PHYS 7810 Extreme Physics Spring 2026. Problem Set 3. Due Feb 19

1. The generation of gravitational waves during inflation — 20 points

This problem walks you through the generation of a primordial power spectrum of gravitational waves from vacuum fluctuations during inflation. As you do the math, please do your best to explain what you are doing and why.

(a) Quantum field operators (3 points)

Quantum field theory postulates destruction and creation operators a_k and $a_k^\dagger$ (lowering and raising operators) that respectively reduce and increase by one the number of particles of 3-momentum k in the system. The destruction and creation operators a_k and $a_k^\dagger$ are postulated to satisfy commutation relations

$$[a_{k'}, a_k^\dagger] = (2\pi)^3 \delta^3(k' - k) \ , \tag{1.1a}$$
$$[a_{k'}, a_k] = [a_{k'}^\dagger, a_k^\dagger] = 0 \ . \tag{1.1b}$$

A generic field operator $\hat{\varphi}_k(t)$ for a bosonic field of mass m, with a hat on $\hat{\varphi}$ to emphasize that it is a quantum mechanical operator rather than a classical field, is a sum of a forward-in-time destruction operator a_k and a backward-in-time creation operator $a_k^\dagger$, and its conjugate momentum operator $\hat{\pi}_k(t)$ is its time derivative,

$$\hat{\varphi}_k(t) \equiv \sqrt{\frac{1}{2\omega_k}} \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t} \right) \ , \tag{1.2a}$$
$$\hat{\pi}_k(t) = \frac{d\hat{\varphi}_k(t)}{dt} = -i\sqrt{\frac{\omega_k}{2}} \left(a_k e^{-i\omega_k t} - a_k^\dagger e^{i\omega_k t} \right) \ , \tag{1.2b}$$

where $\omega_k \equiv \sqrt{k^2 + m^2}$ is the positive frequency associated with the 3-momentum k. From the commutation relations (1.1) and the definitions (1.2) of $\hat{\varphi}_k$ and $\hat{\pi}_k$, show that they satisfy equal-time quantum commutation relations

$$[\hat{\varphi}_{k'}(t), \hat{\pi}_k(t)] = i\,(2\pi)^3 \delta^3(t, k' - k) \ , \tag{1.3a}$$
$$[\hat{\varphi}_{k'}(t), \hat{\varphi}_k(t)] = [\hat{\pi}_{k'}(t), \hat{\pi}_k(t)] = 0 \ . \tag{1.3b}$$

(b) Power spectrum of vacuum fluctuations (3 points)

Quantum field theory postulates the existence, in Minkowski space, of a vacuum state, the state containing no particles. There is an out-vacuum $\langle 0|$ state and in-vacuum state $|0\rangle$, and the vacuum-to-vacuum amplitude is declared to be one, $\langle 0|0\rangle = 1$. The destruction operator a_k is defined to yield zero acting on the in-vacuum, while the creation operator $a_k^\dagger$ yields zero acting on the out-vacuum,

$$\langle 0|a_k^\dagger = 0 \ , \quad a_k|0\rangle = 0 \ . \tag{1.4}$$

Show that field $\hat{\varphi}$ defined by equation (1.2a) has zero vacuum expectation value

$$\langle 0|\hat{\varphi}_k(t)|0\rangle = 0 \ , \tag{1.5}$$

but its variance has a nonzero expectation value

$$\langle 0|\hat{\varphi}_{k'}(t)\hat{\varphi}_k(t)|0\rangle = \frac{(2\pi)^3}{2\omega_k}\delta^3(k'-k) \ . \tag{1.6}$$

The power spectrum $P(k)$ is defined to be the coefficient of the Dirac delta function $(2\pi)^3\delta^3(k'-k)$ in equation (1.6),

$$P_{\hat{\varphi}}(k) = \frac{1}{2\omega_k} \ . \tag{1.7}$$

For a massless field, $\omega_k = |k|$.

