II. GENERAL RELATIVITY

Einstein (Nov 1915).

Musings:

- If absolute spacetime does not exist, what gives it its flat Euclidean geometry?
- Can the equations of motion for accelerated observers somehow be the same as those of unaccelerated observers (= "general covariance")?
- Could acceleration and gravity be the same thing?

Postulates of GR

0. Spacetime forms a 4-dimensional continuum. See page 1·22.
1. Existence of locally inertial frames.
2. Principle of Equivalence of gravity and acceleration.
3. Einstein's equations relate geometry of spacetime to its energy-momentum content.

1. Existence of locally inertial frames

Statement:

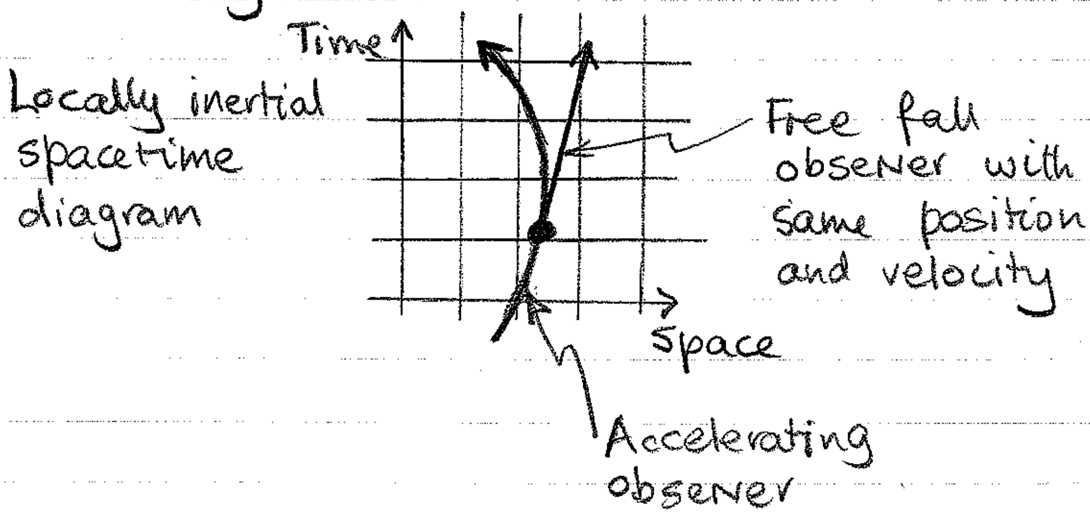
(a) At every point of spacetime there exists a "free-fall", or "locally inertial" frame of reference, in a sufficiently small neighborhood of which objects move according to special relativity,

 (i) objects move in straight lines in the absence of other (e.g. electromagnetic) forces;

 (ii) the speed of light is c;

 (iii) Maxwell's electromagnetic equations operate.

(b) The proper times and distances measured by an accelerated observer are the same as those of an unaccelerated observer instantaneously at the same position and velocity.



- Why do locally inertial frames exist?
  Andrew's opinion:
  Because observers exist, and observers have a locally flat notion of spacetime.

2. Equivalence Principle

(a) Weak Equivalence Principle

Statement:
Gravitational mass = inertial mass

Problem:
$E = mc^2$. So how do different forms of energy gravitate?

(b) Strong Equivalence Principle

Statement:
The effects of gravity are indistinguishable from those of acceleration, in sufficiently small regions of spacetime.

- Postulate of existence of locally inertial frames is often taken to be part of the Strong E.P.

3. Einstein's equations

are :

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{"Einstein tensor" = compressive part of curvature tensor}} = 8\pi G T_{\mu\nu}$$ 10 equations

Newton's gravitational constant

energy-momentum tensor

Each side is a symmetric 4×4 tensor.

- Generalize the Poisson equation of Newtonian gravity

$$\nabla^2 \phi = 4\pi G \rho$$ 1 equation

Newtonian gravitational potential

mass density

$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is "Laplacian"

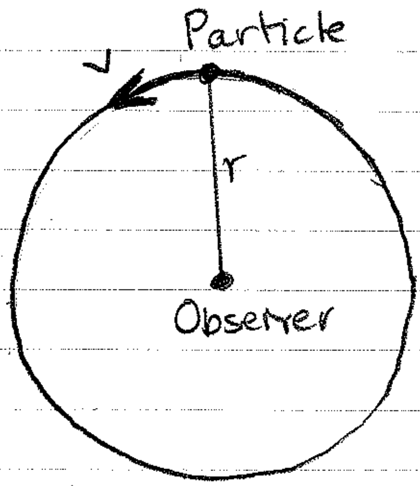
- Einstein's equations do NOT follow from Principle of Equivalence.
  But it is the simplest generally covariant equation consistent with Strong P.E. (Hilbert 1915).

Consequences of Equivalence Principle

1. Gravitational redshift.
2. Gravitational bending of light.
3. Gravity is equivalent to curvature of spacetime.

1. Gravitational redshift

Thought experiment (a).



(Charged) particle in particle accelerator goes in circles at velocity

To observer at center, particle appears time-dilated (slowed) by factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\approx 1 + \frac{1}{2}\frac{v^2}{c^2} \quad \text{for small } v.$$

So particle appears to observer redshifted by $1 + z = \gamma$ transverse Doppler shift

(note no extra light travel time effects, since distance r is constant).

Hence

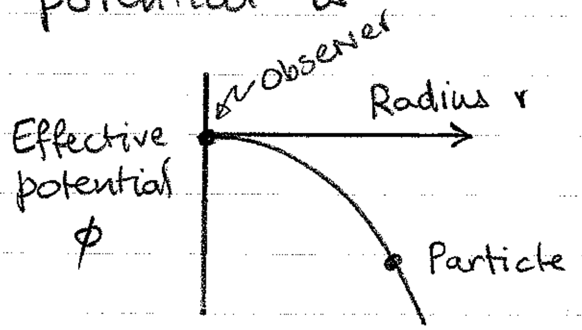
$$z \approx \frac{1}{2}\frac{v^2}{c^2}$$

for small v
(usual transverse Doppler shift for small v).

Accelerating particle experiences "centrifugal force" which by Equivalence Principle is identical to a gravitational force.
The effective (centrifugal) potential is

$$\phi = -\frac{1}{2}\omega^2 r^2$$

ω = angular frequency
(radians / time)



ie $\phi = -\frac{1}{2} v^2$.

Hence

$$z \approx -\frac{\phi}{c^2}$$

More explicitly

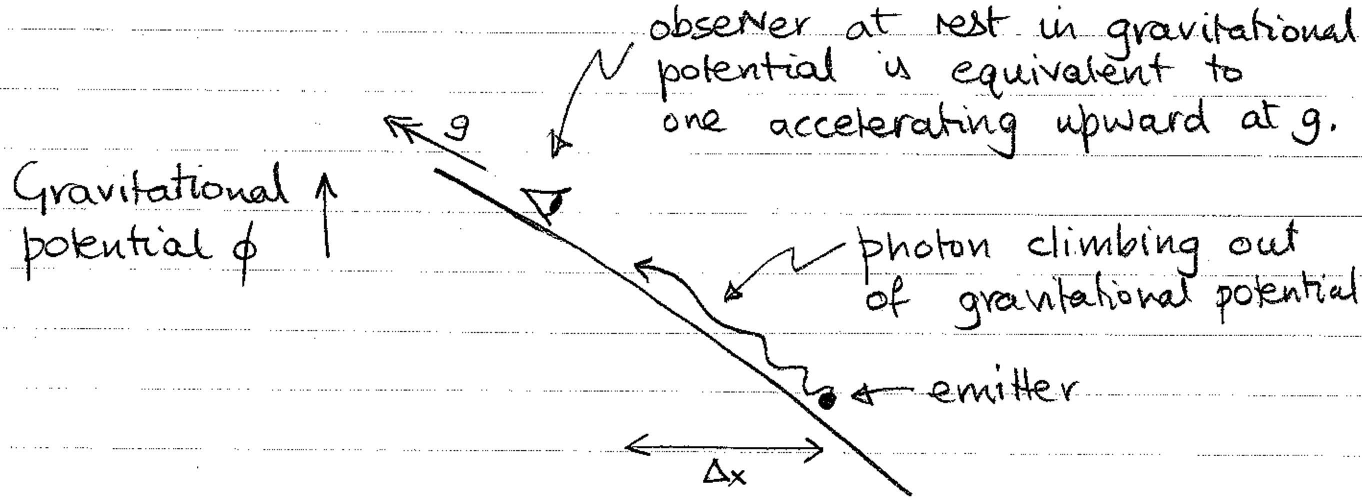
potential at observer
potential at emitter

$$\boxed{z \approx \frac{\phi_{obs} - \phi_{emit}}{c^2}}$$

formula for gravitational redshift in a weak gravitational field.

Gravitational redshift (contd)

Thought experiment (b)



Photon moves distance

$$\Delta x = c\,\Delta t$$

in small time interval Δt.

Observer stationary in gravitational field

$\equiv$ observer accelerating upward at $g = \frac{d\phi}{dx}$

During time Δt, equivalent observer accelerates to

$$v = g\,\Delta t = \frac{d\phi}{dx}\,.\,\Delta t$$

$$= \frac{d\phi}{dx}\,\frac{\Delta x}{c} = \frac{\Delta\phi}{c}$$

So photon appears redshifted by

$$z \approx \frac{v}{c} = \frac{\Delta\phi}{c^2}$$

non-relativistic Doppler shift; OK for weak gravity

ie $\quad z \approx \dfrac{\phi_{obs} - \phi_{emit}}{c^2}$ as before ✓

Recall usual relation between redshift and observed (through telescope) time dilation:

$$1 + z = \frac{\nu_{em}}{\nu_{obs}}$$

$$= \frac{\text{frequency of emitting clock from its own point of view}}{\text{frequency of emitting clock observed through telescope}}$$

(Photons are good clocks!).

So gravitational redshift also implies

$\boxed{\text{clocks run slow in a gravitational field, as perceived by an outside observer}}$

- It is as if photons lose energy climbing out of a gravitational potential

<u>Gravitation redshift of Sun</u>

$$z_\odot = \frac{\phi_\infty - \phi_\odot}{c^2} = \frac{G M_\odot}{c^2 R_\odot} \approx 2 \times 10^{-6}$$

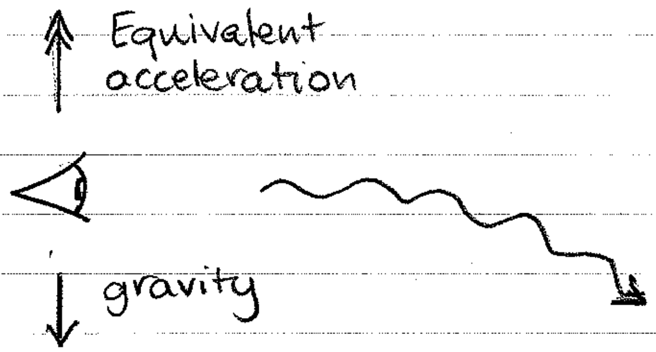
Observed (1991, Astrophysical Journal 376, 757)

$$z_\odot = \frac{655 \text{ m/s}}{299{,}792{,}458 \text{ m/s}} = 2.18 \times 10^{-6}$$

2. Gravitational Bending of Light

Free-fall observer thinks light moves in straight lines:

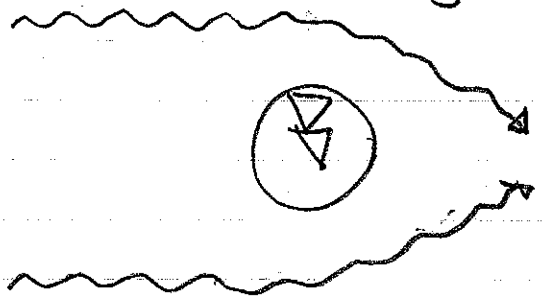
Observer in gravitational field is equivalent to accelerating observer who thinks light travels curved path:



3. Gravity = curvature of spacetime

Equivalence Principle says gravity can be eliminated by moving into free-fall frame; and objects then move on "straight lines" in the free-fall frame. So gravity is "equivalent" to spacetime geometry

But evidently objects moving initially parallel can be caused to converge as they pass by a gravitating mass.

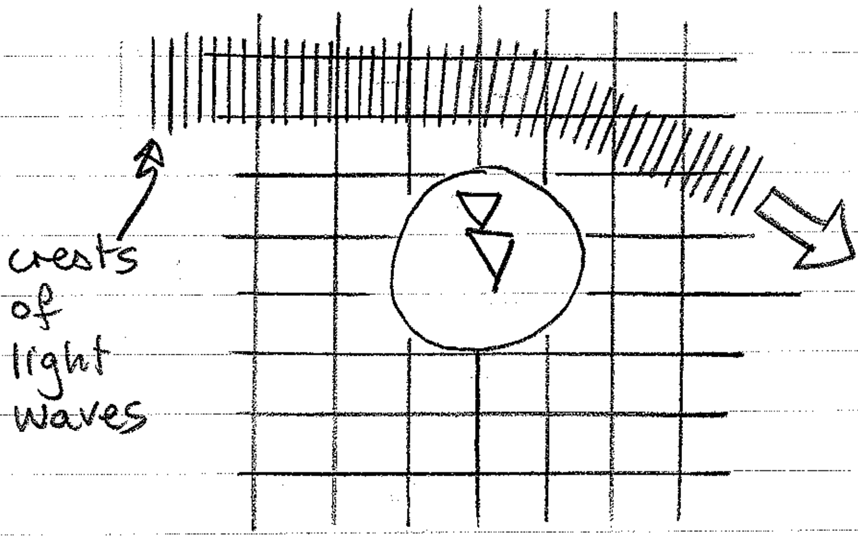


So the geometry <u>isn't</u> (in general)
the geometry of Euclid.

Question remains:
what is geometry of spacetime around a gravitating mass?

1. Einstein's first (wrong) idea

Space is flat, and only time is curved.



Argument:
Time dilation in gravitational field
⇒ Light appears (to observer at infinity) to go slow near gravitating mass
⇒ Light bends around mass.

Problem:
Gives special status to "time" dimension.

2. Einstein's eventual (correct) idea

- Space as well as time is curved.

- Gravitational equations are fully "generally covariant"
  – they take the same form in any frame of reference, accelerating or not, in presence of gravity or not.

- Momentum as well as energy gravitates.

- For photons,

$$|\mathbf{p}| = E$$

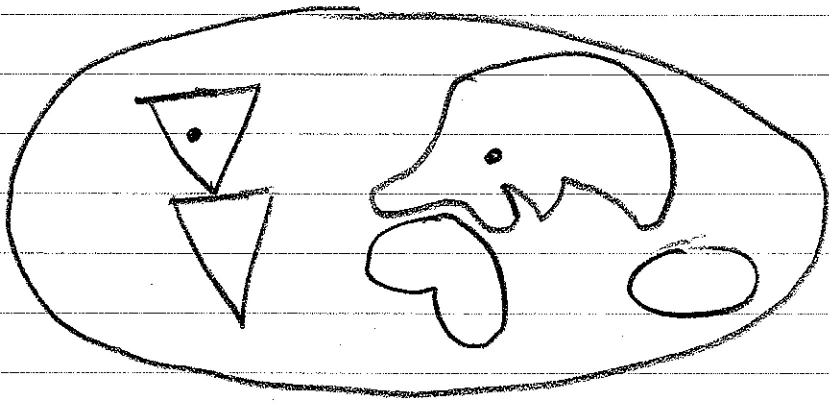
  so photons gravitate $\boxed{\text{twice}}$ as much as ordinary matter.

