | \psi_{\underline{i}} \rangle \in [0, 1]$$

  where $\underline{i} = (i_1 \ldots i_n)$, $p_{\underline{i}} = p_{i_1} \ldots p_{i_n}$, $|\psi_{\underline{i}}\rangle = |\psi_{i_1}\rangle \otimes \ldots \otimes |\psi_{i_n}\rangle \in \mathcal{H}^{\otimes n}$.
- **Rate** R of the source code $(\mathcal{E}, \mathfrak{D}, M_n)$:

$$R := \lim_{n \to \infty} \frac{\log M_n}{n}$$

- $R \geq 0$ is called an *achievable rate* if there exists a code $(\mathcal{E}, \mathfrak{D}, M_n)$ with rate R and $\lim_{n \to \infty} \bar{F}_n = 1$.

Definition

Minimum compression length $m(\rho)$:= smallest achievable rate

Optimal rates in Quantum Information Theory

- **Source Coding Theorem** (Schumacher 1995):
  Minimum compression length is given by **von Neumann entropy**

  $$S(\rho) = -\operatorname{Tr}(\rho \log \rho)$$

  of the source ρ.
- Proof consists of two parts:
  - **Achievability:** *There is* a code that achieves a rate $R = S(\rho)$.
  - **Converse:** *Every* code satisfies $R \geq S(\rho)$
- Coding theorems (such as the above) give the optimal rate of an information-theoretic task in the **asymptotic limit** $n \to \infty$.

Questions

- *How well can we perform the task for finite n?*
- *How fast is the convergence in n?*

One-shot setting

- Assume that we perform the source coding task only *once*:

$$\rho^{\otimes n} \xrightarrow{n=1} \rho = \sum_i p_i \psi_i$$

- Need to allow for an **error** $\epsilon \in (0,1)$ **in the protocol**:

$$\bar{F}(\mathcal{E}, \mathfrak{D}, M) \geq 1 - \epsilon$$

Definition: ϵ-error minimum compression length

$$m^{\epsilon}(\rho) := \inf\{\log M \mid \exists \text{ code } (\mathcal{E}, \mathfrak{D}, M) \text{ with } \bar{F}(\mathcal{E}, \mathfrak{D}, M) \geq 1 - \epsilon\}$$

- (Asymptotic) minimum compression length $m(\rho)$ is recovered by

$$m(\rho) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} m^{\epsilon}(\rho^{\otimes n})$$

- **Goal:** Bound $m^{\epsilon}(\rho)$ in terms of entropic quantities

One-shot source coding

Theorem (Datta, FL 2015)

Let $\rho \in \mathcal{D}(\mathcal{H})$ be the source state of a quantum source, and let $\epsilon \in (0,1)$. Then we have

$$m^\epsilon(\rho) \approx H_s^\epsilon(\rho)$$

where the information spectrum entropy $H_s^\epsilon(\rho)$ *is defined as (Tomamichel and Hayashi 2013)*

$$H_s^\epsilon(\rho) := \inf\{\gamma \in \mathbb{R} \mid \operatorname{Tr}\left[\rho\{\rho \leq 2^{-\gamma} I\}\right] \leq \epsilon\}.$$

- Notation: For Hermitian operator A with spectral decomposition $A = \sum_i \lambda_i E_i$, we define

$$\{A \geq 0\} := \sum_{i:\ \lambda_i > 0} E_i.$$

Information spectrum entropy

$$H_s^\epsilon(\rho) := \inf\{\gamma \in \mathbb{R} \mid \operatorname{Tr}[\rho\{\rho \leq 2^{-\gamma} I\}] \leq \epsilon\}$$

- Intuition behind $H_s^\epsilon(\rho)$: Cut off small eigenvalues of ρ such that the sum of the remaining eigenvalues is ϵ-close to 1 (recall $\operatorname{Tr}\rho = 1$).
- **Second order asymptotic expansion**

$$H_s^\epsilon(\rho^{\otimes n}) = nS(\rho) - \sqrt{n}\,\sigma(\rho)\Phi^{-1}(\epsilon) + \mathcal{O}(1)$$

- First order n: von Neumann entropy $S(\rho)$
- Second order $\sqrt{n}$:
  - quantum information variance $\sigma(\rho)^2 := \operatorname{Tr}\left[\rho(\log\rho)^2\right] - S(\rho)^2$
  - inverse c.d.f. of a standard normal distribution $\Phi^{-1}(\epsilon) := \sup\{x \in \mathbb{R} \mid \Phi(x) \leq \epsilon\}$
- Main proof ingredient: Berry-Esseen Theorem

