ASTR 3740 Relativity & Cosmology Spring 2025. Problem Set 6.
Due Wed 16 Apr

1. Anti-gravity

(a) (5 points) Condition for an accelerating Universe

Suppose that the Universe contains only matter energy (M) and vacuum energy (a cosmological constant Λ), and that it is geometrically flat

$$\Omega_M + \Omega_\Lambda = 1 \tag{1.1}$$

where $\Omega_M \equiv \rho_M/\rho_c$ and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$ are the contributions to Omega in matter and vacuum. How big must Ω_Λ be for the Universe to be accelerating? [Hint: Friedmann's equation for the acceleration $\ddot{a} \equiv d^2a/dt^2$ of the cosmic scale factor $a(t)$ is

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \tag{1.2}$$

which shows that the Universe is accelerating if $\rho+3p < 0$. Ordinary matter has mass-energy density ρ_M but essentially no pressure, $p_M = 0$, while vacuum has negative pressure equal to its mass-energy density, $p_\Lambda = -\rho_\Lambda$.]

(b) (5 points) Draw your own conclusion

The final (2018) analysis of data from the Planck satellite, coupled with other CMB data, supernovae, galaxy clustering, and other astrophysical data, indicates $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$ (https://arxiv.org/abs/1807.06209). Is our Universe accelerating?

2. Solutions to Friedmann's equations in a Flat Universe

Suppose that the Universe is flat, $\kappa = 0$, so that Friedmann's energy equation reduces to

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho \ . \tag{2.1}$$

Suppose further that the Universe is dominated by stuff whose mass-energy density ρ varies with cosmic scale factor a as

$$\rho \propto a^{-n} \tag{2.2}$$

as the Universe expands, with n a constant. For example, $n = 3$ for ordinary matter, $n = 4$ for radiation, and $n = 0$ for vacuum energy.

(a) (5 points) Case $n \neq 0$

Solve Friedmann's equation to show that, for $n \neq 0$,

$$a \propto t^{2/n} \ . \tag{2.3}$$

[Hint: You should find that Friedmann's equation can be recast in the form $t = \int f(a)da$ where $f(a)$ is some function of cosmic scale factor a. You may set $a = 0$ at $t = 0$, which says that the Universe had zero size at zero age.]

(b) (5 points) Deceleration or acceleration?

For what range of n is the Universe decelerating ($\ddot{a} < 0$) or accelerating ($\ddot{a} > 0$)? Is the Universe decelerating or accelerating in the particular cases of a matter-dominated ($n = 3$) or radiation-dominated ($n = 4$) Universe?

(c) (5 points) Case $n = 0$

The case $n = 0$ corresponds to vacuum density, which remains constant as the Universe expands. Solve Friedmann's equation for this case to show that

$$a \propto e^{Ht} \tag{2.4}$$

where $H \equiv \dot{a}/a$, the Hubble constant, is in this case a constant in time as well as space. What is the Hubble constant H here in terms of the vacuum energy ρ_Λ?

(d) For your information (no credit)

You may be wondering whether there is a relation between the index n in this question and the pressure p in the Anti-Gravity question. The answer is yes. It is straightforward to show (but I'm not asking you to do this) from the energy equation $d(\rho a^3) + pd(a^3) = 0$ (which you may recognize as the equation $dE + pdV = 0$ of thermodynamics) that

$$n = 3\left(1 + \frac{p}{\rho}\right) \ . \tag{2.5}$$

3. (5 points) Phyics Nobel Prize in Astrophysics & Cosmology

Has the Physics Nobel Prize ever been awarded for work in astrophysics or cosmology? If so, to whom, and for what? [Look it up on the web — and don't forget to refererence your sources.]