- Result:

  $\boxed{\text{Einstein's equations}}$

Geodesics in curved 2D space

= "straight lines"
= shortest distance between two points



What is geodesic between Boulder and Dushanbe, Tajikistan over surface of $\oplus$?

2D metric

How are distances measured in curved space?

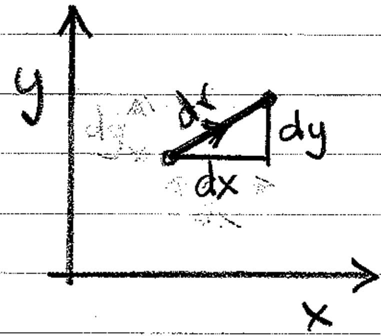
Consider first two infinitesimally separated points.

Example (a) : Flat surface

Choose Cartesian coordinates x, y.
Then distance dr between points

separated by dx, dy is given by

$$dr^2 = dx^2 + dy^2$$



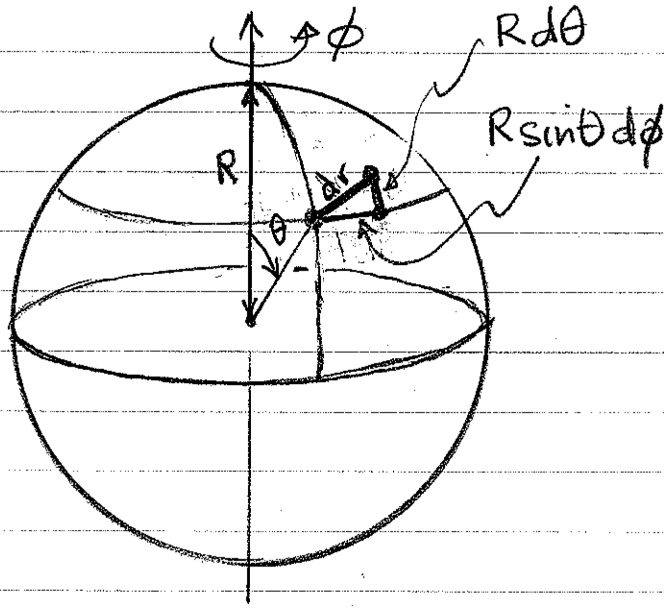
Example (b): Surface of sphere

Choose polar coordinates θ, ϕ.
Distance dr between points
separated by $d\theta$, $d\phi$ is given by

$$dr^2 = R^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

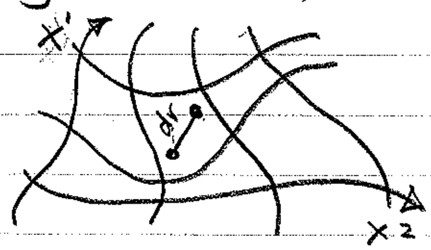
R = radius of sphere

notice this varies with θ



In general 2D curved space:

Choose arbitrary coordinate system x^1, x^2.
Then distance dr between
points separated by dx^1, dx^2
given by



$$dr^2 = g_{11}\,dx^1 dx^1 + 2 g_{12}\,dx^1 dx^2 + g_{22}\,dx^2 dx^2$$

$$= g_{ij}\,dx^i dx^j \qquad \text{(implicit sum over } i,j = 1,2 \text{)}$$

Quantity g_{ij} is the $\boxed{\text{metric}}$.

- g_{ij} is a symmetric ($g_{ij} = g_{ji}$) 2×2 matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ in 2D
- in general, values of g_{ij} at different points (x^1, x^2) are different.

Example (a): Flat surface

If take Cartesian coordinates $x^1 = x$, $x^2 = y$
then

$$g_{11} = g_{xx} = 1$$
$$g_{22} = g_{yy} = 1$$
$$g_{12} = g_{xy} = 0$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ is unit matrix

Example (b): Surface of sphere

If take polar coordinates $x^1 = \theta$, $x^2 = \phi$
then

$$g_{11} = g_{\theta\theta} = R^2$$
$$g_{22} = g_{\phi\phi} = R^2 \sin^2\theta$$
$$g_{12} = g_{\theta\phi} = 0$$

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

General coordinate transformations

There are (infinitely) many ways to choose coordinates on a surface. But the scalar distance dr is by definition invariant.

If have two coordinate systems x^i and x'^i say then by usual chain rule of calculus

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \quad \text{(implicit summation over } j = 1,2\text{)}$$

The scalar distance dr is given by

$$dr^2 = g'_{ij}\, dx'_i\, dx'_j$$

metric in primed frame

$$= g'_{ij} \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_\ell} dx^k dx^\ell$$

$$= g_{k\ell}\, dx^k dx^\ell$$

metric in unprimed frame

whence

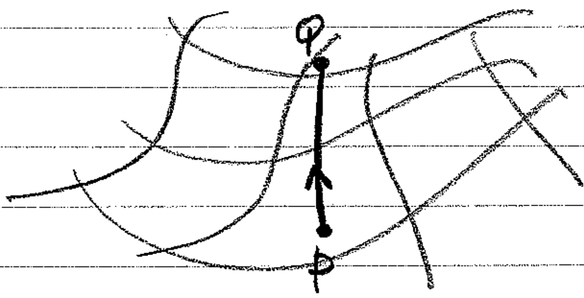
$$g_{k\ell} = g'_{ij} \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_\ell}$$

relates metrics in unprimed and primed coordinate systems.

Back to geodesics on 2D surface

A geodesic between 2 fixed points P & Q is that line which minimizes the distance

$$r = \int_P^Q dr$$



The distance r is a **scalar**, independent of coordinate system.

so this is a generally covariant statem

The metric g_{ij} provides the mathematical means

Comments on 2D metric

- Metric translates weird coordinates into observables : proper distances.
- Can use any coordinate system; but some show more clearly what's going on.
- Can't flatten a sphere; can't choose coordinates x, y on surface of a sphere such that $dr^2 = dx^2 + dy^2$ everywhere; though can do that locally.
- Metric is an intrinsic property of surface; flatlanders confined to surface of sphere can determine metric, without leaving surface (without going into 3D).

Geodesics and the metric in Relativity

Scalar distance dr in ordinary space must be replaced by a scalar spacetime distance ds in relativity.

Special Relativity (flat spacetime)

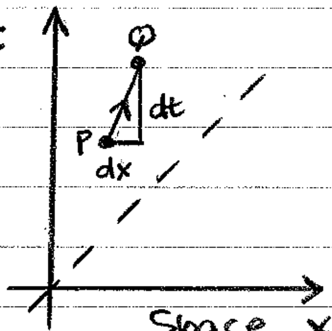
Scalar spacetime distance ds between two points separated by infinitesimal dt, dx, dy, dz is given by

$$ds^2 = -dt^2 + dr^2$$
$$= -dt^2 + dx^2 + dy^2 + dz^2$$

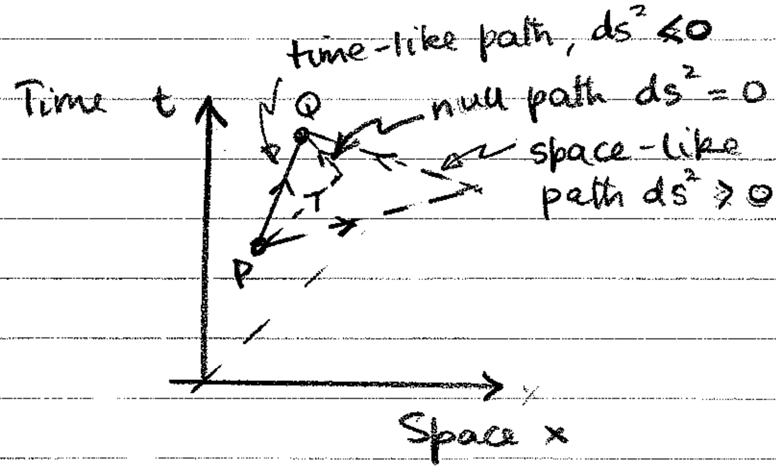
Particles move between two fixed events P, Q in such a way as to minimize

$$-ds^2 = -dt^2 + dr^2$$
$$= -dt^2 + dx^2 + dy^2 + dz^2$$

or equivalently to maximize $-ds^2$.



"The static person ages most".



General Relativity

Choose arbitrary coordinate system $x^\lambda = (x^0, x^1, x^2, x^3)$.
Scalar spacetime distance ds between two points separated by infinitesimal $dx^\lambda = (dx^0, dx^1, dx^2, dx^3)$ is given by

$$\boxed{ds^2 = g_{\lambda\mu}\, dx^\lambda dx^\mu}$$

(implicit summation over $\lambda, \mu = 0,1,2,3$).

Quantity $g_{\lambda\mu}$ is the $\boxed{\text{metric}}$

- $g_{\lambda\mu}$ is symmetric ($g_{\lambda\mu} = g_{\mu\lambda}$) 4×4 matrix

$$g_{\lambda\mu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$

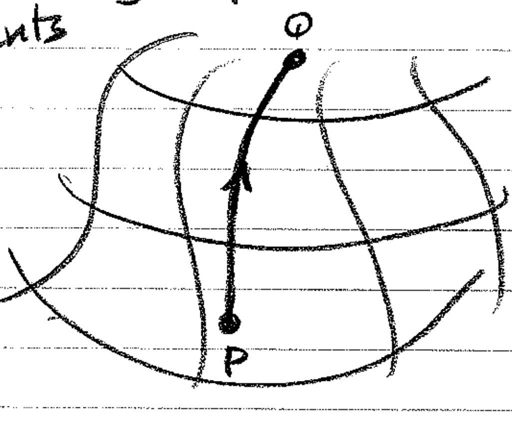
- It has 10 independent components.

Particles move on geodesics

In general (and special) relativity, particles move between two fixed events P & Q so as to maximize

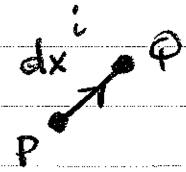
$$S = \int_P^Q \sqrt{-ds^2}$$

Such lines are geodesics in GR



Proper time and distance in GR

Consider a 4-interval dx^i between two infinitesimal separated events P & Q.



As usual, $ds^2 = g_{\lambda\mu}\, dx^\lambda dx^\mu$.

Exactly as in SR:

$$ds^2 \begin{cases} < & & \text{time-like interval} \\ = & 0 & \text{null, or light-like interval} \\ > & & \text{space-like interval} \end{cases}$$

Timelike interval: $(ds^2 < 0)$

$|ds| = \sqrt{-ds^2}$ = proper time between P & Q
= time experienced by observer (freely-falling) between P & Q

Spacelike interval $(ds^2 > 0)$

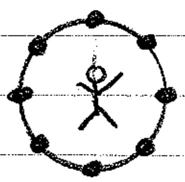
$|ds|$ = proper distance between P & Q measured by (freely-falling) observer for whom P & Q happen simultaneously.

CURVATURE

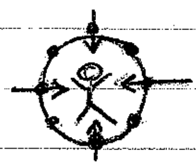
Q: Can space be curved inside a region where there is no mass (there may be mass outside the region)?

How can you tell if you are in gravitating system?

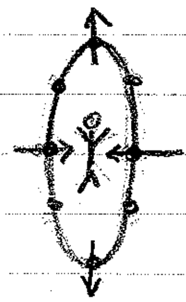
Go into free fall frame.
Set up small sphere of particles in free fall around you, all initially with zero velocity

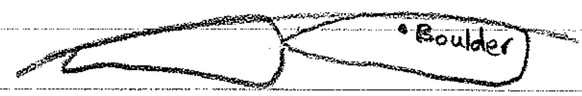


(a) If sphere starts to contract, you conclude it contains mass.

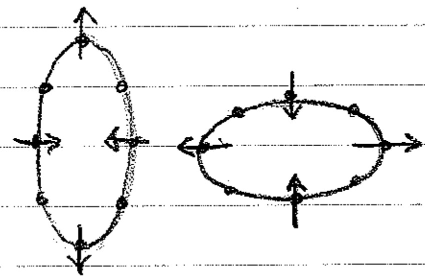


(b) If sphere starts to distort tidally, without changing volume, you conclude there is mass off to one side.





(b′) If sphere oscillates tidally, you conclude a gravitational wave is going by.



How do linelanders (1D space + 1D time) know they are in a gravitating system?

They do the same thing – in lineland.

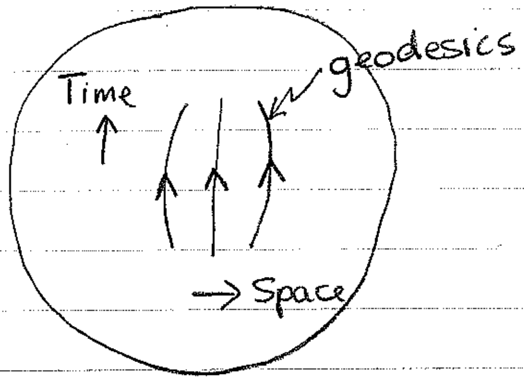
Linelander in free fall sets up 1D sphere (ie 2 particles) initially at rest about him.

(a) If "sphere" of 2 particles starts to contract, linelander concludes sphere contains mass.

(b) No tides in lineland.

Spacetime diagram of lineland gravitating system

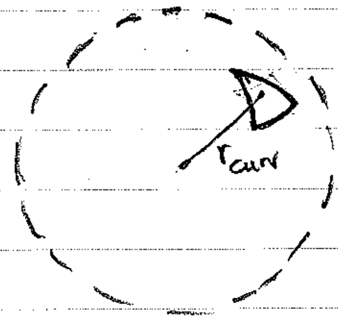
Convergence of geodesics looks like curvature of spacetime.

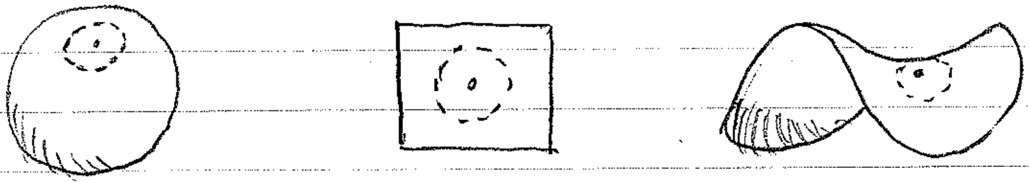


The curvature of a 2D surface at any point is characterized by single quantity:

$$\text{Curvature} = \frac{1}{r_{curv}^2}$$

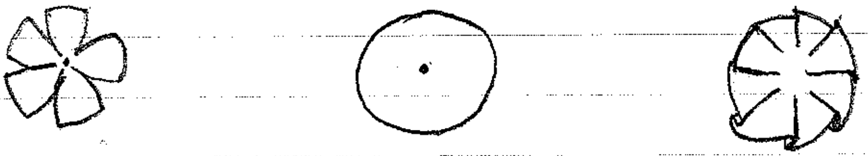
r_{curv} = radius of curvature





Positive curvature | Flat | Negative curvature

Person confined to surface can measure its curvature without leaving surface



$< 2\pi$ | $= 2\pi$ | $> 2\pi$

from $\dfrac{\text{circumference}}{\text{radius}}$ of small circle.

Curvature in 4D

Existence of locally inertial frames.
$\Longrightarrow$ at any point there exists a frame where
(a) metric is flat, $g_{\kappa\lambda} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) the derivatives of the metric are zero
$$\frac{\partial g_{\kappa\lambda}}{\partial x^{\mu}} = 0$$
(to ensure that the metric is flat "in a small neighborhood" of the point".

