ASTR 3740 Relativity & Cosmology Spring 2007. Problem Set 6.
Due Wed 18 Apr

This problem set may take you some time to complete, so please do not wait until the last day to start it.

1. Condition for an accelerating Universe

Suppose that the Universe contains only matter energy (M) and vacuum energy (a cosmological constant Λ), and that it is geometrically flat

$$\Omega_M + \Omega_\Lambda = 1 \tag{1.1}$$

where $\Omega_M \equiv \rho_M/\rho_c$ and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$ are the contributions to Omega in matter and vacuum. How big must Ω_Λ be for the Universe to be accelerating? [Hint: Friedmann's equation for the acceleration $\ddot{a} \equiv d^2a/dt^2$ of the cosmic scale factor $a(t)$ is

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \tag{1.2}$$

which shows that the Universe is accelerating if $\rho+3p < 0$. Ordinary matter has mass-energy density ρ_M but essentially no pressure, $p_M = 0$, while vacuum has negative pressure equal to its mass-energy density, $p_\Lambda = -\rho_\Lambda$.]

2. Solutions to Friedmann's equations in a Flat Universe

Suppose that the Universe is flat, $\kappa = 0$, so that Friedmann's energy equation reduces to

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho \ . \tag{2.1}$$

Suppose further that the Universe is dominated by stuff whose mass-energy density ρ varies with cosmic scale factor a as

$$\rho \propto a^{-n} \tag{2.2}$$

as the Universe expands, with n a constant. For example, $n = 3$ for ordinary matter, $n = 4$ for radiation, and $n = 0$ for vacuum energy.

(a) Case $n \neq 0$

Solve Friedmann's equation to show that, for $n \neq 0$,

$$a \propto t^{2/n} \ . \tag{2.3}$$

[Hint: You should find that Friedmann's equation can be recast in the form $t = \int f(a)da$ where $f(a)$ is some function of cosmic scale factor a. You may set $a = 0$ at $t = 0$, which says that the Universe had zero size at zero age.]

(b) Deceleration or acceleration?

For what range of n is the Universe decelerating ($\ddot{a} < 0$) or accelerating ($\ddot{a} > 0$)? Is the Universe decelerating or accelerating in the particular cases of a matter-dominated ($n = 3$) or radiation-dominated ($n = 4$) Universe?

(c) Case $n = 0$

The case $n = 0$ corresponds to vacuum density, which remains constant as the Universe expands. Solve Friedmann's equation for this case to show that

$$a \propto e^{Ht} \tag{2.4}$$

where $H \equiv \dot{a}/a$, the Hubble constant, is in this case a constant in time as well as space. What is the Hubble constant H here in terms of the vacuum energy ρ_Λ?

(d) For your information (no credit)

You may be wondering whether there is a relation between the index n in this question and the pressure p in the Anti-Gravity question. The answer is yes. It is straightforward to show (but I'm not asking you to do this) from the energy equation $d(\rho a^3) + p d(a^3) = 0$ (which you may recognize as the equation $dE + pdV = 0$ of thermodynamics) that

$$n = 3\left(1 + \frac{p}{\rho}\right) . \tag{2.5}$$

3. Flatness problem

An amusing statement of this cosmological problem can be found on Ned Wright's graph at `http://www.astro.ucla.edu/~wright/cosmo_03.htm#FO` .

Suppose that the temperature at the moment of the Big Bang was about the Planck temperature $\sim 10^{32}$ K, and suppose for simplicity that from that time to the present, when the temperature is about the CMB temperature 3 K, the Universe has been radiation-dominated, so that the density ρ has been declining with cosmic scale factor a as $\rho \propto a^{-4}$. As you have learned in the lectures, curvature can be described by a "curvature density" ρ_K which declines as $\rho_K \propto a^{-2}$. Observations of the CMB today indicate that the curvature today is quite small, so ρ_K/ρ is a small number, at most 0.01 or so. Given the above rough assumptions, roughly how big, at most, was the ratio ρ_K/ρ of curvature to total density at the moment of the Big Bang? What value of $\Omega_K \equiv \rho_K/(\rho + \rho_K)$ does that translate into? [Hint: recall that temperature T varies with cosmic scale factor a as $T \propto a^{-1}$.]