ASTR 3740 Relativity & Cosmology Spring 2025. Problem Set 7.
Due Wed 23 Apr

1. Horizon Problem

(a) (3 points) Expansion factor

The temperature of the CMB today is $T_0 \approx 3\,\mathrm{K}$. By what factor has the Universe expanded (i.e. what is a_0/a) since the temperature was the GUT (Grand Unified Theory) temperature $T \approx 10^{29}\,\mathrm{K}$? [Hint: Argue that $T \propto a^{-1}$ during the expansion of the Universe.]

(b) (3 points) Hubble distance

By what factor has the Hubble distance c/H increased during the expansion of part (a)? Assume that the Universe has been mainly radiation-dominated during this period, and that the Universe is flat. [Hint: For a flat Universe $H^2 \propto \rho$, and for radiation-dominated Universe $\rho \propto a^{-4}$.]

(c) (3 points) Comoving Hubble distance

Hence determine by what factor the comoving Hubble distance $x_H = c/(aH)$ has increased during the expansion of part (a).

(d) (3 points) Comoving Hubble distance during inflation

During inflation the Hubble distance c/H remained constant, while the cosmic scale factor a expanded exponentially. What is the relation between the comoving Hubble distance $x_H = c/(aH)$ and cosmic scale factor a during inflation? [You should obtain an answer of the form $x_H \propto a^?$.]

(e) (3 points) e-foldings to solve the Horizon Problem

By how many e-foldings must the Universe have inflated in order to solve the Horizon Problem? Assume again, as in part (a), that the Universe has been mainly radiation-dominated during expansion from the GUT temperature to the current temperature, and that this radiation-dominated epoch was immediately preceded by a period of inflation. [Hint: Inflation solves the Horizon Problem if the currently observable Universe was within the Hubble distance at the beginning of inflation, i.e. if the comoving $x_{H,0}$ now is less than the comoving Hubble distance $x_{H,\mathrm{i}}$ at the beginning of inflation. The 'number of e-foldings' is $\ln(a_\mathrm{f}/a_\mathrm{i})$, where ln is the natural logarithm, and a_i and a_f are the cosmic scale factors at the beginning (i for initial) and end (f for final) of inflation.]

2. Flatness Problem

An amusing statement of this cosmological problem can be found on Ned Wright's graph at https://astro.ucla.edu/~wright/cosmo_03.htm#FO .

(a) (3 points) Friedmann's equation

Use the definitions $H^2 = \frac{8}{3}\pi G\rho_c$ of the critical density ρ_c, and $\Omega \equiv \rho/\rho_c$ of Omega, to show that Friedmann's equation (including the curvature term)

$$H^2 = \frac{8}{3}\pi G\rho - \frac{\kappa c^2}{a^2} \tag{2.1}$$

can be rewritten in the form

$$\frac{\Omega - 1}{\Omega} = \frac{3\kappa c^2}{8\pi G\rho a^2} \; . \tag{2.2}$$

(b) (3 points) Evolution of Ω with a

Suppose once again that $\rho \propto a^{-n}$. Show that (a simple consequence of eq. (2.2))

$$\frac{\Omega - 1}{\Omega} \propto a^{?} \tag{2.3}$$

where ? is an exponent which you should derive (in terms of n).

(c) (3 points) Here's the flatness problem

Suppose that the temperature at the moment of the Big Bang was about the GUT temperature $\sim 10^{29}$ K. The radiation temperature of the Universe today is of course that of the CMB, about 3 K. If Ω_0 (subscript 0 means the present time) is of order, but not equal to, one at the present time ($\Omega_0 \sim 0.3$, say), roughly how close to one was Ω at the Big Bang? [Hint: Define the small quantity $\epsilon \equiv \Omega - 1$, and use (2.3) to estimate ϵ at the Big Bang. Note that for tiny ϵ, you can approximate $1 + \epsilon \approx 1$. Assume that $T \propto a^{-1}$ during the expansion of the Universe, and for simplicity that the Universe has been radiation-dominated for most of that expansion, so that $n \approx 4$.]

(d) (3 points) Relation between Horizon and Flatness Problems

Show that Friedmann's equation (2.1) can be written in the form

$$\Omega - 1 = \kappa x_H^2 \tag{2.4}$$

where $x_H \equiv c/(aH)$ is the comoving Hubble distance. Use this equation to argue in your own words how the horizon and flatness problems are related. [The main part of this question is not the math but the explanation. You should convince the grader that you understand physically what is going on.]