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ASTR 5770 Cosmology Fall 2023. Problem Set 2.

1 Equations of motion in weak gravity

This is *the end of* Exercise 2.18. Consider the Newtonian metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) \ , \tag{1}$$

where $\Phi(x, y, z)$ is the familiar Newtonian gravitational potential, a function only of the spatial coordinates x, y, z, not of time t.

1.1 Equation of motion of a massless particle

From the geodesic equation

$$\frac{dv^\kappa}{d\lambda} + \Gamma^\kappa_{\mu\nu} v^\mu v^\nu = 0 \tag{2}$$

show that the spatial components $\boldsymbol{v} \equiv \{v^x, v^y, v^z\}$ satisfy

$$\frac{d\boldsymbol{v}}{d\lambda} = 2\,\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{\nabla}\Phi) \tag{3}$$

where boldface symbols represent 3D vectors, and in particular $\boldsymbol{\nabla}\Phi$ is the spatial 3D gradient $\boldsymbol{\nabla}\Phi \equiv \partial\Phi/\partial x^\alpha = \{\partial\Phi/\partial x, \partial\Phi/\partial y, \partial\Phi/\partial z\}$. [Hint: Recall that the 3D vector triple product satisfies $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.]

1.2 Interpret

Interpret your answer, equation (3). In what ways does this equation for the acceleration of photons differ from the equation governing the acceleration of massive particles? [Hint: Without loss of generality, the affine parameter can be normalized so that the photon speed is one, $v = 1$, so that $\boldsymbol{v}$ is a unit vector representing the direction of the photon.]

2 Deflection of light by the Sun

This is Exercise 2.18.

2.1 Born approximation

Consider light that passes by a spherical mass M sufficiently far away that the potential Φ is always weak. The potential at distance r from the spherical mass can be approximated by the Newtonian potential

$$\Phi = -\frac{GM}{r} \ . \tag{4}$$

Approximate the unperturbed path of light past the mass as a straight line. The plan is to calculate the deflection as a perturbation to the straight line (physicists call this the Born approximation). For definiteness, take the light to be moving in the x-direction, offset by a constant amount y away from the mass in the y-direction (so y is the impact parameter, or periapsis). Argue that equation (3) becomes

$$\frac{dv^y}{d\lambda} = v^x \frac{dv^y}{dx} = -2\,(v^x)^2 \frac{\partial \Phi}{\partial y} \ . \tag{5}$$

Integrate this equation to show that

$$\frac{\Delta v^y}{v^x} = -\frac{4GM}{y} \ . \tag{6}$$

Argue that this equals the deflection angle $\Delta\phi$.

2.2 Deflection of light by the Sun

Calculate the predicted deflection angle $\Delta\phi$ in arcseconds for light that just grazes the limb of the Sun.

2.3 Compare to observations

A remarkable test of the prediction was accomplished recently from observations of the 21 August 2017 Eclipse by amateur astronomer and CU alumnus Don Bruns (Physics & Mathematics 1974), as reported in Physics Today, August 2018. Briefly, what did Don Bruns find?

3 Gravitational lensing

In Exercise 2 you found that, in the weak field limit, light passing a spherical mass M at impact parameter y is deflected by angle

$$\Delta\phi = \frac{4GM}{yc^2} \ . \tag{7}$$

1. **Lensing equation.**

   Argue that the deflection angle $\Delta\phi$ is related to the angles α and β illustrated in the lensing diagram in Figure 1 by

$$\alpha D_{\mathrm{S}} = \beta D_{\mathrm{S}} + \Delta\phi D_{\mathrm{LS}} \ . \tag{8}$$

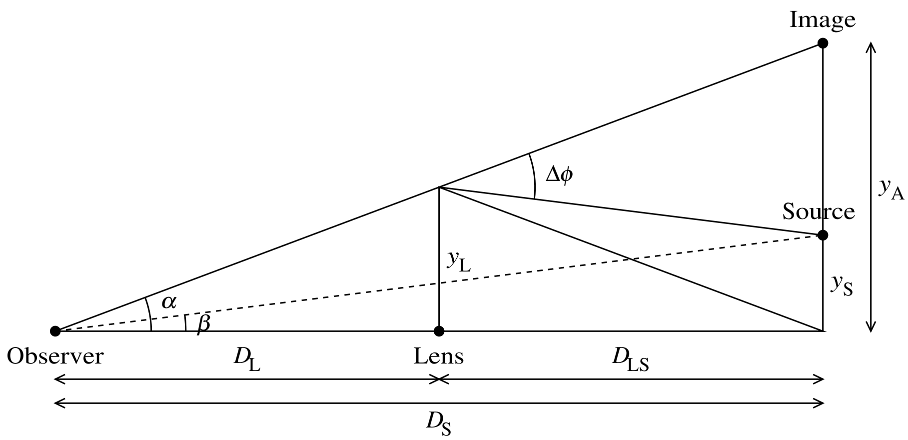

Figure 1: Lensing diagram.

Hence or otherwise obtain the "lensing equation" in the form commonly used by astronomers

$$\beta = \alpha - \frac{\alpha_{\mathrm{E}}^2}{\alpha} \ , \tag{9}$$

where

$$\alpha_{\mathrm{E}} = \left(\frac{4GM}{c^2} \frac{D_{\mathrm{LS}}}{D_{\mathrm{L}} D_{\mathrm{S}}} \right)^{1/2} \quad . \tag{10}$$

2. **Solutions.** Equation (9) has two solutions for the apparent angles α in terms of β. What are they? Sketch both solutions on a lensing diagram similar to Figure 1.
3. **Magnification.** Figure 2 illustrates the appearance of a finite-sized source lensed by a point gravitational lens. If the source is far from the lens, then the source redshift is unchanged by the gravitational lensing. But the distortion changes the apparent brightness of the source by a magnification μ equal to the ratio of the apparent area of the lensed source to that of the unlensed source. For a small source, the ratio of areas is

$$\mu = \frac{y_{\mathrm{A}} \, dy_{\mathrm{A}}}{y_{\mathrm{S}} \, dy_{\mathrm{S}}} \ . \tag{11}$$

What is the magnification of a small source in terms of α and α_{E}? When is the magnification largest?
4. **Einstein ring around the Sun?** The case $\alpha = \alpha_{\mathrm{E}}$ evidently corresponds to the case where the source is exactly behind the lens, $\beta = 0$. In this case the lensed source appears as an "Einstein ring" of light around the lens. Could there be an Einstein ring around the Sun, as seen from Earth?
5. **Einstein ring around Sgr A*.** What is the maximum possible angular size of an Einstein ring around the $4 \times 10^6 \, \mathrm{M}_\odot$ black hole at the center of our Milky Way, 8 kpc away? Might this be observable?

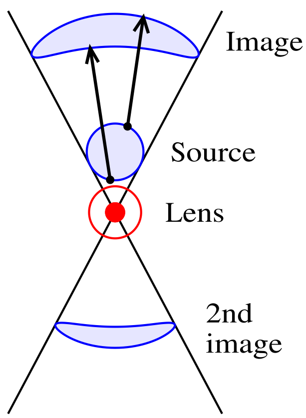

Figure 2: The appearance of a source lensed by a point lens. The lens in this case is a black hole, whose physical size is the filled circle, and whose apparent (lensed) size is the surrounding unfilled circle. However, any mass, not just a black hole, will lens a background source.