The curvature is related (but not equal) to the matrix of second derivatives $\frac{\partial^2 g_{\kappa\lambda}}{\partial x^{\mu}\partial x^{\nu}}$ of metric.

The RIEMANN curvature tensor $R_{\kappa\lambda\mu\nu}$ ~ units $\frac{1}{\text{length}^2}$ is the unique generally covariant tensor which can be constructed from the metric and its first and second derivatives
(Want a more precise statement? It's the commutator of the covariant derivative.)

Related tensors:

RICCI tensor $R_{\lambda\nu} \equiv R^{\kappa}{}_{\lambda\kappa\nu}$

Curvature scalar $R \equiv R^{\lambda}{}_{\nu}$

Can show:

(a) RIEMANN $\overset{\text{essentially}}{=}$ RICCI + WEYL

RIEMANN	RICCI	WEYL
	compressive part	tidal part
20 components	10 cpts	10 cpts

(b) RICCI $\Longleftrightarrow$ WEYL

are related by "Bianchi identities", which are relations between derivatives of RIEMANN.

Einstein Field Equations

Schematically,

RICCI = G × ENERGY-MOMENTUM DENSITY

Newton's gravitational constant

compressive part of curvature

10 equations

Important point to understand:

$$\boxed{\text{Geometry of spacetime} \Longleftrightarrow \text{energy-momentum content of spacetime}}$$

Recap of fundamental assertions of GR

Q: How do objects move in a gravitational field?

A: They go on "straight lines" = <u>geodesics</u> in free fall frames.

Q: What determines the gravitational field?

A: Einstein's equations.

One of the most important examples of a nontrivial geometry in GR is

The Karl Schwarzschild (1916) geometry

describes the geometry of empty space outside a spherically symmetric mass M.

Schwarzschild metric is

$$\boxed{ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

where r_s is the Schwarzschild radius

$$\boxed{r_s = \frac{2GM}{c^2}}$$

Q: How do you derive this Schwarzschild?

A: Solve Einstein's equations – but not now.

Wrong – but easy to remember – derivation of Schwarzschild radius (John Michell 1783)

Classically

$$\frac{1}{2} v_{esc}^2 = \phi = \frac{GM}{r}$$

v_{esc} = grav. escape velocity

gravitational potential radius r from mass M

Set $v_{esc} = c$, then

$$\frac{1}{2}c^2 = \frac{GM}{r_s}$$

$$\Rightarrow \quad r_s = \frac{2GM}{c^2} \quad \text{as claimed.}$$

Let's explore what Schwarzschild metric means...

1. Schw geometry is stationary — doesn't change with time

How do you know that?
Because the metric coefficients

$$g_{tt} = -\left(1 - \frac{r_s}{r}\right), \quad g_{rr} = \frac{1}{1 - \frac{r_s}{r}}$$

$$g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta$$

$$g_{other} = 0$$

are all independent of time t.

2. Schw geometry is spherically symmetric

How do you know that?
Because

- recognize $r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$ part of metric as the metric of the (2D) surface of a (3D) sphere of radius r;
- metric coefficients are otherwise independent of angular coordinate θ, ϕ.

3. Proper time of observers at rest in Schw geometry

Observers at rest in Schw geometry non-freely-falling follow wordlines with $dr = d\theta = d\phi = 0$.
Proper time measured by these observers is

$$d\tau^2 = -ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 \quad \begin{cases} \to dt^2 & \text{as } r \to \infty \\ > 0 & r > r_s \\ = 0 & r = r_s \\ < 0 & r < r_s \end{cases}$$

$$= \left(1 - \frac{r_s}{r}\right)^{1/2} dt$$

Notice

- as $r \to \infty$, $d\tau \to dt$
  ie. Schwarzschild time t is the proper time of observers at rest infinitely far from the mass;
- the proper time of observers at rest runs slower, by $\left(1 - \frac{r_s}{r}\right)^{1/2}$, nearer mass.
- as $r \to r_s$
  $d\tau \to 0$!
  time appears to stop for objects at the Schwarzschild radius, as observed by observers at rest outside the Schwarzschild radius.

Q: Does it make sense to talk about observers "at rest" here?

4. Redshift of objects at rest in Schw geometry

As usual

$$1+z = \frac{\nu_{em}}{\nu_{obs}} = \frac{dt_{obs}}{dt_{em}}$$

Take observer at infinity, so $dt_{obs} = dt$
emitter at radius r, so $dt_{em} = d\tau$

then

$$\boxed{1+z = \frac{dt}{d\tau} = \frac{1}{\left(1-\frac{r_s}{r}\right)^{1/2}}}$$

ie. object at rest at r appears redshifted to observer at rest at infinity;
conversely object at rest at infinity appears blueshifted to observer at rest at r.

Notice

- as $r \to \infty$,

$$1+z \approx 1 + \frac{1}{2}\frac{r_s}{r}$$

$$= 1 + \frac{1}{2}\frac{2GM}{c^2 r} = 1 + \frac{GM}{c^2 r}$$

$$= 1 + \frac{\phi_\infty - \phi}{c^2}$$ Newtonian potential

is usual gravitational redshift for weak field;

- as $r \to r_s$,

$$1+z \to \infty \quad !$$

redshift $\to \infty$ at Schwarzschild radius.

5. Circumference vs. radius in Schw geometry

The Schwarzschild radial coordinate r is chosen so that the proper circumference measured by observers at rest in Schw geometry is

$$\boxed{\begin{array}{l}\text{Proper circumference} \\ \text{at radial position } r\end{array} = 2\pi r}$$

(Didn't have to choose r that way; but it's obviously a good choice)

Proper radial interval dl measured by observer at rest at radius r is

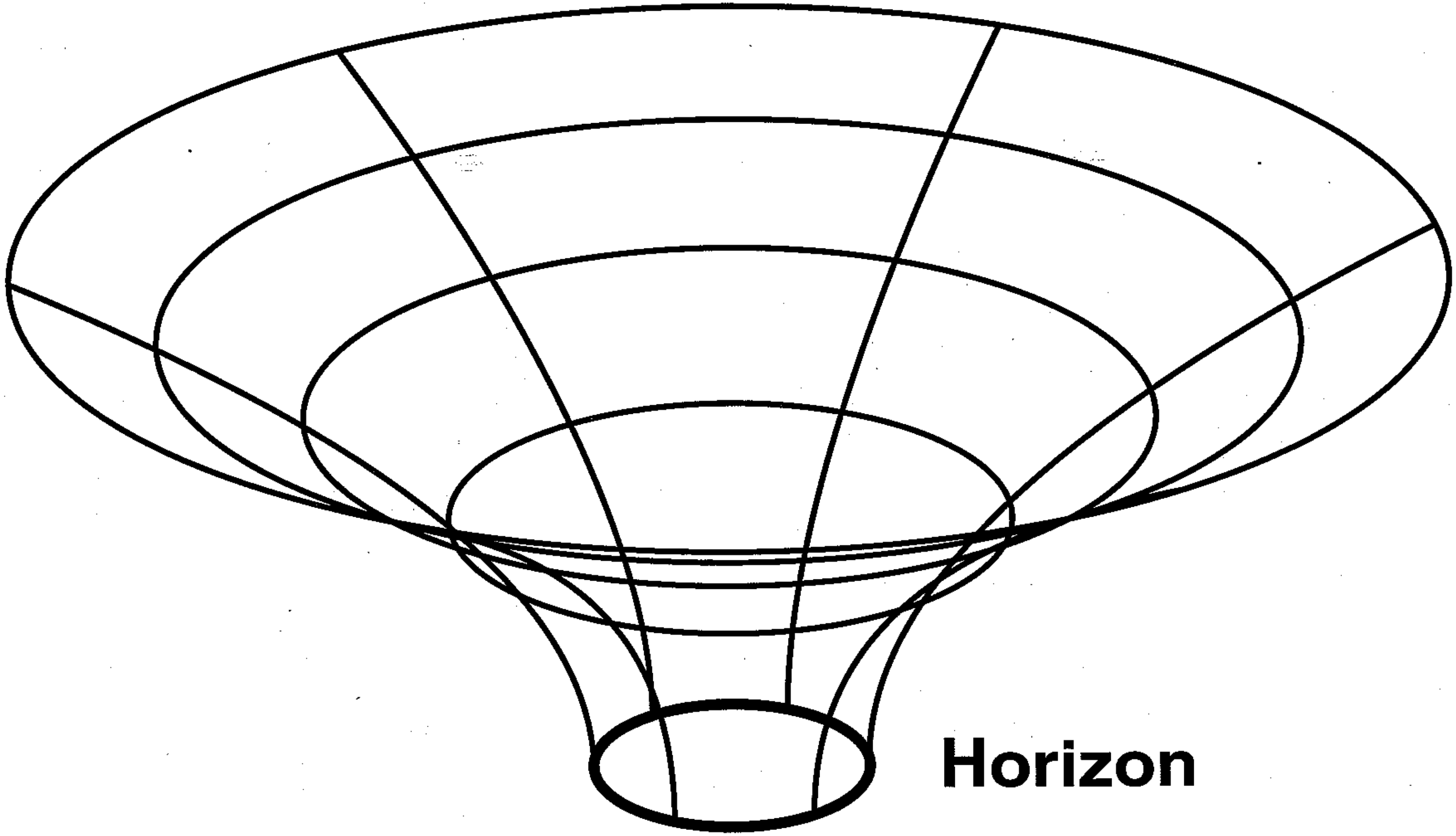
$$dl = \left[\frac{dr^2}{\left(1 - \frac{r_s}{r}\right)}\right]^{1/2}$$

$$= \frac{dr}{\left(1 - \frac{r_s}{r}\right)^{\frac{1}{2}}}$$

Notice:

- as $r \to \infty$, $dl \to dr$
  so recover usual flat-space behavior at large r
- $dl > dr$ for $r_s < r < \infty$
- as $r \to r_s$, $dl \to \infty$!

Embedding diagram of Schw geometry outside Schw radius



6. <u>What happens inside Schwarzschild radius?</u>

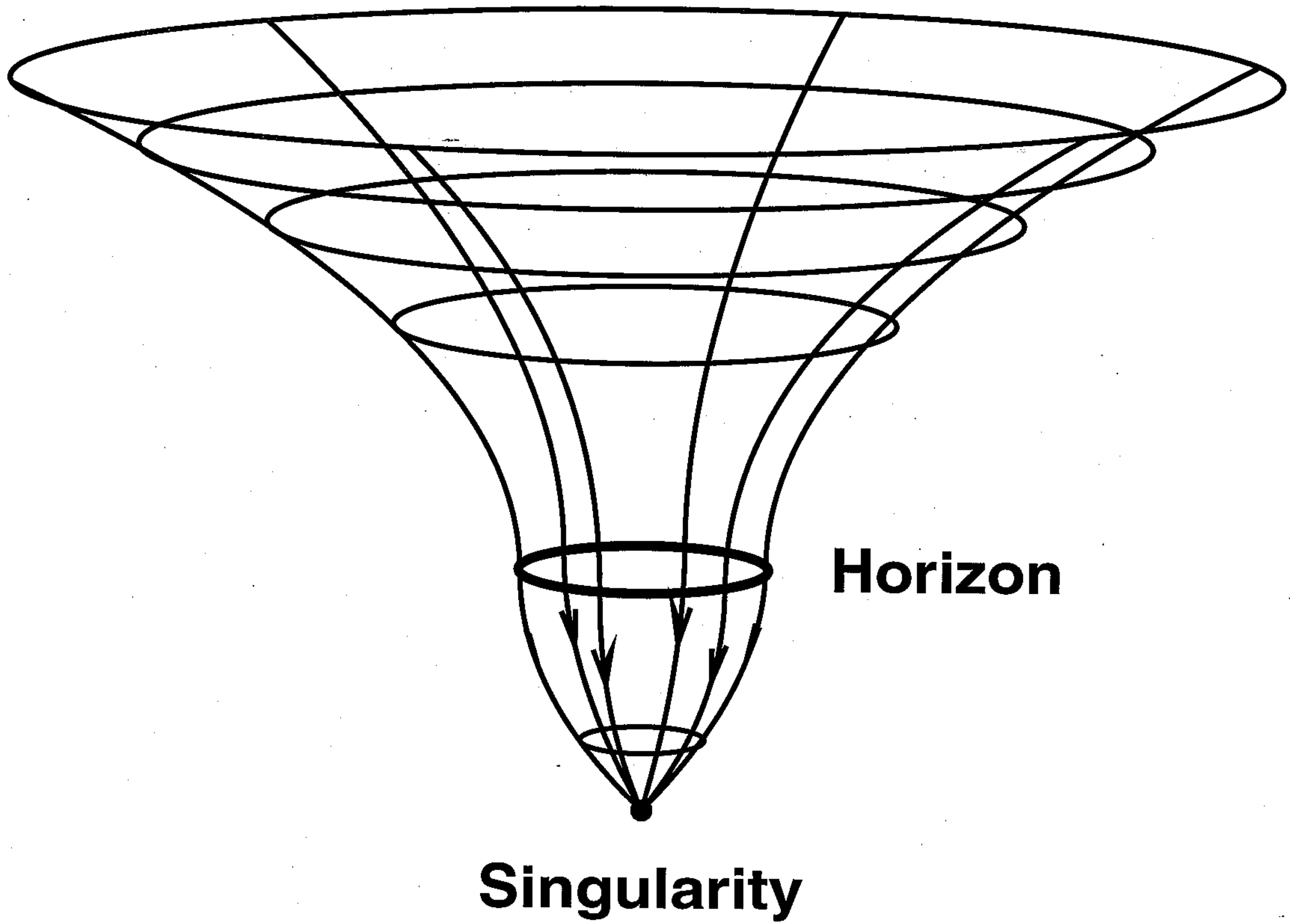
Consider the worldlines of "stationary" observers, who follow $dr = d\theta = d\phi = 0$
Interval along worldline satisfies
$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2$$
$$> 0 \qquad \text{for } r < r_s$$

so "worldlines" are space-like, not time-like.
So they can't be worldlines of real observers at all — they correspond to faster than light motion!

Consider alternatively radial lines at fixed t, θ, ϕ. Intervals along these lines satisfy
$$ds^2 = \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)}$$
$$< 0 \qquad \text{for } r > r_s$$

which are time-like inside r_s.
So <u>these</u> lines (at constant Schw time t) <u>are</u> worldlines of real observers!

What does this mean?
No stationary observers possible inside r_s.