(c) Cosmic scale factor and conformal time during inflation (2 points)

The power spectrum of the CMB indicates that the Universe today is close to being spatially flat, which justifies treating the unperturbed Universe as spatially flat during inflation. When dealing with perturbations to an FLRW universe, it is convenient to use as coordinates conformal time η, related to proper time t by $a\,d\eta = dt$, and comoving spatial coordinates $\{x, y, z\}$, so that the unperturbed FLRW line-element is conformally Minkowski,

$$ds^2 = a^2(-\,d\eta^2 + dx^2 + dy^2 + dz^2) \ , \tag{1.8}$$

where $a(\eta)$ is the cosmic scale factor. During inflation, the cosmic scale factor increases with cosmic time t as $a \propto e^{Ht}$ with constant Hubble parameter H. Show that the cosmic scale factor varies with conformal time η as

$$a = \frac{1}{H|\eta|} \ , \tag{1.9}$$

with η normalized so that $\eta = -1/H$ at $a = 1$. Notice that η is negative, and increasing from $-\infty$ to 0 as a increases from 0 to ∞.

(d) Wave equation for gravitational waves (3 points)

Perturbations to an FLRW universe comprise 6 physical degrees of freedom, consisting of 2 spin-0 "scalars" Ψ and Φ, 2 right- and left-handed spin-1 "vectors" W_+ and W_-, and 2 right- and left-handed spin-2 "tensors" h_{++} and h_{--}. Of these, only the tensors h_{ab} satisfy a propagating wave equation $\Box h_{ab} = 0$ (in the absence of a tensor source of energy-momentum). These represent gravitational waves. In an FLRW universe, the wave equation for gravitational waves is

$$-a^2\Box h_{ab} = \left(\frac{\partial^2}{\partial\eta^2} + 2\frac{\dot{a}}{a}\frac{\partial}{\partial\eta} - \nabla^2\right)h_{ab} = 0 \ , \tag{1.10}$$

where the dot denotes derivative with respect to conformal time, $\dot{a} \equiv da/d\eta$, and ∇^2 is the comoving spatial Laplacian. Show that during inflation the wave equation (1.10) has solutions

$$h_{ab} \propto e^{\pm i(-k\eta + k_\alpha x^\alpha)}\big(1 \mp ik|\eta|\big) \ , \tag{1.11}$$

where $k = \sqrt{k_\alpha k^\alpha}$ is a positive comoving wavenumber. Argue that this implies that a primordial gravitational wave inside the horizon, satisfying $k|\eta| \gg 1$, tends to redshift to nothing during inflation, but that the wave freezes out once it exits the horizon, $k|\eta| \ll 1$.

(e) Wave equation suitable for quantization (2 points)

A basic feature of quantum field theory is that fields to be quantized must satisfy a wave equation that, in the frame of the observer (here a comoving observer), looks like the wave equation of a simple harmonic oscillator with a possibly variable mass term. Show that the wave equation (1.10) can be recast as

$$\left(\frac{\partial^2}{\partial\eta^2} - \frac{\ddot{a}}{a} - \nabla^2\right)(a h_{ab}) = 0 \ . \tag{1.12}$$

Conclude that the field to be quantized is not h_{ab} itself, but rather $a h_{ab}$.

(f) Quanta of the gravitational field (0 points)

A generic feature of a bosonic field φ is that its conjugate momentum π is a first spacetime derivative of the field. Since the Dirac delta-function on the right hand side of equation (1.3a) has units length^3, it follows that the field $\hat{\varphi}_k(t)$ must have units length^2 while its conjugate momentum $\hat{\pi}_k(t)$ must have units length. If the field is expanded in real space instead of Fourier space, then the field $\hat{\varphi}_x(t)$ must have units length^{-1} while its conjugate momentum $\hat{\pi}_x(t)$ must have units length^{-2}. The tensor perturbation $h_{ab}(t, x)$ in real space is dimensionless (as is $a h_{ab}$), so h_{ab} must be divided by something with units of length to bring it to a form that can be quantized. The correct normalization can be deduced from the fact that the lowest (quadratic) order contribution to the Hilbert Lagrangian L_h from the tensor perturbation h_{ab} is (R_h is the contribution of h_{ab} to the Ricci scalar R)

