PHYS 7810 Extreme Physics Spring 2026. Problem Set 4. Due Mar 12

1. Geometric Algebras and Spinors — 20 points

Spinors, discovered by Cartan (1913), constitute the fundamental spin $\frac{1}{2}$ representation of the rotation group $\mathrm{Spin}(K, M)$ (the covering group of the special orthogonal group $\mathrm{SO}(K, M)$) in $K{+}M$ spacetime dimensions. The fermions of the Standard Model (SM) are spinors, and the symmetries of the SM, including the Lorentz group of spacetime, are symmetries of spinors. The Brauer-Weyl (1935) theorem states that the algebra of outer (tensor) products of spinors in $K{+}M$ spacetime dimensions is isomorphic to the geometric (Clifford) algebra $\mathrm{Cl}(K, M)$ of multivectors of all grades. Spinors and the associated geometric algebra form a powerful and unifying structure to comprehend all the observed particles and forces of Nature. This problem set acquaints you with this structure. One way to proceed would be to follow the Brauer-Weyl path, starting with spinors and proceeding to the geometric algebra. This problem set follows the inverse (historical) path, starting with the geometric algebra and proceeding to spinors.

(a) Geometric product (2 points)

Let $\boldsymbol{\gamma}_a$, $a = 1, ..., N$ be a basis of orthonormal vectors in N-dimensional Euclidean space. Their inner products form the Euclidean metric $\boldsymbol{\gamma}_a \cdot \boldsymbol{\gamma}_b = \delta_{ab}$. A general **vector**, a **multivector** of **grade** 1, is a real linear combination of basis vectors, $\boldsymbol{a} = a^a \boldsymbol{\gamma}_a$. In mathematics, an algebra is a set of elements over which addition and multiplication are defined. Grassmann (1877) and Clifford (1878)'s insight was to elevate vectors $\boldsymbol{a}$ into an algebra by defining multiplication of basis vectors by

$$\boldsymbol{\gamma}_a \boldsymbol{\gamma}_b = \begin{cases} \delta_{ab} & (a = b) \ , \\ -\boldsymbol{\gamma}_a \boldsymbol{\gamma}_b & (a \neq b) \ , \end{cases} \tag{1.1}$$

and imposing distributivity. The product of two different basis vectors is antisymmetric, and often denoted with a wedge sign,

$$\boldsymbol{\gamma}_a \boldsymbol{\gamma}_b = \boldsymbol{\gamma}_a \wedge \boldsymbol{\gamma}_b \quad (a \neq b) \ . \tag{1.2}$$

Show that if $\boldsymbol{a} = a(\cos\alpha \boldsymbol{\gamma}_1 + \sin\alpha \boldsymbol{\gamma}_2)$ and $\boldsymbol{b} = b(\cos\beta \boldsymbol{\gamma}_1 + \sin\beta \boldsymbol{\gamma}_2)$, then

$$\boldsymbol{ab} \equiv \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \ , \tag{1.3}$$

where

$$\boldsymbol{a} \cdot \boldsymbol{b} = \tfrac{1}{2}\{\boldsymbol{a}, \boldsymbol{b}\} = ab\cos(\beta - \alpha) \ , \quad \boldsymbol{a} \wedge \boldsymbol{b} = \tfrac{1}{2}[\boldsymbol{a}, \boldsymbol{b}] = ab\sin(\beta - \alpha)\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \ . \tag{1.4}$$

Conclude that $\boldsymbol{a} \wedge \boldsymbol{b}$ physically represents the directed two-dimensional surface element generated by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, with magnitude equal to the area of the surface element. Wedge products $\boldsymbol{a} \wedge \boldsymbol{b}$ of vectors are called **bivectors**, multivectors of grade 2.