Horizon
Singularity

7. Schwarzschild spacetime diagram

Light rays follow null geodesics (as always)

$$ds = 0 .$$

Consider radial null geodesics. Given by

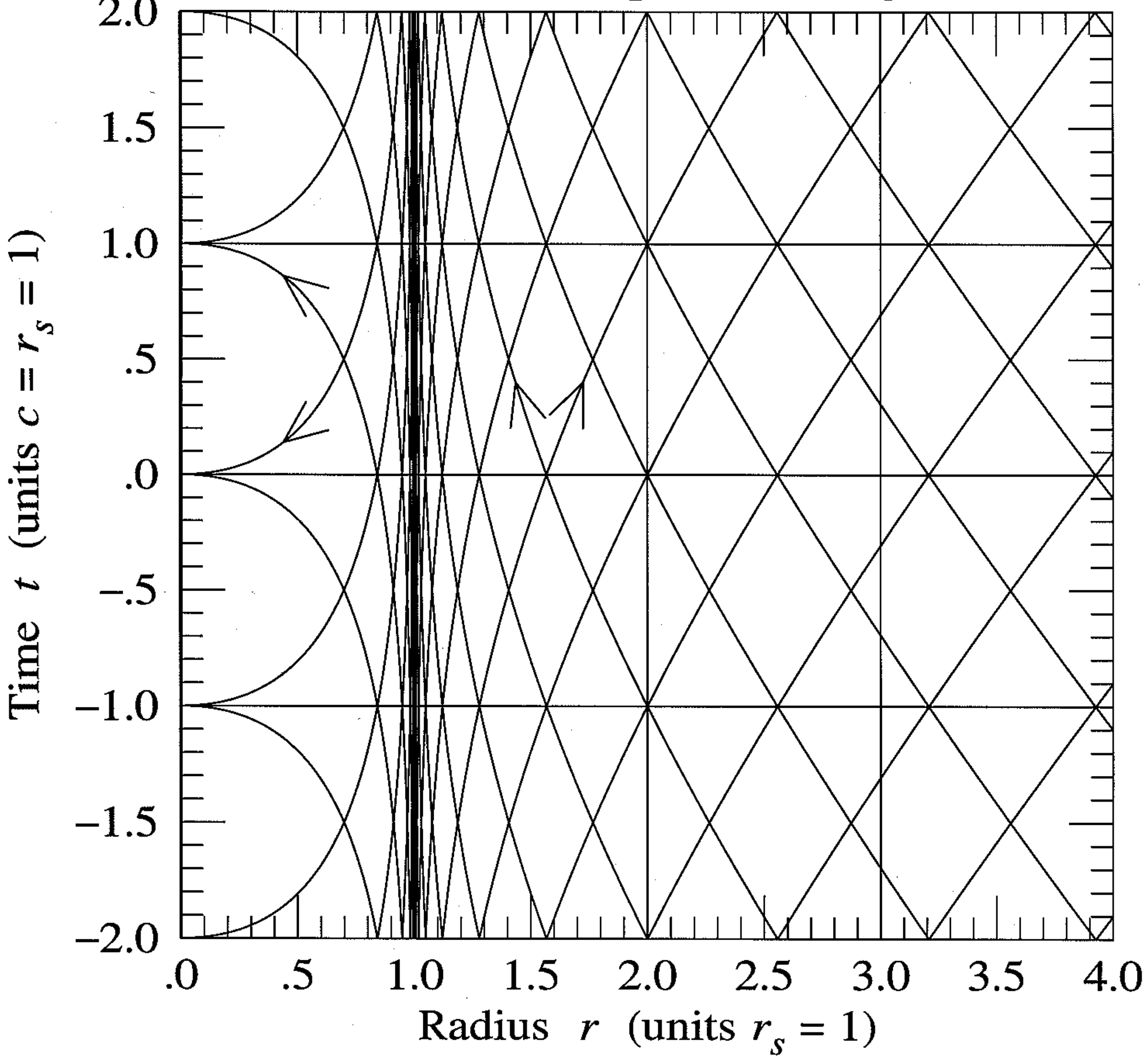
$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} = 0$$

ie $\dfrac{dr}{dt} = \pm \left(1 - \dfrac{r_s}{r}\right)$ $\quad$ + = outgoing, − = infalling

ie
$$t = \pm \int \frac{r\,dr}{r - r_s}$$
$$= \pm \left[r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right]$$

The paths of light rays seem to go to $t = \pm \infty$ at horizon $r \to r_s$.
Does this mean infalling light rays never reach the horizon? NO!
It just means light (and anything else) never falls in from the point of view of an outside observer.

Schwarzschild spacetime diagram



8. Eddington - Finkelstein coordinates

Chose time coordinate t_F so infalling light rays move inward at 45° in spacetime diagram.

For infalling light rays

$$t = -r - r_s \ln\left|\frac{r}{r_s} - 1\right| + \text{constant}$$

ie

$$t + r_s \ln\left|\frac{r}{r_s} - 1\right| = -r + \text{constant}$$

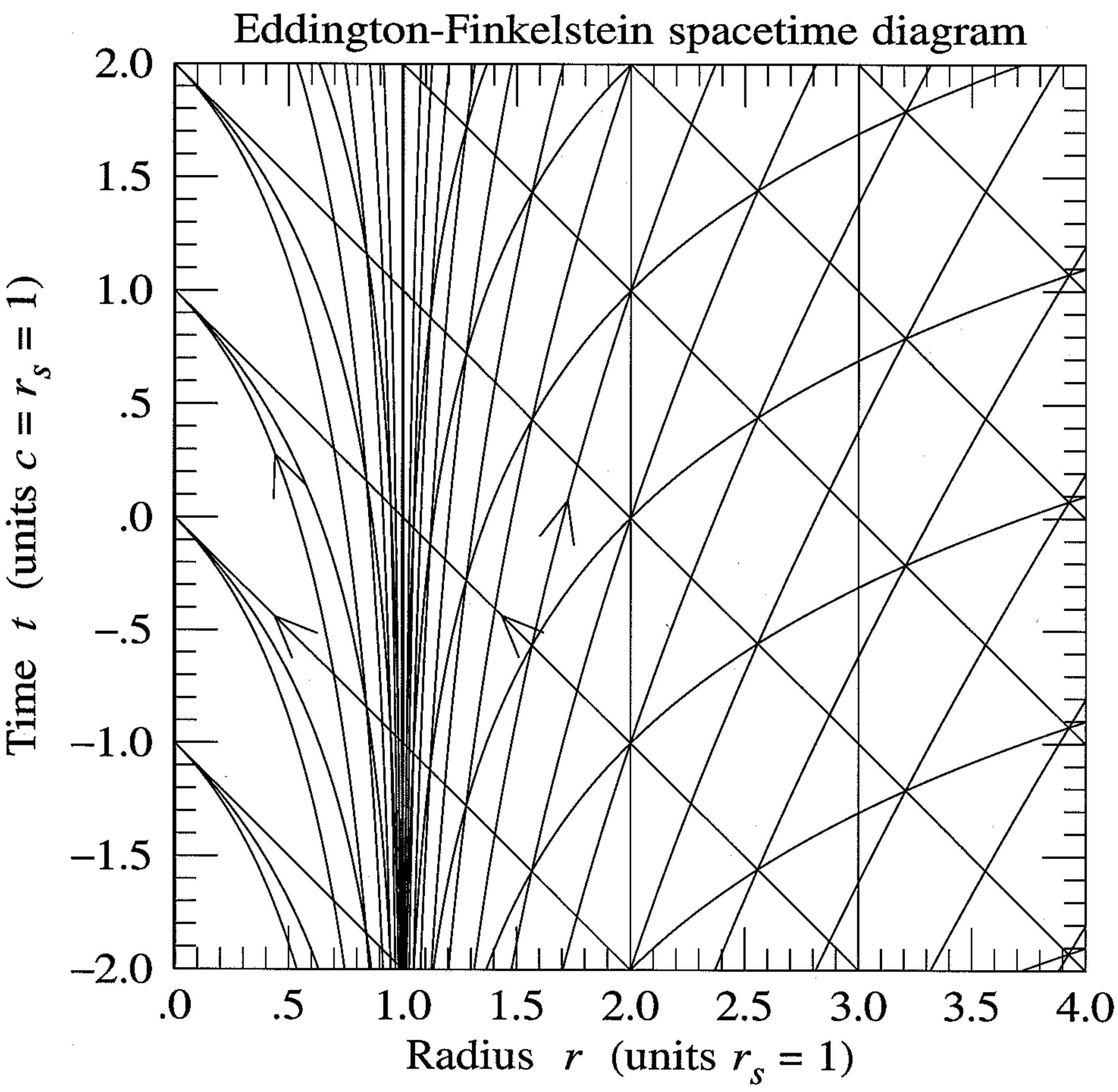
Define Finkelstein time by

$$t_F \equiv t + r_s \ln\left|\frac{r}{r_s} - 1\right|$$

then infalling light rays follow

$$t_F = -r + \text{constant}$$

ie they go inward at 45°, as designed.


Eddington-Finkelstein spacetime diagram
Time t (units $c = r_s = 1$)
2.0
1.5
1.0
.5
.0
−.5
−1.0
−1.5
−2.0
.0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Radius r (units $r_s = 1$)

2·39

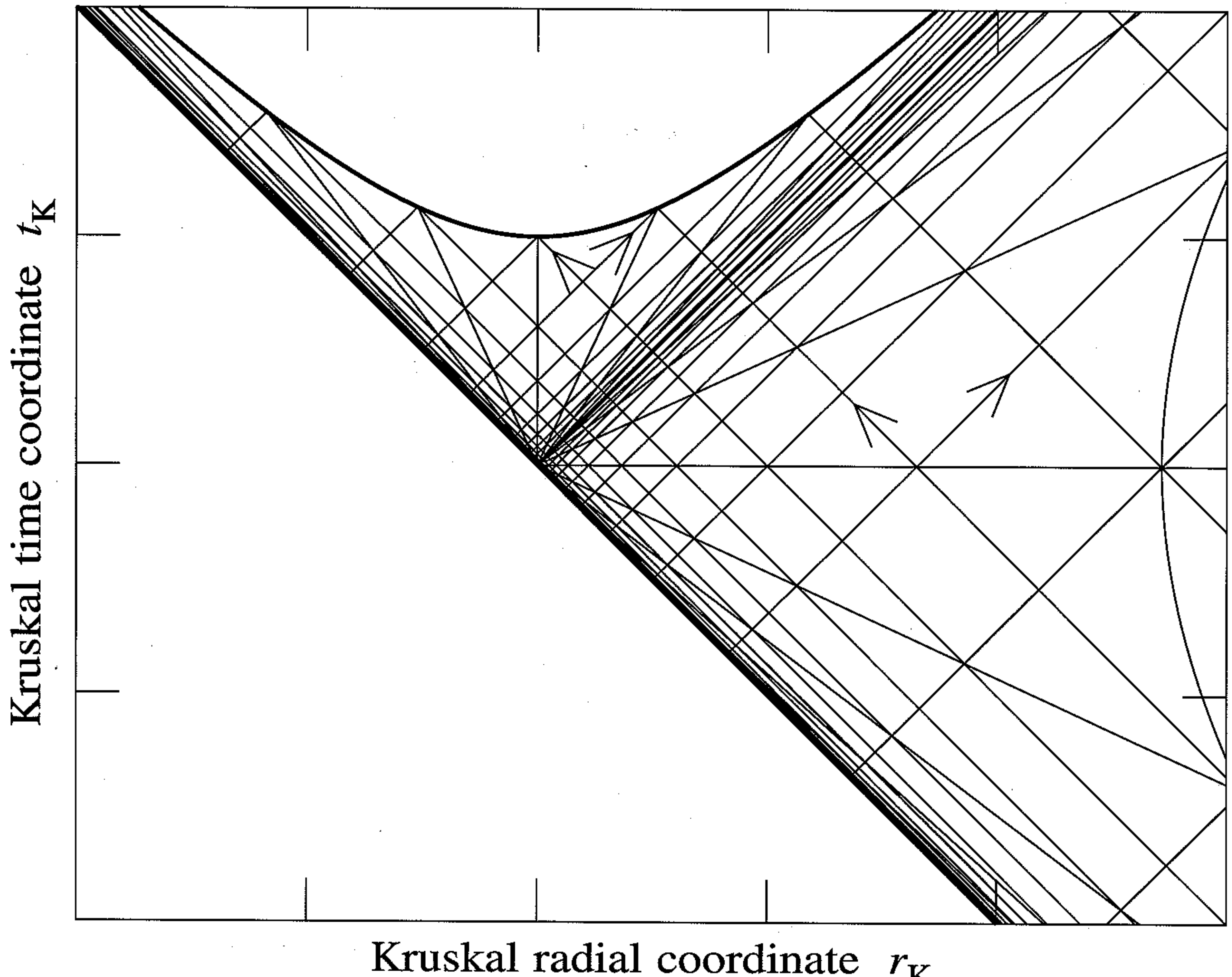
9. Kruskal - Szekeres coordinates

Choose space r_K at time t_K coordinates so that both outgoing and infalling lightrays are at 45° in spacetime diagram.

See "Falling into a Black Hole" web page for more details.

These coordinates reveal transparently the causal structure of the Schwarzschild geometry
— in particular there is NO singularity at the horizon $r = r_s$.

Kruskal-Szekeres spacetime diagram



10. Schwarzschild white hole, wormhole, black hole

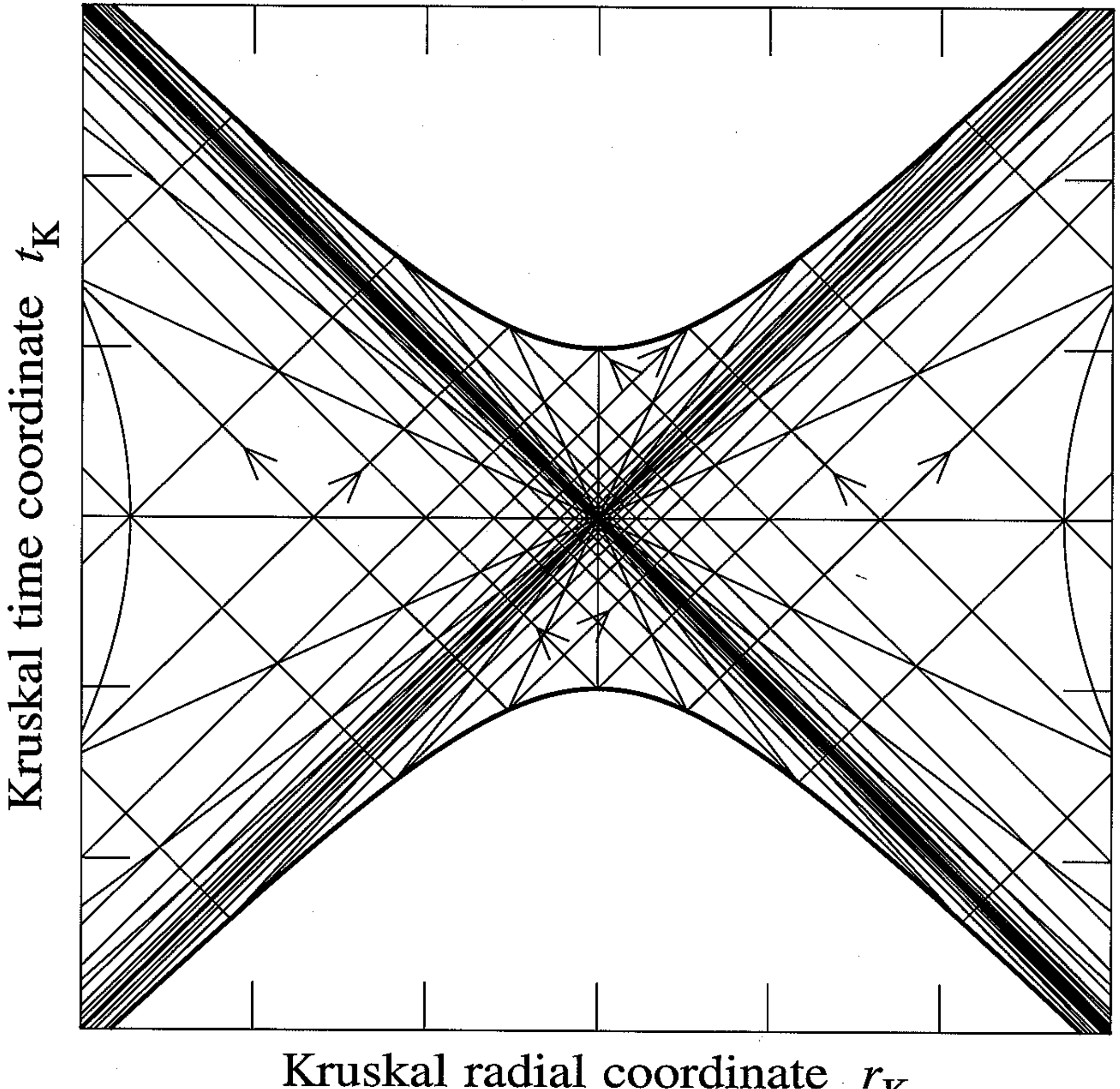
Full Schwarzschild geometry consists of
- white hole (= black hole going backwards in time)
- black hole
- connected by wormhole

called "Einstein-Rosen bridge"

Problems:
- existence of white hole violates 2nd law of thermodynamics;
- the realistic collapse of a star does not produce a white hole or wormhole, only a black hole;
- even if wormhole formed, you could not get through it.