$$L_h \equiv \frac{1}{16\pi G} R_h = -\frac{1}{16\pi G}\eta^{\mu\nu}\partial_\mu h_{ab}\partial_\nu h^{ab} = -\frac{1}{16\pi G}\eta^{\mu\nu}\left(\partial_\mu h_{++}\partial_\nu h_{--} + \partial_\mu h_{--}\partial_\nu h_{++}\right) \ . \tag{1.13}$$

By comparison, the Lagrangian of a real scalar field φ is

$$L_\varphi = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \ . \tag{1.14}$$

Thus the gravitational Lagrangian (1.13) looks like the Lagrangian of a pair (one for each polarization) of real scalar fields $h_{ab}/\sqrt{8\pi G}$.

(g) Gravitational power spectrum (3 points)

From parts (e) and (f) you have found that the gravitational field to be quantized is

$$\frac{a h_{ab}}{\sqrt{8\pi G}} \ . \tag{1.15}$$

From equation (1.2), coupled with the argument that the FLRW spacetime can be well-approximated as Minkowski spacetime for modes well inside the horizon, $k|\eta| \gg 1$, conclude that the correct normalization of the forward-in-time contribution to the gravitational field operator is

$$\frac{a h_{ab}}{\sqrt{8\pi G}} = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 + \frac{i}{k|\eta|} \right) \ , \tag{1.16}$$

while the backward-in-time contribution is the complex conjugate of this. Conclude that once a mode has exited the horizon, $|k\eta| \ll 1$, the classical tensor field h_{ab} at wavenumber k asymptotes to

$$h_{ab} \to i \sqrt{\frac{8\pi G H^2}{2k^3}} e^{-ik\eta} \ . \tag{1.17}$$

Conclude that the power spectrum $P_h(k)$ of gravitational waves, summed over the two polarizations, is

$$P_h(k) = \frac{8\pi G H^2}{k^3} \ . \tag{1.18}$$

The corresponding dimensionless power spectrum $\Delta_h^2(k)$, obtained by multiplying by $4\pi k^3/(2\pi)^3$, is

$$\Delta_h^2(k) = \frac{4 G H^2}{\pi} \ . \tag{1.19}$$

(h) Dependence on the potential energy of the inflaton field (2 points)

During early inflation, the energy density in the inflaton field φ equals the potential energy of the field, $\rho = V(\varphi)$. How does the power spectrum (1.19) of tensor modes depend on the potential energy V of the inflaton field during inflation? [Hint: Recall that the Hubble parameter satisfies $H^2 = 8\pi G\rho/3$.]

(i) Tensor tilt (2 points)

It is conventional in the cosmological community to define the slow-roll inflationary parameter $\epsilon_{\rm i}$ by the logarithmic slope of the Hubble parameter H during inflation,

$$\epsilon_{\rm i} \equiv -\frac{d \ln H}{d \ln a} \ . \tag{1.20}$$

The parameter $\epsilon_{\rm i}$ depends on the unknown shape of the potential $V(\varphi)$ of the scalar inflation field φ that drove inflation. To solve the horizon problem, the Universe must inflate for many e-folds before inflation comes to an end, so the parameter $\epsilon_{\rm i}$ must be small, and probably positive as long as the inflationary potential bends downward. The tensor tilt $n_{\rm t}$ is defined to be the logarithmic slope of the dimensionless power spectrum with respect to wavenumber k at the approximate scale at which modes being observed exit the horizon,

$$n_{\rm t} \equiv \frac{d \ln \Delta_h^2(k)}{d \ln k} \ . \tag{1.21}$$

Argue that

$$n_{\rm t} = -2\epsilon_{\rm i} \ , \tag{1.22}$$

which is expected to be small and probably negative.