ASTR 5770 Cosmology Fall 2025. Problem Set 7. Due Wed Oct 22

1. Neutron freeze-out (12 points)

This problem is essentially Dodelson & Schmidt (2020) "Modern Cosmology" Ch.4 Exercises 4.4 and 4.5. You are welcome to try going through their problem 4.4, which guides you through the derivation of the rate coefficient (1.12) below (their eq. (4.60)), which involves approximating electrons as relativistic and protons and neutrons as non-relativistic, and integrating the Boltzmann equation over momenta of the particles, but this problem forgoes those tricky details in the interests of seeing how the big picture works out.

The dominant reactions connecting neutrons n and protons p in the early universe were

$$
\begin{aligned}
n + \nu &\leftrightarrow p + e \ , && (1.1a) \\
n + \bar{e} &\leftrightarrow p + \bar{\nu} \ , && (1.1b) \\
n &\leftrightarrow p + e + \bar{\nu} \ , && (1.1c) \\
n + \bar{e} + \nu &\leftrightarrow p \ . && (1.1d)
\end{aligned}
$$

For all these reactions, the quantum-mechanical amplitude-squared $|\mathcal{M}|^2$ is the dimensionless quantity (Hayes (2012) "Neutron Beta-Decay")

$$
|\mathcal{M}|^2 \approx 32 G_F^2 (1 + 3 g_A^2) E_n E_p E_e E_\nu \ , \tag{1.2}
$$

(replace E_e by $E_{\bar{e}}$ and E_ν by $E_{\bar{\nu}}$ as appropriate) where $G_F = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi coupling constant, $g_A = 1.27$ is the axial-vector coupling, and, for non-relativistic neutrons and protons, $E_n \approx E_p \approx m_p = 938\,\mathrm{MeV}$ is the proton mass. At low densities, collisions involving 3 particles are rare, and the third process (1.1c) leads to free neutron decay, which occurs at a rate $dn_n/dt = -n_n/\tau_n$ with experimental neutron lifetime (Callahan 2018 arXiv:1810.00958)

$$
\tau_n = (877.7 \pm 1)\,\mathrm{s} \ . \tag{1.3}
$$

In fact the neutron lifetime (1.3) provides the most precise measurement of the constant $G_F^2(1 + 3g_A^2)$ that goes into amplitude-squared (1.2). Because all four processes (1.1) are related, all rate coefficients can be expressed in terms of the neutron lifetime τ_n.

The mass difference Q between neutrons and protons,

$$
Q = m_n - m_p = 1.2933\,\mathrm{MeV} \ , \tag{1.4}
$$

set the approximate temperature at which nucleosynthesis began. At that temperature the Universe was radiation-dominated, with an effective entropy-weighted number of relativistic particle species, Table 10.4,

$$
g_s = 2 + \tfrac{7}{8} 10 = 10.75 \ , \tag{1.5}
$$

the relativistic species consisting of 2 bosons (photons) plus 10 fermions (6 neutrinos, 2 electrons, 2 positrons). The Hubble parameter H at temperature $T = Q$ was (go ahead, check it)

$$
H(Q) = \sqrt{\frac{4\pi^3 G Q^4 g_s}{45 c^5 \hbar^3}} = 1.13\,\mathrm{s}^{-1} \ . \tag{1.6}
$$

(a) Neutron fraction

The expansion rate (1.6) at the onset of nucleosynthesis was fast enough compared the neutron decay time (1.3) that during this time the first two reactions (1.1a) and (1.1b) dominated. For these two reactions, integrating the Boltzmann equation over neutron and proton momenta yields an evolution equation for the proper neutron number density n_n,

$$\frac{1}{a^3}\frac{dn_n a^3}{dt} = -\, n_n \lambda_{np}(T) + n_p \lambda_{pn}(T) \ , \tag{1.7}$$

in which the inverse rate λ_{pn} for $p \to n$ is related to the rate λ_{np} for $n \to p$ by detailed balance,

$$\frac{\lambda_{pn}(T)}{\lambda_{np}(T)} = \left.\frac{n_n}{n_p}\right|_{\mathrm{TE}} = \frac{g_n}{g_p} e^{-Q/T} = e^{-Q/T} \ . \tag{1.8}$$

Define the neutron number fraction X_n by

$$X_n \equiv \frac{n_n}{n_p + n_n} \ . \tag{1.9}$$

Introduce

$$a \equiv \frac{Q}{T} \ , \tag{1.10}$$

which is the cosmic scale factor normalized to 1 at a temperature $T = Q$. Assume that the universe is radiation-dominated. Argue that equation (1.7) becomes

$$\frac{dX_n}{da} = \frac{a \lambda_{np}(Q/a)}{H(Q)} \left[-\, X_n + (1 - X_n) e^{-a} \right] \ . \tag{1.11}$$

(b) Integrate numerically

Dodelson & Schmidt, eq. (4.60), obtain an approximate value for the rate λ_{np} in the approximation that not only neutrinos but also electrons are relativistic,

$$\lambda_{np} \approx \frac{255}{\tau_n a^5}(12 + 6a + a^2) \ \mathrm{s}^{-1} \ . \tag{1.12}$$

Integrate equation (1.11) numerically to find X_n as a function of a. What is the value of X_n at large a? [Comment: This calculation neglects free neutron decay (1.1c), which modifies the result somewhat. The justification for the neglect of free neutron decay is that the free neutrons are mostly synthesized into ^{4}He before they can decay.]

2. He-4 mass fraction (8 points)

(a) Relation between He-4 mass fraction and neutron number fraction

Define the ^{4}He mass fraction Y by

$$Y \equiv \frac{4 n_{^4\mathrm{He}}}{n_{\mathrm{H}} + 4 n_{^4\mathrm{He}}} \ . \tag{2.1}$$

Assume that H and ^{4}He dominate the abundance of elements after nucleosynthesis, and that all neutrons ultimately go into ^{4}He. What is the relation between Y and the neutron number fraction X_n defined by equation (1.9)? What is the ^{4}He mass fraction predicted by the neutron fraction you obtained in question 1(b)?

(b) Compare to observed He-4 mass fraction

Compare to the observed ^{4}He mass fraction obtained by Cyburt et al. (2015) "Big Bang Nucleosynthesis: 2015" arXiv.1505.01076. You should find that your estimate is a bit higher; the difference comes from your neglect of free neutron decays, which reduce the actual neutron density compared to your estimate.