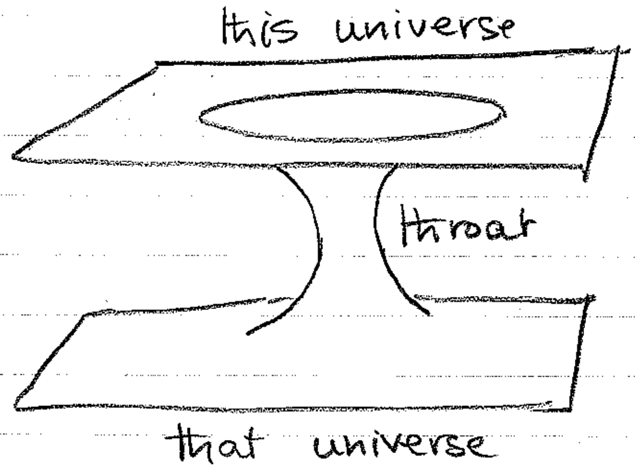
See "Falling into a Black Hole" for more details.

Kruskal-Szekeres spacetime diagram



Traversable wormholes

You are free to choose whatever spacetime geometry you like.



But Einstein's equations then specify what energy-momentum content you need to attain that geometry.

Problem:

- A traversable wormhole requires negative mass material at its throat (this is a general theorem).

  Associated with fact that wormholes cause light rays to diverge, not converge.

Big issue up to 1950s:
<u>Does a black hole ever really collapse?</u>

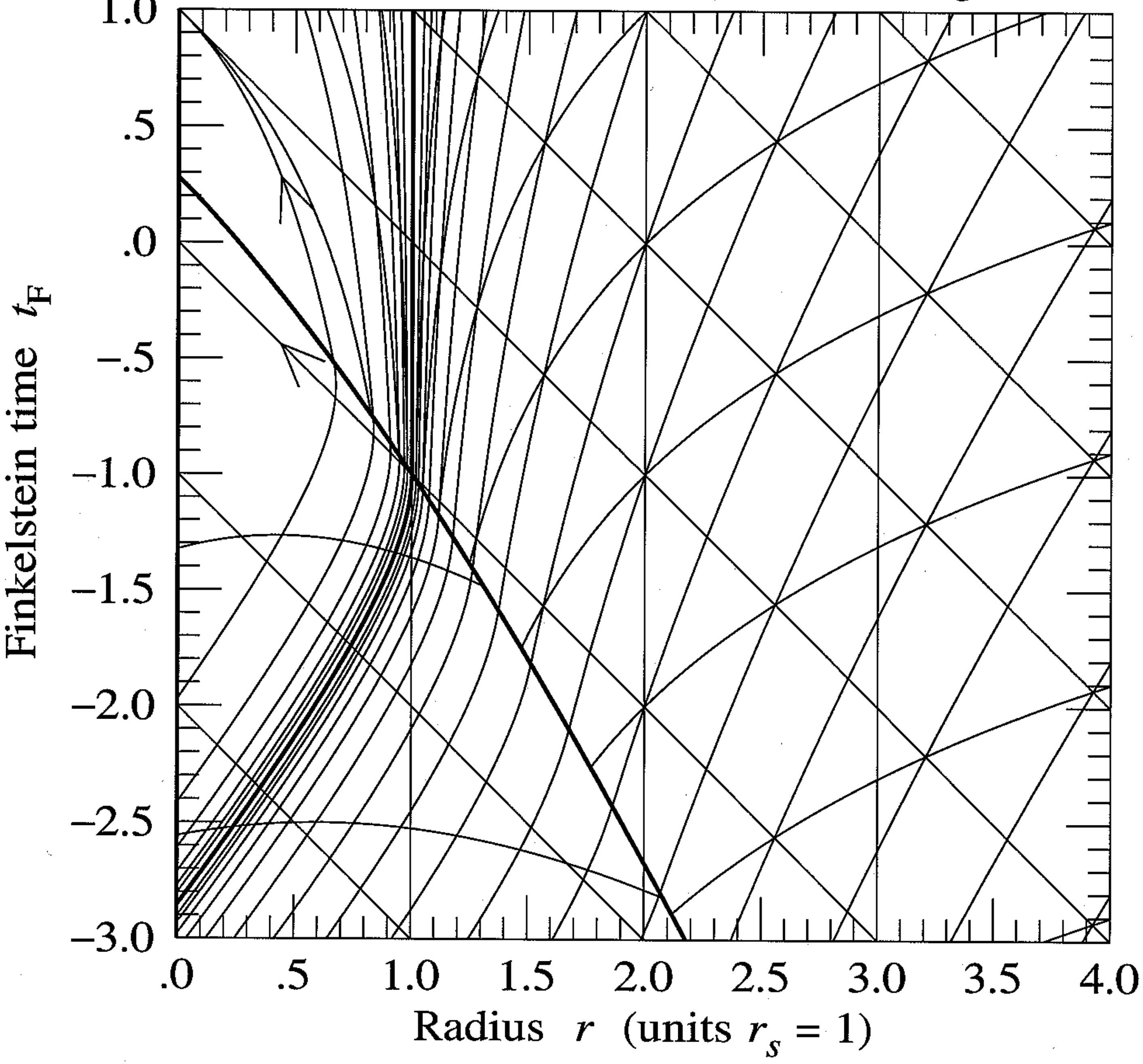
Pre - 1967 terms: "frozen star"
"collapsar".

Ans: Yes, it does collapse!
- In its own frame it collapses in a finite time to the singularity.
- It just takes an infinite time for light near horizon to reach an outside observer.

John Wheeler (1967) coins term "Black Hole".

See "Falling into a Black Hole" website
→ collapse

Eddington-Finkelstein spacetime diagram



River Model of Schwarzschild geometry

Metric is

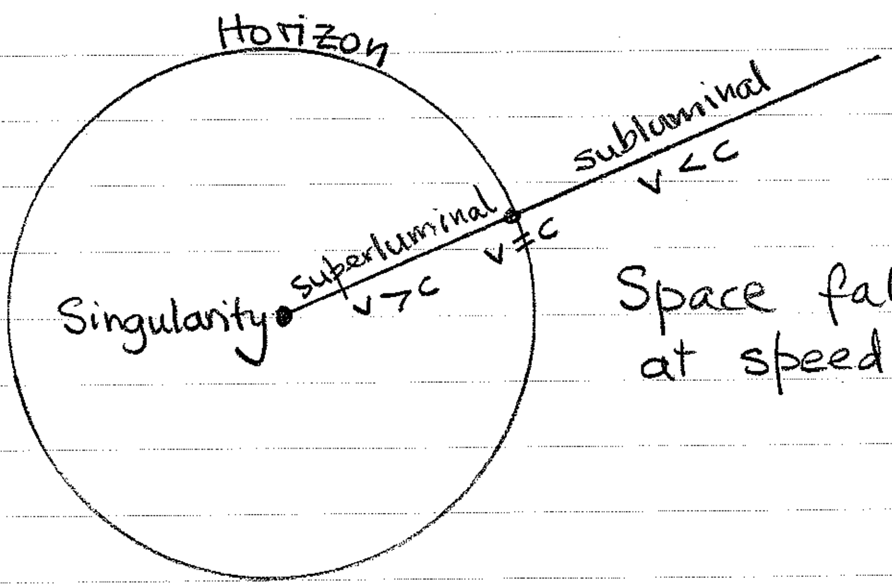
$$ds^2 = -dt_{ff}^2 + (dr + v\,dt_{ff})^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where $v = \sqrt{\frac{2M}{r}}$ happens to be the Newtonian escape velocity

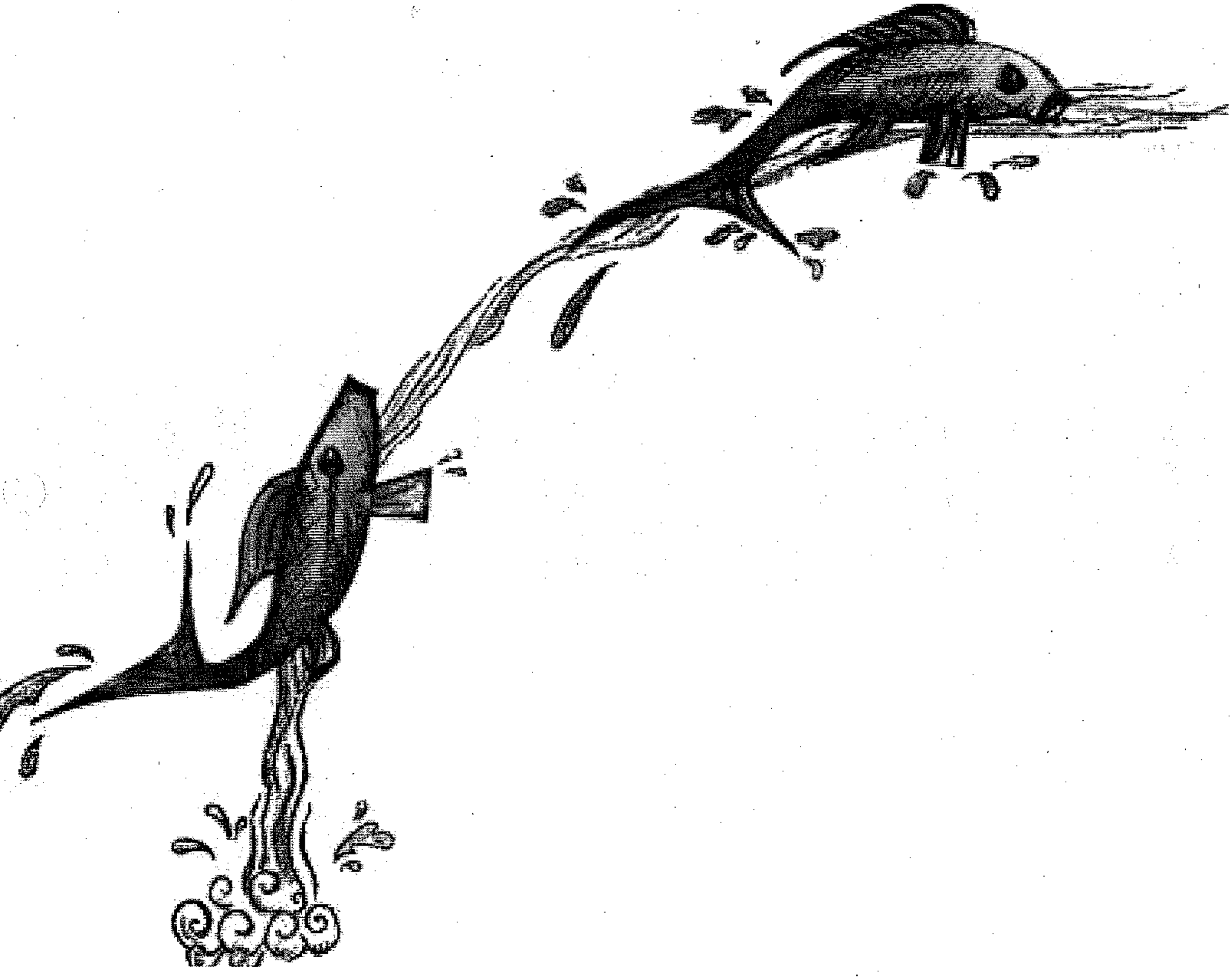
and

$$t_{ff} = \underbrace{t}_{\text{Schwarzschild time}} - \int_r^\infty \frac{v}{1-v^2}\,dr$$

is the proper time experienced by observers who free-fall radially from zero velocity at infinity.



Space falls in at speed v

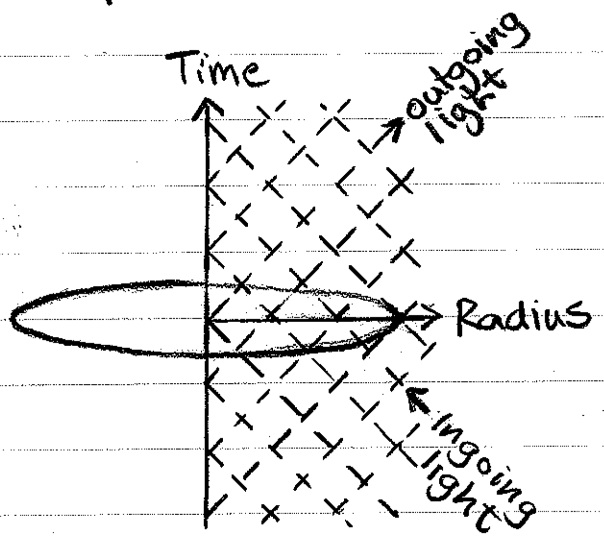


The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall.

Penrose Diagrams

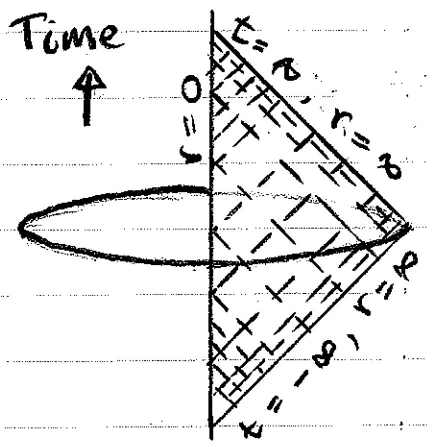
are spacetime diagrams constructed so
(a) Light moves at 45° from vertical everywhere (just as in special relativity), and
(b) Spatial and temporal infinity
$r = \infty \qquad t = \pm\infty$
are brought to a finite position on the diagram.

Motivation: who cares about coordinates; the important thing is the causal structure of the spacetime.