(b) Multivectors (2 points)

Grassmann and Clifford extended multiplication by defining basis multivectors of grade p to be the antisymmetric product of p basis vectors,

$$\boldsymbol{\gamma}_a \boldsymbol{\gamma}_b ... \boldsymbol{\gamma}_c \equiv \boldsymbol{\gamma}_a \wedge \boldsymbol{\gamma}_b \wedge ... \wedge \boldsymbol{\gamma}_c \quad (a, b, ...c \text{ all unequal}) , \tag{1.5}$$

and imposing distributivity and associativity. The basis multivector of grade 0 is denoted 1, the **scalar**, and multiplication by it leaves all elements unchanged. Show that in N dimensions the number of grade-p basis multivectors equals the binomial factor

$$\binom{N}{p} , \tag{1.6}$$

and that the total number of basis multivectors of all grades, from 0 to N, is

$$2^N . \tag{1.7}$$

The wedge product of p vectors

$$\boldsymbol{a} \wedge \boldsymbol{b} \wedge ... \wedge \boldsymbol{c} \quad (p \text{ vectors}) \tag{1.8}$$

is a multivector of grade p, and physically represents a directed p-volume element generated by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, ..., $\boldsymbol{c}$, with magnitude equal to the p-volume of the volume element.

(c) Reverse (2 points)

The **reverse** of a basis multivector, denoted with a tilde, is defined to be the reversed product

$$\widetilde{\boldsymbol{\gamma}_a \wedge \boldsymbol{\gamma}_b \wedge ... \wedge \boldsymbol{\gamma}_c} \equiv \boldsymbol{\gamma}_c \wedge ... \wedge \boldsymbol{\gamma}_b \wedge \boldsymbol{\gamma}_a . \tag{1.9}$$

The reverse of a general multivector $\boldsymbol{a}$ is the multivector obtained by reversing each of its components. Show that reversion satisfies

$$\widetilde{\boldsymbol{a} + \boldsymbol{b}} = \widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}} , \tag{1.10a}$$

$$\widetilde{\boldsymbol{a}\boldsymbol{b}} = \widetilde{\boldsymbol{b}}\widetilde{\boldsymbol{a}} . \tag{1.10b}$$

Show that the reverse of a multivector of pure grade p satisfies

$$\widetilde{\boldsymbol{a}} = (-)^{[p/2]} \boldsymbol{a} , \tag{1.11}$$

where $[p/2]$ signifies the largest integer less than or equal to $p/2$. Thus reversion leaves unchanged the sign of multivectors of grade 0 or 1 (mod 4), and flips the sign of multivectors of grade 2 or 3 (mod 4).

(d) Transformations (2 points)

Let R be an element of some subset of elements of the geometric algebra that form a group, and consider a transformation that transforms all multivectors $\boldsymbol{a}$ by

$$R : \ \boldsymbol{a} \to R \boldsymbol{a} R^{-1} . \tag{1.12}$$

Show that the transformation (1.12) preserves addition and multiplication in the geometric algebra, that is, the transform of the sum or product of two multivectors equals the sum or product of the transforms of the two multivectors. Show that scalars, multivectors of grade 0, are invariant under transformations (1.12), and therefore that scalar products of multivectors are invariant under transformations (1.12).

(e) Commutators (2 points)

Elements of the group can be obtained by exponentiating infinitesimal transformations. Show that if $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are two not necessarily commuting multivectors (of any grade), then to second order in a Taylor expansion the logarithm of the product of their exponentials is

$$\ln(e^{\boldsymbol{\theta}} e^{\boldsymbol{\phi}}) = \boldsymbol{\theta} + \boldsymbol{\phi} + \tfrac{1}{2}[\boldsymbol{\theta}, \boldsymbol{\phi}] + \cdots \ , \tag{1.13}$$

where $[\boldsymbol{\theta}, \boldsymbol{\phi}] \equiv \boldsymbol{\theta}\boldsymbol{\phi} - \boldsymbol{\phi}\boldsymbol{\theta}$ is the commutator of $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$. The complete formula for $\ln(e^{\boldsymbol{\theta}} e^{\boldsymbol{\phi}})$ in terms of repeated commutators is known as the Baker-Campbell-Hausdorff formula.

