ASTR 5770 Cosmology Fall 2025. Problem Set 8. Due Wed Nov 5

1. Behavior of radiation (20 points)

(a) Generic behaviour of radiation

This is essentially Exercise 30.9 in the book. Before recombination, photons are tightly coupled to baryons through non-relativistic electron-photon (Thomson) scattering. The photon-baryon fluid thus behaves as a single energy-momentum conserving fluid. In the simple limit of negligible baryon density, the photon-baryon fluid can be treated as a relativistic fluid with $p/\rho = 1/3$. The sound speed c_s in the relativistic fluid is

$$c_s = \sqrt{\frac{p}{\rho}} = \sqrt{\frac{1}{3}} \ . \tag{1.1}$$

The sound horizon distance η_s is defined in terms of the horizon distance η to be

$$\eta_s \equiv c_s \eta = \frac{\eta}{\sqrt{3}} \ . \tag{1.2}$$

The equations of conservation of energy and momentum of the radiation lead to the result that, in terms of the sound horizon distance η_s, the monopole radiation fluctuation Θ_0 at comoving wavenumber k driven by scalar gravitational potentials Ψ and Φ satisfies the second order differential equation

$$\left(\frac{d^2}{d\eta_s^2} + k^2 \right) (\Theta_0 - \Phi) = - k^2 (\Psi + \Phi) \ . \tag{1.3}$$

Find the homogeneous solutions of equation (1.3), that is, the solutions for $\Theta_0 - \Phi$ with zero source on the right hand side of equation (1.3). Hence find the retarded Green's function of the equation. Write down the general solution of equation (1.3) for $\Theta_0 - \Phi$ as an integral over the Green's function of the source $- k^2(\Psi + \Phi)$. [Hint: You might like to use abbreviated symbols. I used $\Theta \equiv \Theta_0 - \Phi$, $y \equiv k\eta_s$, and $F \equiv \Psi + \Phi$.]

(b) Radiation-dominated epoch

Prior to matter-radiation equality, there is a regime where radiation dominates both the mean energy density and its fluctuation from the mean. During this regime, the quadrupole pressure is small, so the two scalar gravitational potentials are equal to a good approximation,

$$\Psi = \Phi \ . \tag{1.4}$$

The Einstein equations coupled to the radiation yield a second order differential equation for the potential Φ,

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0 \ . \tag{1.5}$$

Confirm that the growing and decaying solutions to equation (1.5) are

$$\Phi_{\rm grow} = \frac{3j_1(y)}{y} = \frac{3(\sin y - y\cos y)}{y^3} \ , \tag{1.6a}$$

$$\Phi_{\rm decay} = -\frac{j_{-2}(y)}{y} = \frac{\cos y + y\sin y}{y^3} \ , \tag{1.6b}$$

where $y \equiv k\eta_s$, and $j_\ell(y) \equiv \sqrt{\pi/(2y)}J_{\ell+1/2}(y)$ are spherical Bessel functions.

(c) Radiation in the radiation-dominated epoch

The Einstein energy equation implies the initial condition that

$$\Theta_0(0) = -\tfrac{1}{2}\Phi(0) \ . \tag{1.7}$$

From your Green's function solution in part (a), find the solution for $\Theta_0 - \Phi$ that is finite at $\eta = 0$. Plot your solution for $\Theta_0 - \Phi$ as a function of y. Also plot the source -2Φ. Your plot should reveal the fact that $\Theta_0 - \Phi$ oscillates about -2Φ.

(d) Comment

Comment on your result. How do the gravitational potential Φ and temperature monopole Θ_0 evolve once a mode is inside the horizon? Can you come up with a physical explanation of what is going on?