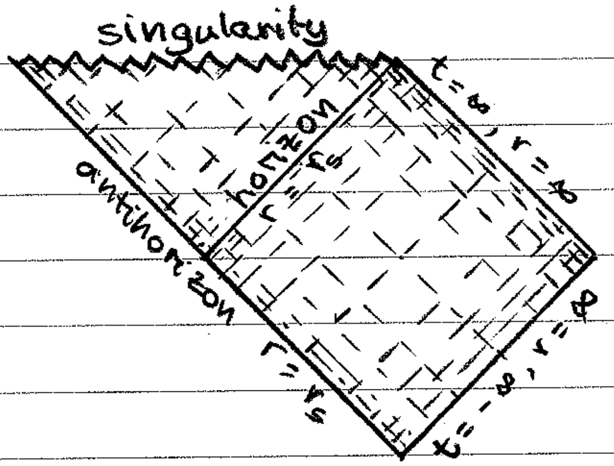
Simplest example: ordinary flat space (the domain of special relativity)

Standard spacetime diagram

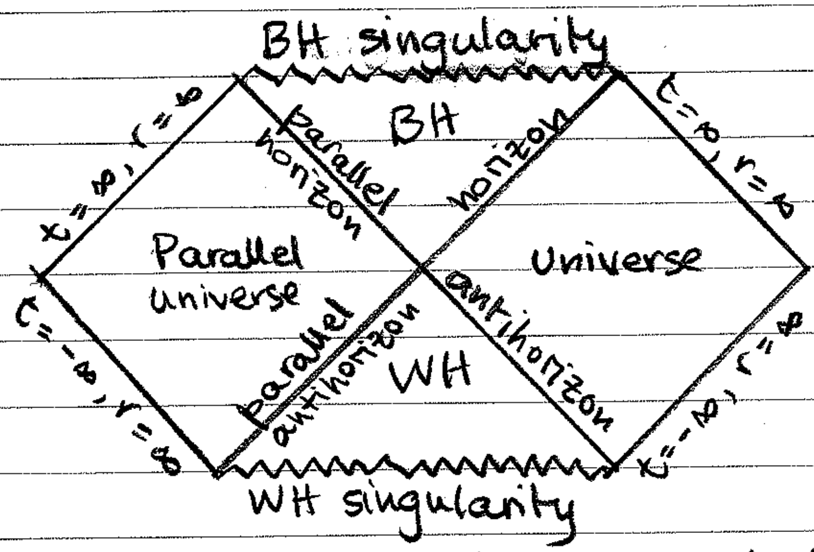


Penrose diagram

Penrose diagram of the Schwarzschild geometry



Penrose diagram of the complete analytically extended Schw geometry



includes White Hole and Parallel Universe as well as Black Hole and Universe.

No-Hair Theorem (Israel 1967 [Uncharged, non-rotating]; Price, Carter, Hawking, etc 1970s [General proof])

The geometry outside an isolated black hole evolves to a state characterized by just 3 quantities
(1) Mass
(2) Electric Charge
(3) Angular Momentum (Spin)

- Isolated means not accreting, and far from other gravitating objects.

- A BH loses its hair by gravitational radiation over the course of several light crossing times.

- The theorem does NOT apply to the deep interiors of BHs.

$\Rightarrow$ BHs are simple !

Reissner - Nordström geometry

Geometry of empty space around a spherical mass M with electric charge Q

$$ds^2 = -B\,dt^2 + \frac{dr^2}{B} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where $B = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ (units $c = G = 1$)

The RN geometry horizons where $B = 0$:

outer horizon

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

inner horizon

Two horizons !?

The River Model of the Reissner-Nordström BH

The metric is the same as river metric of Schw BH:

$$ds^2 = -dt_{ff}^2 + (dr + v\,dt_{ff})^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

but with $v = \sqrt{\dfrac{2M(r)}{r}}$

where

$$M(r) = M - \frac{Q^2}{2r}$$

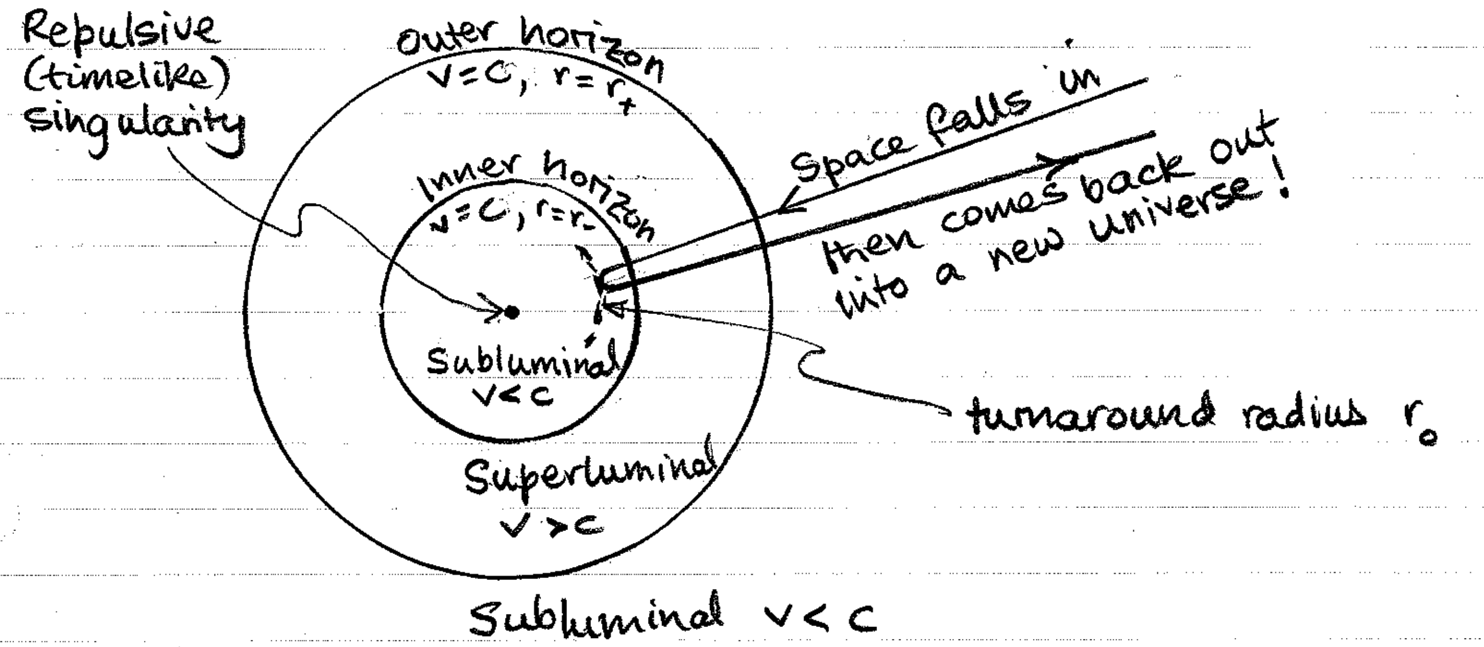
$M(r)$: Mass interior to r

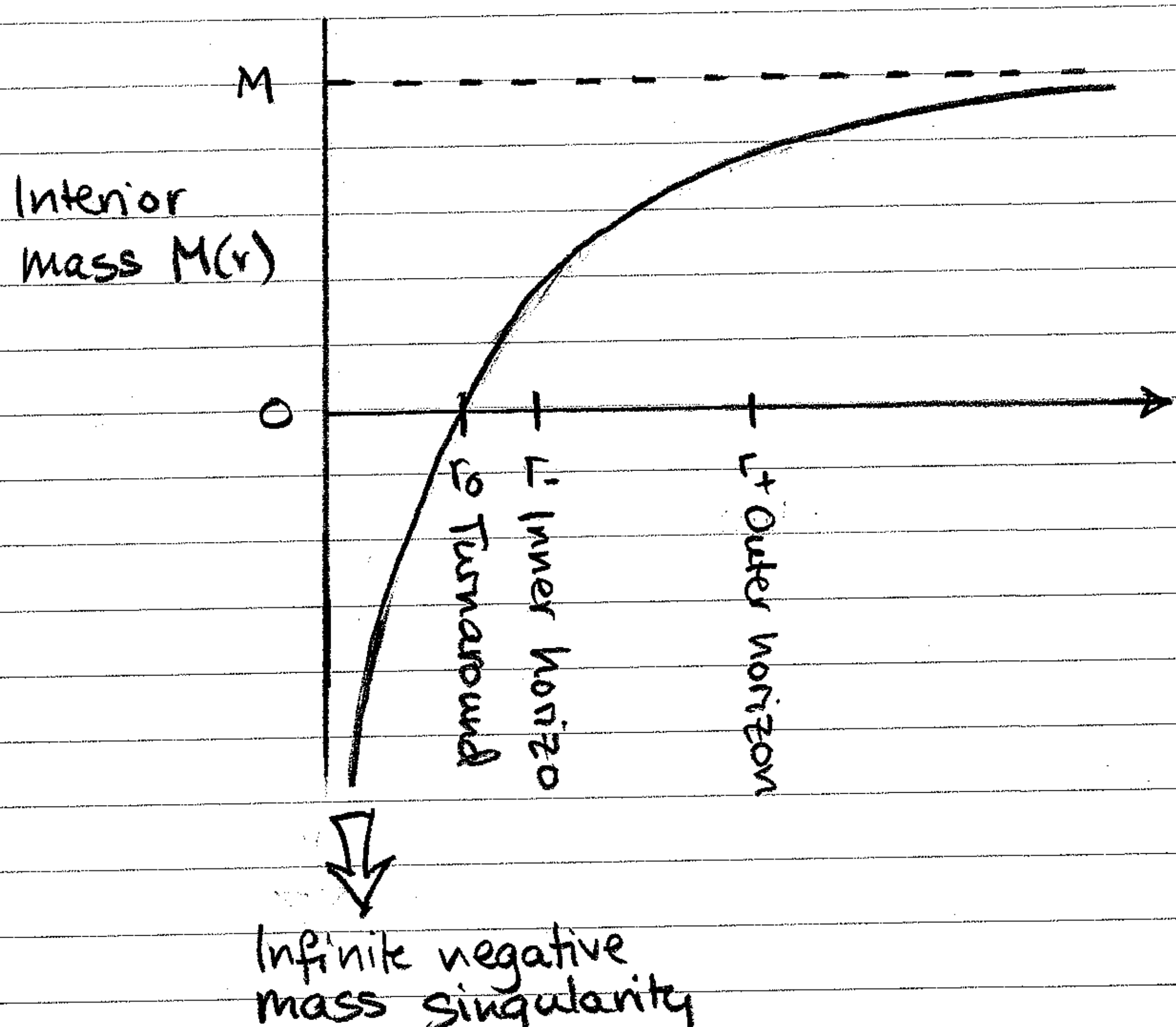
M: Mass at $r \to \infty$

$$\frac{Q^2}{2r} = \int_r^\infty \frac{E^2}{8\pi} 4\pi r^2\, dr$$

$\frac{E^2}{8\pi}$: Energy in electric field

$$E = \frac{Q}{r^2}$$





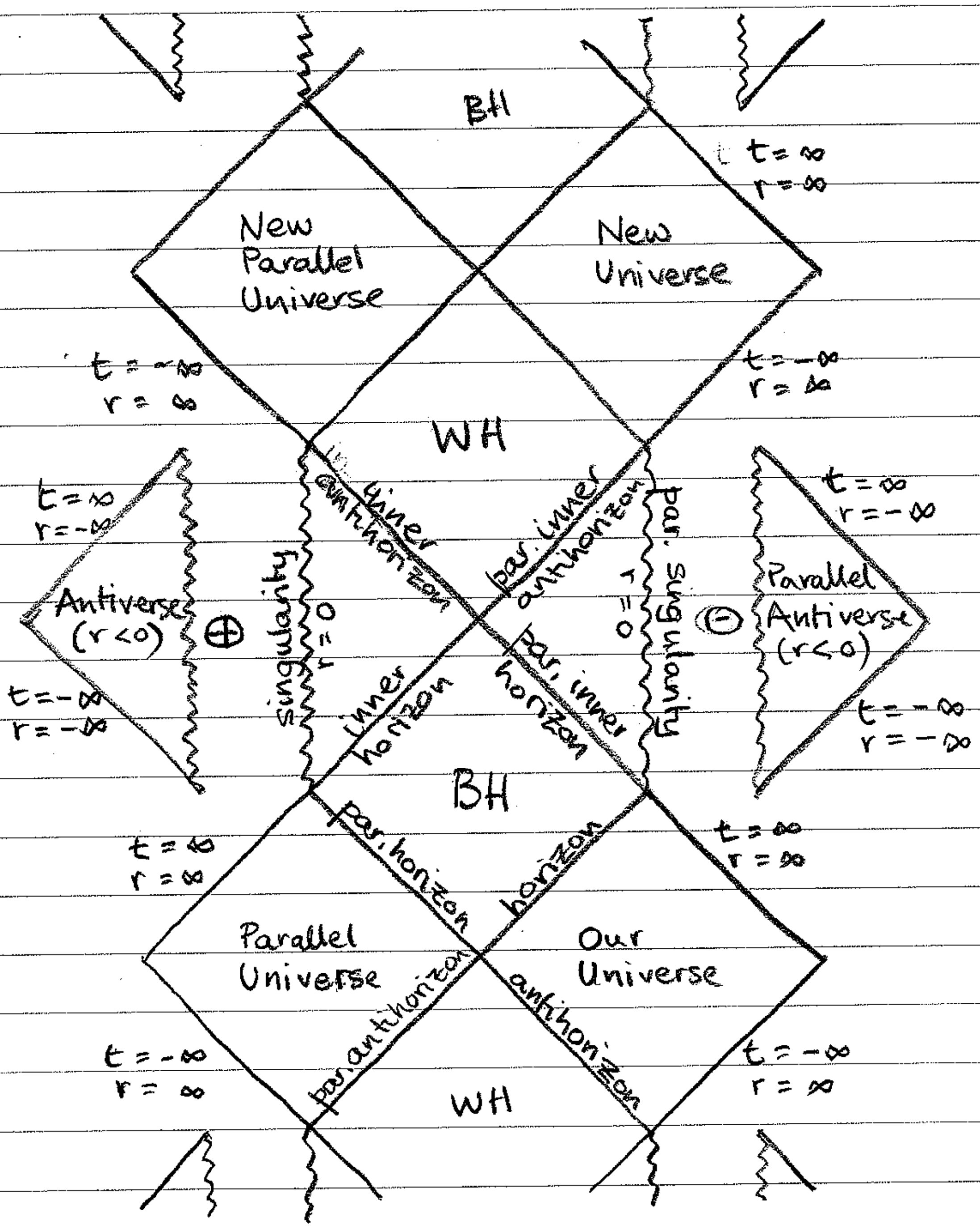
Is it real?

(1) RN geometry apparently describes point singularity of infinite negative mass, sheathed in an electric field containing infinite positive mass, balanced so mass is finite (M) at infinity.

(2) RN solution is inconsistent.
It predicts that infalling material will collect inside the inner horizon, contradicting the assumption that the space is empty.