(f) Generators (2 points)

The generators of a group comprise a set $\boldsymbol{\theta}_i$, $i = 1, ..., n$ of elements of the algebra which close under commutation,

$$[\boldsymbol{\theta}_i, \boldsymbol{\theta}_j] = f_{ijk} \boldsymbol{\theta}_k \tag{1.14}$$

for some real coefficients f_{ijk}. Real linear combinations of generators $\boldsymbol{\theta}_i$ form a Lie algebra. A Lie group is a group that has, as here, a geometric as well as a group structure. The tangent space of a Lie group defines its associated Lie algebra. The coefficients f_{ijk} are called the **structure coefficients** of the Lie algebra. The number n of generators is called the **dimension** of the group. Show that the commutator of a bivector with a multivector of grade p is another multivector of grade p. Conclude in particular that the commutator of two bivectors is a bivector.

(g) Rotation group (2 points)

It can be shown (Exercise 13.6) that the smallest nontrivial set of elements of the geometric algebra that close under commutation is the algebra of bivectors. The group generated by the algebra of bivectors is the rotation group. The elements R of the rotation group are called **rotors**

$$R = e^{-\frac{\boldsymbol{\theta}}{2}} \ , \tag{1.15}$$

where $\boldsymbol{\theta}$ is a bivector. For example, a right-handed rotation by angle θ in the $\boldsymbol{\gamma}_1$–$\boldsymbol{\gamma}_2$ plane is generated by $\frac{1}{2}\boldsymbol{\theta} = \frac{1}{2}\theta \boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2$. Show that the corresponding rotor is

$$R = \cos\tfrac{\theta}{2} - \sin\tfrac{\theta}{2} \boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \ . \tag{1.16}$$

Show that the inverse of the rotor R is its reverse, $R^{-1} = \widetilde{R}$. Show that under a rotation by the rotor (1.16),

$$\boldsymbol{\gamma}_a \to R \boldsymbol{\gamma}_a \widetilde{R} \ , \tag{1.17}$$

the vectors $\boldsymbol{\gamma}_a$ of geometric algebra transform as

$$\boldsymbol{\gamma}_1 \to \cos\theta \boldsymbol{\gamma}_1 + \sin\theta \boldsymbol{\gamma}_2 \ , \quad \boldsymbol{\gamma}_2 \to \cos\theta \boldsymbol{\gamma}_2 - \sin\theta \boldsymbol{\gamma}_1 \ , \quad \boldsymbol{\gamma}_a \to \boldsymbol{\gamma}_a \quad (a \neq 1, 2) \ , \tag{1.18}$$

in agreement with the usual behavior of vectors under rotations.

(h) Pseudoscalar (2 points)

The **pseudoscalar** I_N is defined to be the highest grade element of the geometric algebra, the multivector of grade N,

$$I_N \equiv \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2...\boldsymbol{\gamma}_N \equiv \boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge ... \wedge \boldsymbol{\gamma}_N \ . \tag{1.19}$$

The reverse of the pseudoscalar is $\widetilde{I_N} = (-)^{[N/2]} I_N$, equation (1.11), and it follows that $(I_N)^2 = (-)^{[N/2]}$. Show that the pseudoscalar I_N commutes with all bivectors, and is therefore invariant under the rotation group. The scalar 1 and the pseudoscalar I_N are the only basis multivectors that are invariant under the rotation group.