Second order asymptotics of source coding

- One-shot bounds for n uses of the source:

$$m^{\epsilon}(\rho^{\otimes n}) \approx H_s^{\epsilon}(\rho^{\otimes n})$$

Theorem (Datta, FL 2015)

$$m^{\epsilon}(\rho^{\otimes n}) = nS(\rho) - \sqrt{n}\,\sigma(\rho)\Phi^{-1}(\epsilon) + \mathcal{O}(\log n)$$

- Corollary: Strong converse of quantum source coding (Winter 1999)

$$\lim_{n\to\infty} \frac{1}{n} m^{\epsilon}(\rho^{\otimes n}) = S(\rho)$$

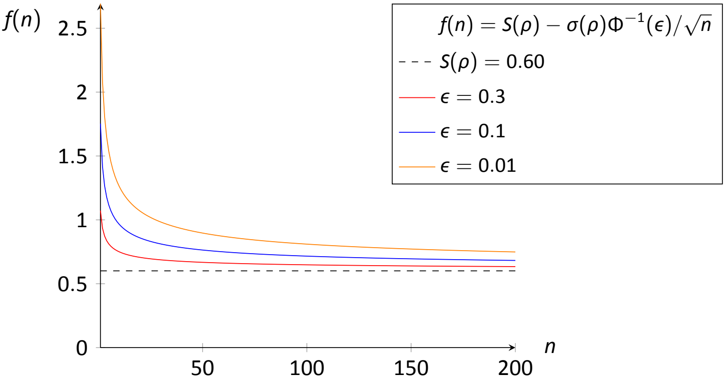
- Compare with:

$$S(\rho) = m(\rho) = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} m^{\epsilon}(\rho^{\otimes n})$$

Convergence to optimal rate

Example: Photon source emitting either $|0\rangle$ or $|+\rangle$ with $p = 0.5$

$$\rho = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \qquad \Longrightarrow S(\rho) = 0.60, \sigma(\rho) = 0.81$$



SOA of information spectrum entropy

Goal: Derive $H_s^\epsilon(\rho^{\otimes n}) = nS(\rho) - \sqrt{n}\,\sigma(\rho)\Phi^{-1}(\epsilon) + \mathcal{O}(\log n)$.

Recall: $H_s^\epsilon(\rho) := -\sup\{\gamma \in \mathbb{R} \mid \operatorname{Tr}[\rho\{\rho \le 2^\gamma I\}] \le \epsilon\}$

Proof idea:

- Consider first the quantity $\operatorname{Tr}[\rho^{\otimes n}\{\rho^{\otimes n} \le 2^\gamma I\}]$.
  - Let ρ have eigenvalues $\{r_i\}_i$ (probability distribution) and define the random variable P_ρ taking values r_i with probability r_i
  - $\operatorname{Tr}[\rho^{\otimes n}\{\rho^{\otimes n} \le 2^\gamma I\}] = \mathbb{P}\{\log P_\rho \le \frac{\gamma}{n}\}$
- Observe:

$$H_s^\epsilon(\rho^{\otimes n}) = -n\sup\{\gamma \in \mathbb{R} \mid \mathbb{P}\{\log P_\rho \le \gamma\} \le \epsilon\} = -nF^{-1}_{\log P_\rho}(\epsilon)$$

where $F^{-1}_{\log P_\rho}(\epsilon)$ is the inverse of the c.d.f. of the RV $\log P_\rho \sim P_\rho$.

SOA of information spectrum entropy

- Then:

$$H_s^\epsilon(\rho^{\otimes n}) = -nF^{-1}_{\log P_\rho}(\epsilon) = nS(\rho) - \sqrt{n}\,\sigma(\rho)F^{-1}_{Y_n}(\epsilon)$$

  where we defined the RV

$$Y_n := \frac{\sqrt{n}}{\sigma(\rho)}(\log P_\rho + S(\rho)).$$

- Central Limit Theorem: $F_{Y_n}(x) \xrightarrow{n\to\infty} \Phi(x)$ for all x.
- Rate of convergence:

Berry-Esseen Theorem

There is a constant $K > 0$ such that for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$|F_{Y_n}(x) - \Phi(x)| \leq \frac{K}{\sqrt{n}}.$$

- Use this to infer $\sqrt{n}\,F^{-1}_{Y_n}(\epsilon) = \sqrt{n}\,\Phi^{-1}(\epsilon) + \mathcal{O}(1)$.

References

Datta, N. and F. Leditzky (2015). *IEEE Transactions on Information Theory* 61.1, pp. 582–608.

Schumacher, B. (1995). *Physical Review A* 51.4, p. 2738.

Tomamichel, M. and M. Hayashi (2013). *IEEE Transactions on Information Theory* 59.11, pp. 7693–7710.

Winter, A. (1999). *IEEE Transactions on Information Theory* 45.7, pp. 2481–2485.

Thank you very much for your attention!