= non-rotating charged BH

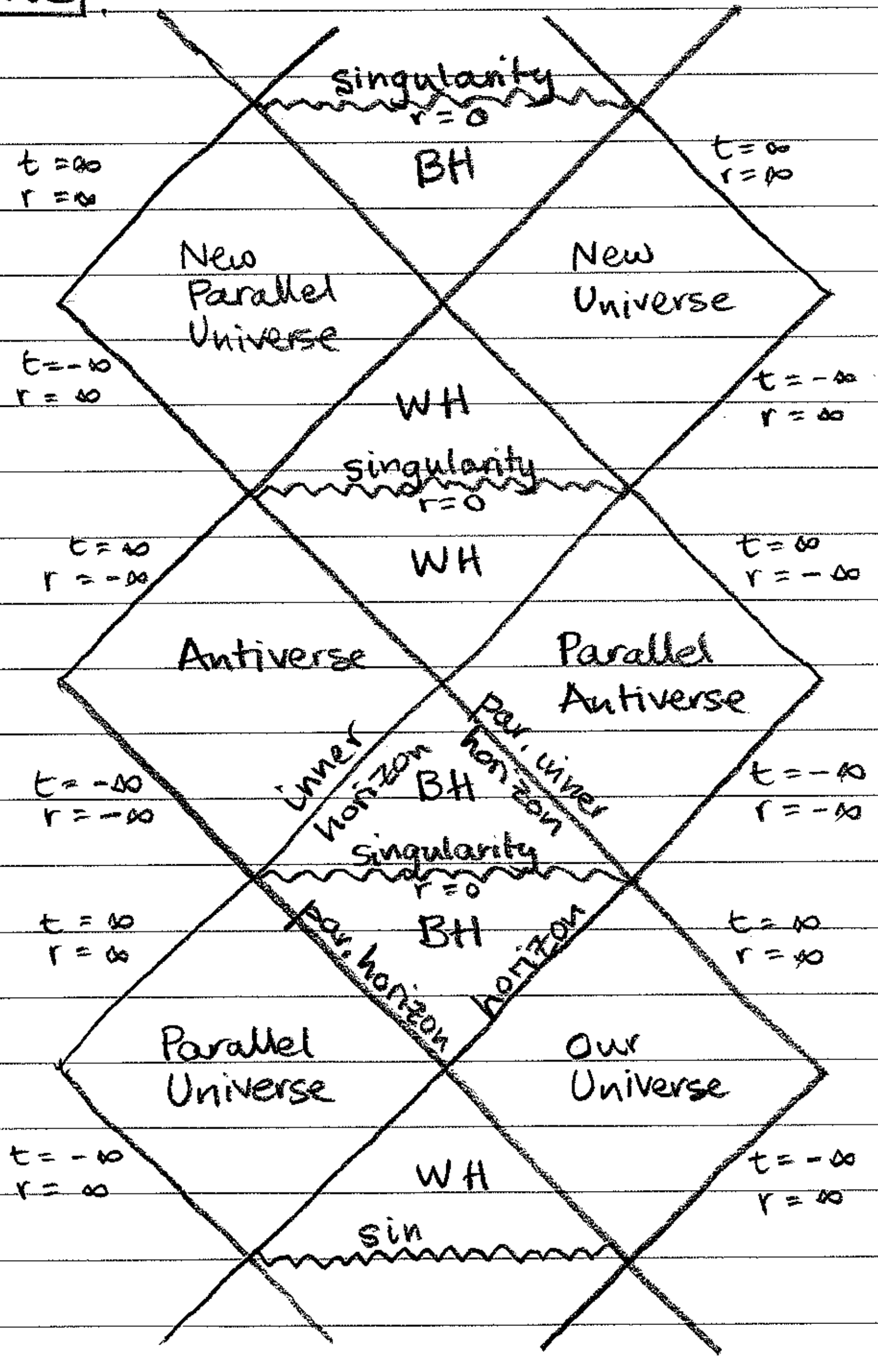
Penrose diagram of Reissner - Nordström geometry



$\sqrt{-1}$

Penrose diagram of Reissner-Nordström geometry with $\boxed{\text{imaginary}}$ charge

An imaginary world where electromagnetic energy is $\boxed{\text{negative}}$.



Rotation + charge

Kerr - Newman

Rotating Black Holes (Kerr 1966)

Geometry of empty space around a rotating mass M with electric charge Q and angular momentum per unit mass a

Boyer - Lindquist metric:

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$+ \frac{R^4 \sin^2\theta}{\rho^2}\left(d\phi - \frac{a}{R^2}dt\right)^2$$

angular velocity of horizon

$$R \equiv \sqrt{r^2 + a^2}$$
$$\rho \equiv \sqrt{r^2 + a^2\cos^2\theta}$$
$$\Delta \equiv R^2 - 2Mr + Q^2$$

Horizons where $\Delta = 0$

ie,

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$$

Ergospheres where $ds = 0$ at $dr = d\theta = d\phi = 0$

ie.

$$r_e = M \pm \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$$

The River Model of the Kerr-Newman BH

See gr-qc/0411060 for details.

The metric is

$$ds^2 = -dt_{ff}^2 + \left[\frac{\rho\, dr}{R} + \frac{vR}{\rho}\left(dt_{ff} - a\sin^2\theta\, d\phi_{ff}\right)\right]^2 + \rho^2 d\theta^2 + R^2 \sin^2\theta\, d\phi_{ff}^2$$

$$\left.\begin{aligned} R &= \sqrt{r^2+a^2} \\ \rho &= \sqrt{r^2+a^2\cos^2\theta} \end{aligned}\right\} \text{ as before}$$

$$v = \frac{(2Mr - Q^2)^{\frac{1}{2}}}{R} \qquad \text{river velocity}$$

Horizons where $v = \pm 1$.

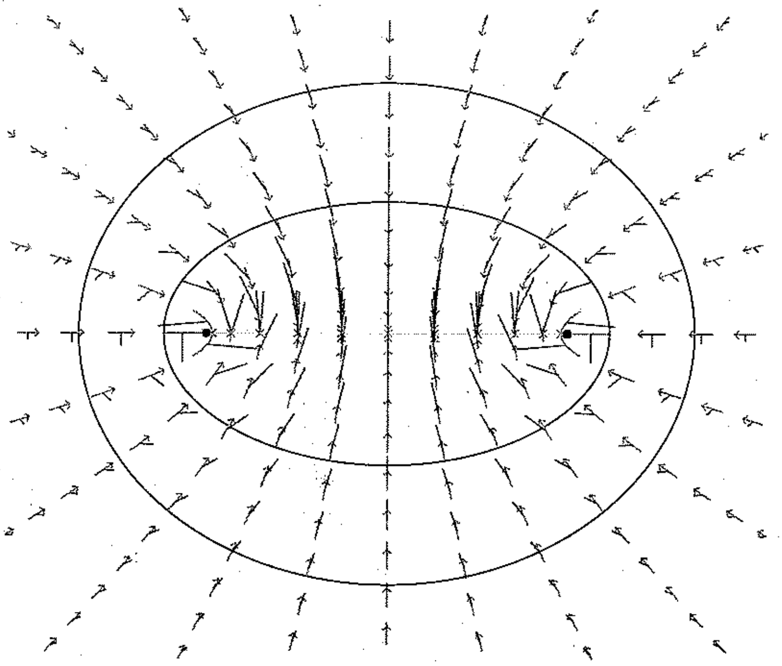


Figure 1: The velocity and twist fields for an uncharged (Kerr) black hole with angular momentum per unit mass $a = 0.95$. The arrowed lines show the magnitude and direction of the river velocity, while the unarrowed lines emerging from the arrowed lines show the magnitude and axis of the river twist. The confocal ellipses show the outer and inner horizons, and the large dots at the foci of the ellipses indicate the ring singularity. In the vacuum Kerr solution, the river velocity goes to zero at the horizontal disc bounded by the ring singularity, then turns around and rebounds through a white hole into a new universe.

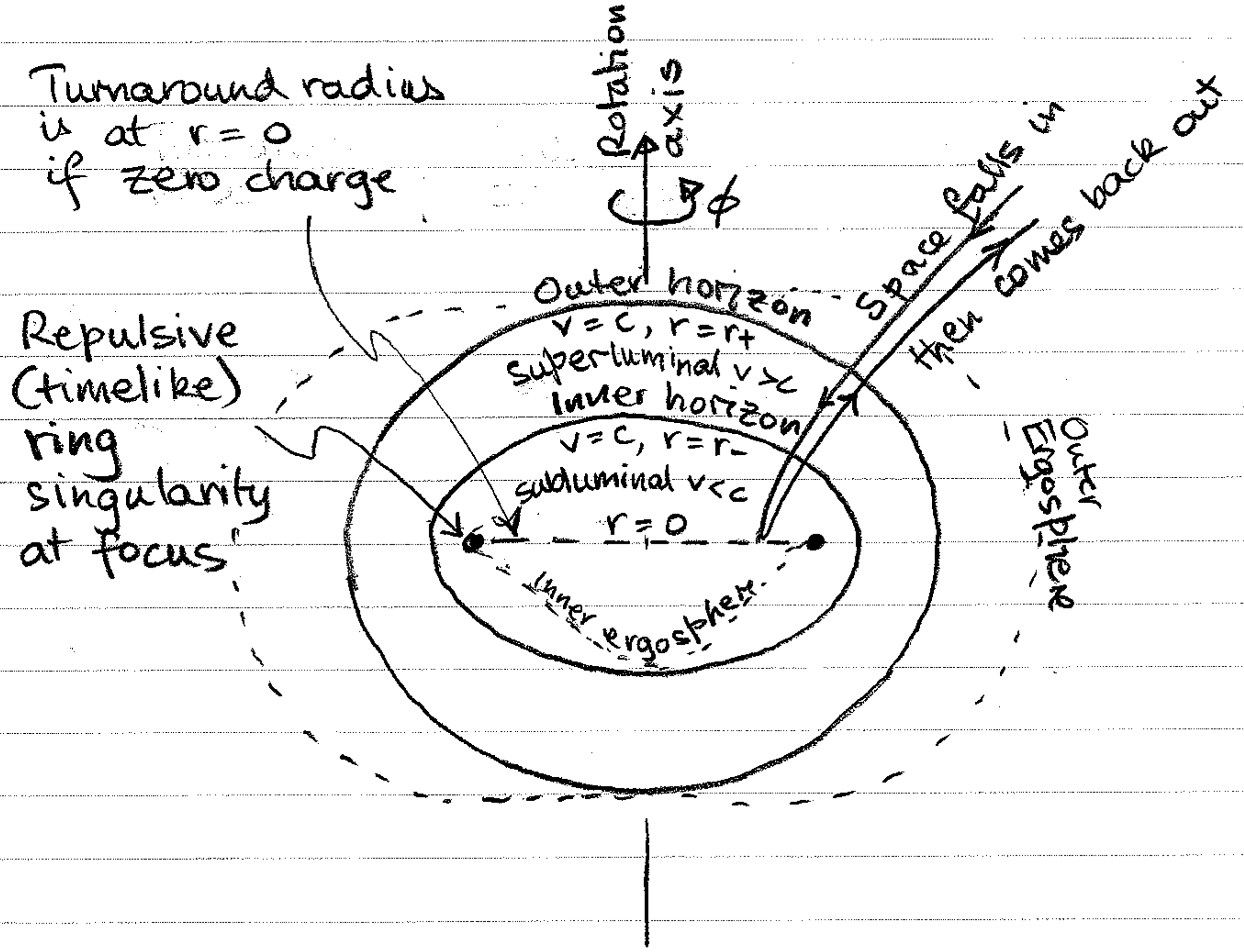
Boyer - Linquist coordinates r, θ, ϕ are <u>oblate spheroidal</u> coordinates (not spherical): Corresponding Cartesian coordinates are

$$x = R \sin\theta \cos\phi$$
$$y = R \sin\theta \sin\phi$$
$$z = r \cos\theta$$

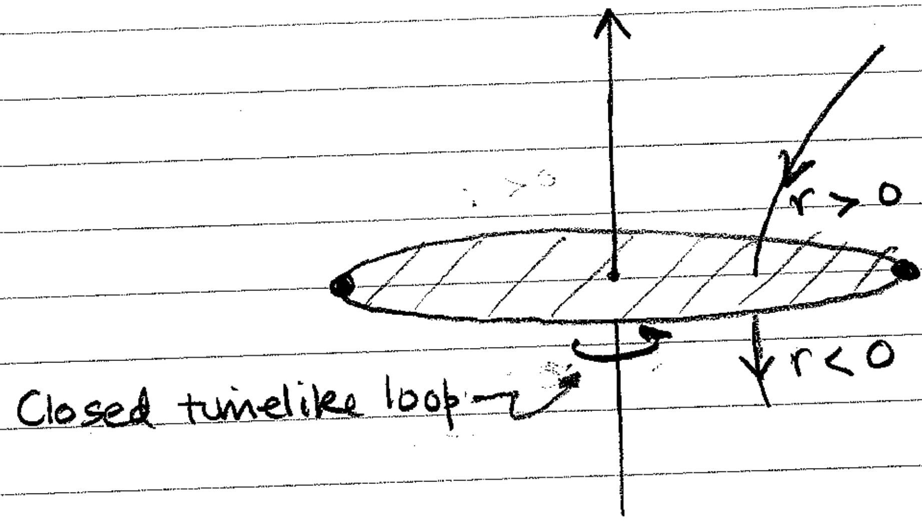
In terms of x, y, z, spheroidal coordinate r is given by

$$r^4 - r^2(x^2+y^2+z^2-a^2) - a^2z^2 = 0.$$

Surfaces of constant r are confocal spheroids.



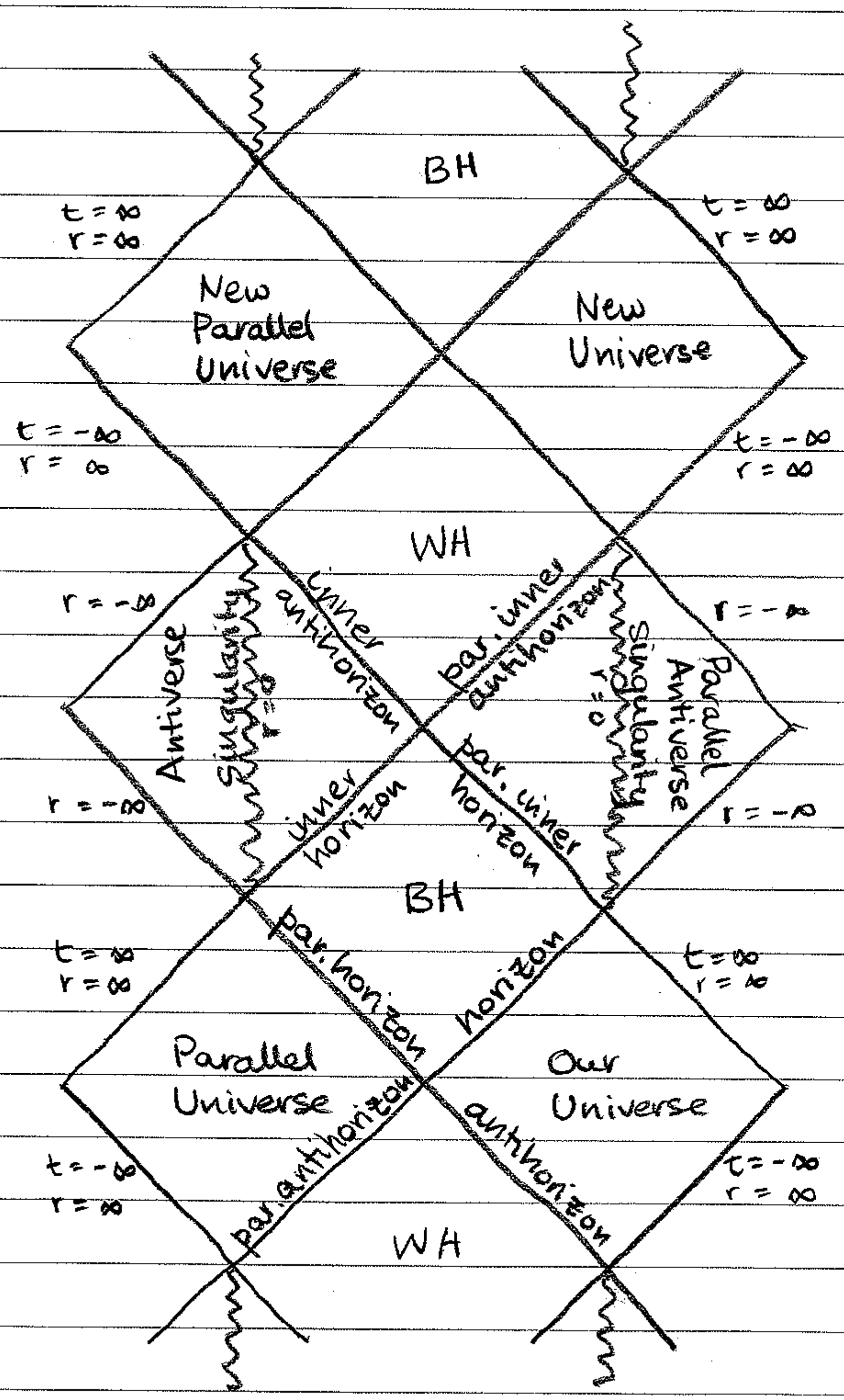
Region at $r<0$



Region at $r<0$ behaves differently from $r>0$. BH viewed from $r<0$ appears to have negative mass $M<0$ and to have no horizons: ring singularity is naked.