(i) Chiral vectors (2 points)

Spinors require a complex structure. It is entirely possible that spinors are the genesis of the complex structure of quantum mechanics. Group the vectors $\boldsymbol{\gamma}_a$ into pairs

$$\boldsymbol{\gamma}_i^+ \equiv \boldsymbol{\gamma}_{2i-1} \ , \quad \boldsymbol{\gamma}_i^- \equiv \boldsymbol{\gamma}_{2i} \ , \tag{1.20}$$

and define complex **chiral** basis vectors by

$$\boldsymbol{\gamma}_i \equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{\gamma}_i^+ + i\boldsymbol{\gamma}_i^-) \ , \quad \boldsymbol{\gamma}_{\bar{\imath}} \equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{\gamma}_i^+ - i\boldsymbol{\gamma}_i^-) \ , \tag{1.21}$$

whose nonvanishing scalar products are $\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}_{\bar{\imath}} = 1$. The chiral vector basis $\boldsymbol{\gamma}_i$ is a complex vector, whose complex conjugate is $\boldsymbol{\gamma}_{\bar{\imath}}$. The orthonormal vectors $\boldsymbol{\gamma}_i^\pm$ constitute, modulo a normalization, the real and imaginary parts of the complex vector $\boldsymbol{\gamma}_i$. Show that under a right-handed rotation by angle θ in the $\boldsymbol{\gamma}_i$ complex plane, the complex vector $\boldsymbol{\gamma}_i$ and its conjugate $\boldsymbol{\gamma}_{\bar{\imath}}$ transform as

$$\boldsymbol{\gamma}_i \to e^{-i\theta}\boldsymbol{\gamma}_i \ , \quad \boldsymbol{\gamma}_{\bar{\imath}} \to e^{i\theta}\boldsymbol{\gamma}_i \ . \tag{1.22}$$

(j) Charges (2 points)

The **rank** of a group is the number of mutually commuting generators of the group. For the rotation group, the mutually commuting generators are the $[N/2]$ bivectors

$$\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_{\bar{\imath}} = \tfrac{1}{2}(\boldsymbol{\gamma}_i\boldsymbol{\gamma}_{\bar{\imath}} - \boldsymbol{\gamma}_{\bar{\imath}}\boldsymbol{\gamma}_i) = -i\boldsymbol{\gamma}_i^+ \wedge \boldsymbol{\gamma}_i^- \ . \tag{1.23}$$

The i-charge λ of a multivector $\boldsymbol{a}$ is the eigenvalue of the i-charge operator $\frac{1}{2}\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_{\bar{\imath}}$ acting by commutation on the multivector,

$$[\tfrac{1}{2}\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_{\bar{\imath}}, \boldsymbol{a}] = \lambda \boldsymbol{a} \ . \tag{1.24}$$

Show that the chiral vector $\boldsymbol{\gamma}_i$ has i-charge 1, and its conjugate $\boldsymbol{\gamma}_{\bar{\imath}}$ has i-charge -1, and that all other chiral basis vectors have charge 0. What is the charge of a product of chiral vectors? [Answer: The charge of a product is the sum of the charges of the factors. For example, a chiral bivector $\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_j$ has i-charge 1 and j-charge 1, and no other charge. A chiral bivector $\boldsymbol{\gamma}_i \wedge \boldsymbol{\gamma}_{\bar{\imath}}$ has zero charge.]

(k) Spinors (optional)

A spinor is a complex (with respect to i) linear combination $\psi \equiv \psi^a \boldsymbol{\epsilon}_a$ of basis spinors $\boldsymbol{\epsilon}_a$. In N dimensions, there are $2^{[N/2]}$ basis spinors $\boldsymbol{\epsilon}_a$,

$$\boldsymbol{\epsilon}_a \equiv \boldsymbol{\epsilon}_{a_1 ... a_{[N/2]}} \ , \tag{1.25}$$