At small $r<0$, circles of ϕ at $dt=dr=d\theta=0$ are timelike. These are CTLs, closed timelike loops that repeat themselves for ever, like Sisyphus.

Penrose diagram of rotating charged Kerr – Newman geometry



Hawking Radiation

1. Thermodynamics

Scientific edifice of 1800s.

2nd law of thermodynamics;

$\boxed{\text{Entropy increases}}$

measure of disorder

Quantum mechanics:

$$S = \ln(\#\text{states})$$

entropy = natural logarithm of the number of states accessed by the system.

2. Paradox

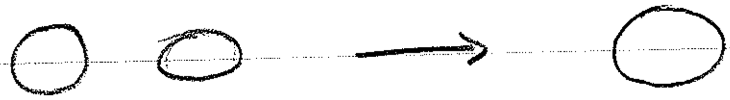
No hair theorem $\Rightarrow$ BHs very simple, ordered, low entropy.

Where does entropy go when stuff falls inside the horizon of BH?

Before early 1970s: entropy disappears!

3. Law of area increase of BHs (Hawking 1970)

When BHs merge,
combined area of horizons always increases.



A_1 A_2 $A \geq A_1 + A_2$

Whereas $M \leq M_1 + M_2$,
because mass is lost
by gravitational radiation.

4. Thermodynamic analogy (Beckenstein 1972)

Beckenstein noticed formal analogy

Thermodynamics		Black Hole
Entropy S	$\longleftrightarrow$	Area A
Temperature T	$\longleftrightarrow$	surface gravity g

Specifically, in units $G = c = 1$,

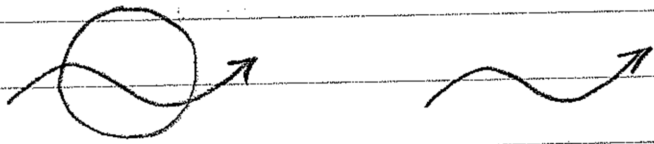
$$T\,dS \longleftrightarrow \frac{g\,dA}{8\pi}$$

5. Stephen Hawking (1974) Effect

Quantum mechanically, BHs do radiate!
Moreover, radiate with blackbody spectrum
(so they have a definite temperature)!

Key concept:

$$\boxed{\text{Peak wavelength} \approx \text{Schw radius}}$$



So BHs look fuzzy in Hawking radiation,
like an atom.
Is a BH a fundamental particle??

Hawking temperature (Planck units $G = c = \hbar = 1$):

$$\boxed{T = \frac{g}{2\pi}} = \frac{r_+ - r_-}{4\pi R_+^2}$$

Hawking entropy:

$$\boxed{S = \frac{A}{4}} = \pi R_+^2$$

where $r_\pm$ = radii of outer/inner horizons.

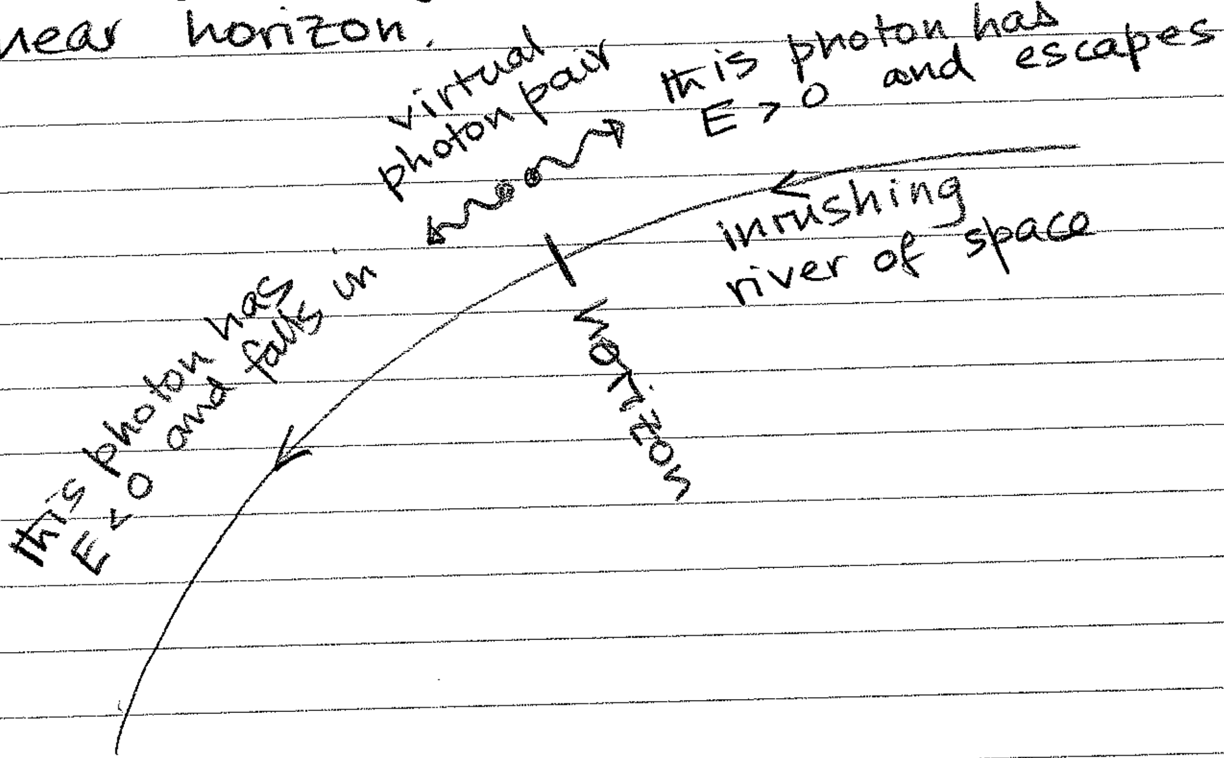
and $R_+^2 = r_+^2 + a^2$.

specific angular momentum

6, <u>How does Hawking effect occur?</u>

Tricky !!
Requires quantum mechanics in curved spacetime, without full quantum GR.

Consequence of vacuum fluctuations near horizon.



7. Hawking evaporation

Area

Hawking temperature

Hawking luminosity:

$$L = \left(\frac{n_{eff}}{2}\right) A \sigma T^4$$

effective number of relativistic particle types

Stephan-Boltzmann constant

$$\left.\begin{aligned} T &\propto \frac{1}{M} \\ L &\propto \frac{1}{M^2} \end{aligned}\right\}$$ Smaller BHs are hotter and more luminous in Hawking radiation

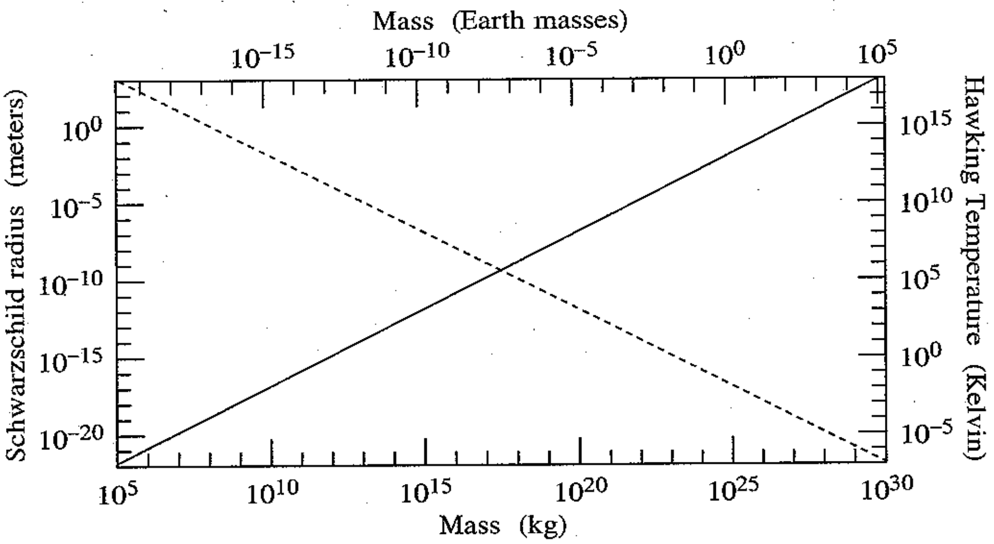
Evaporation formula:

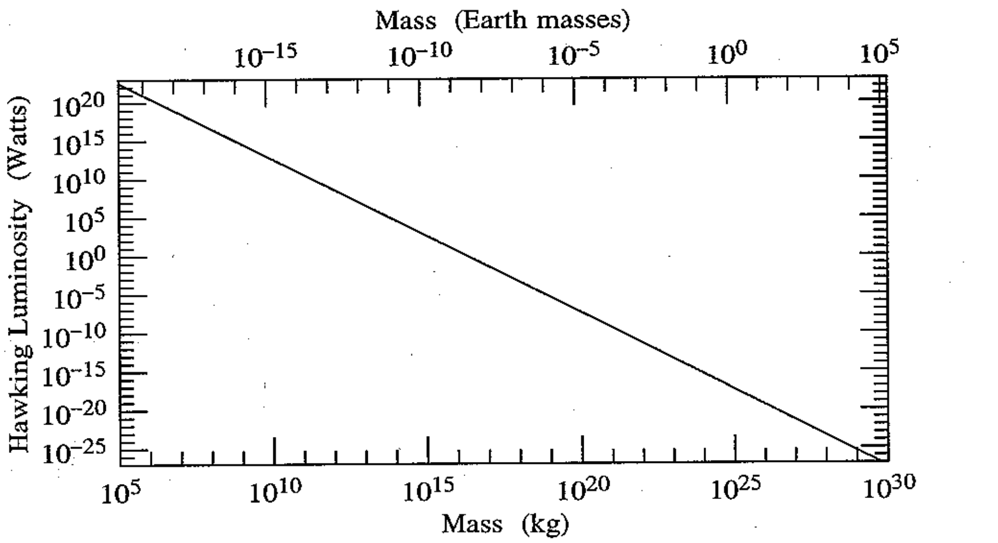
$$\frac{d\,Mc^2}{dt} = -L$$

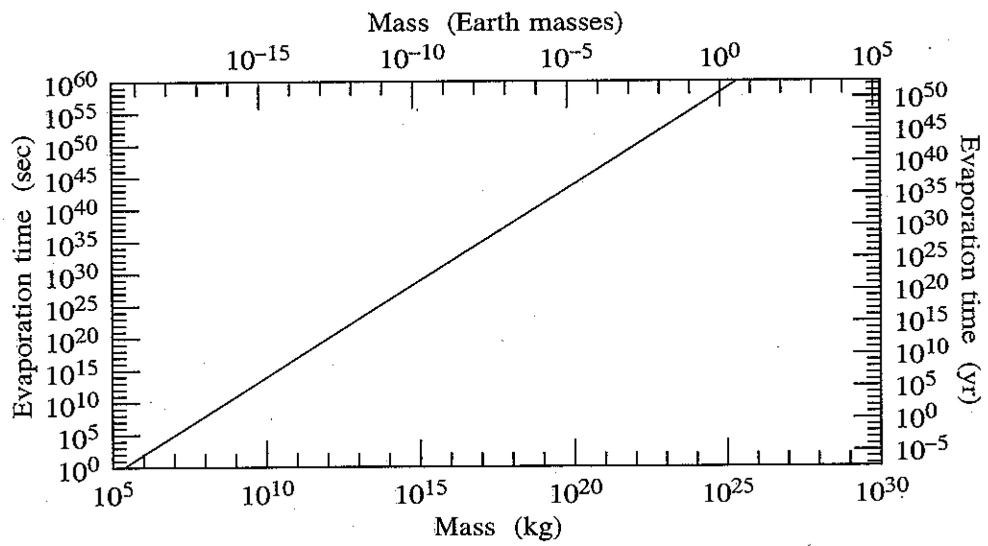
$$\Rightarrow t_{evap} \propto M^3$$

For realistic astronomical BHs, Hawking radiation is miniscule.

2.69


Mass (Earth masses)
10^-15 10^-10 10^-5 10^0 10^5
Schwarzschild radius (meters)
10^0 10^-5 10^-10 10^-15 10^-20
Hawking Temperature (Kelvin)
10^15 10^10 10^5 10^0 10^-5
10^5 10^10 10^15 10^20 10^25 10^30
Mass (kg)


Mass (Earth masses)
10^-15 10^-10 10^-5 10^0 10^5
Hawking Luminosity (Watts)
10^20 10^15 10^10 10^5 10^0 10^-5 10^-10 10^-15 10^-20 10^-25
10^5 10^10 10^15 10^20 10^25 10^30
Mass (kg)


Mass (Earth masses)
10^-15 10^-10 10^-5 10^0 10^5
Evaporation time (sec)
10^60 10^55 10^50 10^45 10^40 10^35 10^30 10^25 10^20 10^15 10^10 10^5 10^0
Evaporation time (yr)
10^50 10^45 10^40 10^35 10^30 10^25 10^20 10^15 10^10 10^5 10^0 10^-5
10^5 10^10 10^15 10^20 10^25 10^30
Mass (kg)

3. Mini-Black Holes

10^8 tonne BH evaporates in age of Universe.

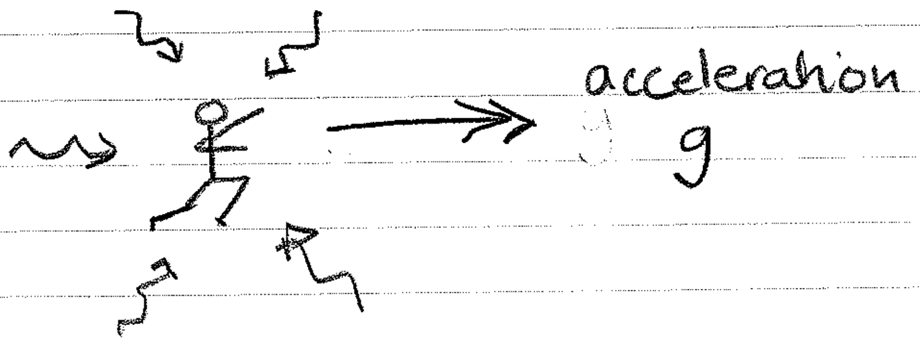
In final second, radiates 10^3 tonne
BANG! V. high energy particles + antiparticles.

Hawking: could mini-BHs have been created in Big Bang?
Depends on equation of state of matter at very high density & temperature.

Maybe, but no astronomical evidence.
Gamma ray bursts are NOT mini-BHs.

1. Unruh (1976) radiation

A person in flat empty space
who accelerates uniformly
sees an isotropic bath of
blackbody radiation in all directions



with Unruh temperature

$$T = \frac{g}{2\pi}$$

eg 4×10^{-20} K at 1 gee.

In quantum GR, "particle"
is not an absolute concept,
but relative to observer's state of motion.

Where does energy of Unruh radiation
come from? From observer's acceleration.

10. Falling into a Black Hole

Image of object falling into a BH
will redshift until
wavelength ≈ Schw radius.
Image will then get lost against
the Hawking noise.

Bet by John Preskill
vs. Stephen Hawking & Kip Thorne:
Is information about objects that fall
in encoded in emitted Hawking radiation?
(Preskill says yes, H & T say no).

If so, then BHs do not destroy
information.

2004: Hawking concedes bet.