where $a_1 ... a_{[N/2]}$ denotes not a set of indices, but rather a bitcode of $[N/2]$ bits specifying the single index a. Each bit can be either up ($\uparrow$) or down ($\downarrow$). Spinors $\psi \equiv \psi^a \boldsymbol{\epsilon}_a$ can be represented as column spinors of length $2^{[N/2]}$, and multivectors $\boldsymbol{a}$ as $2^{[N/2]} \times 2^{[N/2]}$ matrices. A multivector can only pre-multiply a column spinor, in the same way that a matrix can pre-multiply a column vector, but it cannot post-multiply a column vector. The chiral vector $\boldsymbol{\gamma}_i$ raises the i-bit from down to up, while yielding nothing acting on a spinor whose i-bit is already up; while the conjugate vector $\boldsymbol{\gamma}_{\bar{\imath}}$ lowers the i-bit from up to down, while yielding nothing acting on a spinor whose i-bit is already down:

$$\begin{aligned} \boldsymbol{\gamma}_i \, \boldsymbol{\epsilon}_{...\uparrow_i...} &= 0 \ , & \boldsymbol{\gamma}_i \, \boldsymbol{\epsilon}_{...\downarrow_i...} &= \sqrt{2} \, \boldsymbol{\epsilon}_{...\uparrow_i...} \ , \\ \boldsymbol{\gamma}_{\bar{\imath}} \, \boldsymbol{\epsilon}_{...\uparrow_i...} &= \sqrt{2} \, \boldsymbol{\epsilon}_{...\downarrow_i...} \ , & \boldsymbol{\gamma}_{\bar{\imath}} \, \boldsymbol{\epsilon}_{...\downarrow_i...} &= 0 \ . \end{aligned} \tag{1.26}$$

Show from equations (1.26) that the i-charge of a spinor $\boldsymbol{\epsilon}_a$ is $+\frac{1}{2}$ or $-\frac{1}{2}$ as its i-bit is up or down.

(l) Spinor inner and outer products (optional)

There are also row spinors, conveniently denoted

$$\boldsymbol{\epsilon}_a \cdot \equiv \boldsymbol{\epsilon}_a^\top \varepsilon \ , \tag{1.27}$$

where the trailing dot denotes the spinor metric ε, a $2^{[N/2]} \times 2^{[N/2]}$ matrix. The components of the spinor metric are the inner, or scalar, products of row with column spinors,

$$\varepsilon_{ba} \equiv \boldsymbol{\epsilon}_b \cdot \boldsymbol{\epsilon}_a \equiv \boldsymbol{\epsilon}_b^\top \varepsilon \boldsymbol{\epsilon}_a = \pm_a \delta_{\bar{b}a} \ , \tag{1.28}$$

where $\bar{b}$ denotes the bit-flip of b, and $\pm_a$ is the sign of the scalar product $\boldsymbol{\epsilon}_a^\top \varepsilon \boldsymbol{\epsilon}_a = \pm 1$, given by equations (23) or (24) of Hamilton (2023). That the only nonzero components of the inner product are between a spinor and its bit-flip follows from the fact that the scalar product must carry zero charge. A scalar product of a row spinor $\chi \cdot$, a multivector $\boldsymbol{a}$, and a column spinor ψ must remain invariant under a rotation, which implies that the scalar product must transform under a rotor R as

$$R : \ \chi \cdot \boldsymbol{a} \psi \to (\chi \cdot \widetilde{R})(R \boldsymbol{a} \widetilde{R})(R\psi) \ . \tag{1.29}$$

This implies that row spinors $\chi \cdot$ and column spinors ψ must transform under a rotor R as

$$R : \ \chi \cdot \to \chi \cdot \widetilde{R} \ , \quad \psi \to R\psi \ . \tag{1.30}$$

The other way to multiply spinors is their outer product, the product of a column spinor with a row spinor, which yields a matrix (a multivector, in fact),

$$\psi \chi \cdot \ . \tag{1.31}$$

Show that the outer product transforms like a multivector,

$$R: \ \psi\chi\cdot \to R\psi\chi\cdot\widetilde{R}\ . \tag{1.32}$$

Successive outer and inner products are associative, a feature made transparent by the trailing dot notation,

$$\phi\chi\cdot\psi = \phi(\chi\cdot\psi) = (\phi\chi\cdot)\psi\ . \tag{1.33}$$