Fluid eqs 20.1.

Fluid

One of the most common places where one encounters PDEs is the fluid equations. Hydrodynamic codes solve the fluid eqs numerically.

Q: What is a fluid?

A: A system that is locally sufficiently close to thermodynamic equilibrium that it can be characterized locally by thermodynamic variables

density	ρ	number	of each species, in general
bulk velocity	v	momentum	
pressure	p	volume	
internal energy	ε	energy	

Q: What drives a system to be close to thermodynamic equilibrium?

A: Collisions.

Fluid equations

The fluid equations represent the conservation of mass, momentum, and energy:

Mass: $\frac{\partial \rho}{\partial t} + \vec{\nabla}.\rho\vec{v} = 0$ $\quad \vec{\nabla} \equiv \frac{\partial}{\partial \vec{x}}$

Momentum $\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}.(\rho\vec{v}\vec{v}) + \vec{\nabla}p = \underset{\substack{\uparrow \\ \text{external force} \\ \text{e.g. gravity}}}{\vec{F}}$

Energy: $\frac{\partial(\rho\varepsilon)}{\partial t} + \vec{\nabla}.(\rho\varepsilon\vec{v}) + \vec{\nabla}.(p\vec{v}) = \vec{v}.\vec{F} + \underset{\substack{\nearrow \\ \text{heating} \\ -\text{cooling}}}{G}$

For fluid of adiabatic index γ,

$$\underset{\substack{\parallel \\ \text{energy} \\ \text{mass}}}{\varepsilon} = \underbrace{\frac{1}{2}v^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{\gamma-1}\frac{p}{\rho}}_{\text{internal energy}},$$

Q: How are fluid eqs derived?

A: • By waving hands
- From lowest order moments of the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt}\frac{\partial f}{\partial \vec{x}} + \frac{d\vec{p}}{dt}\frac{\partial f}{\partial \vec{p}} = \underset{\text{collision term}}{C[f]}$$

where $f(\vec{x},\vec{p})\, d^3x\, d^3p$ = # particles in interval $d^3x\, d^3p$ of phase space of position $\vec{x}$ and momentum $\vec{p}$.

Momentum eq $- \vec{v}$ times Mass eq
gives alternative version

Momentum: $$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}.\nabla\vec{v}\right) + \vec{\nabla}p = \vec{F}$$

Likewise Energy eq $- \varepsilon$ times Mass eq
gives

Energy: $$\rho\left(\frac{\partial \varepsilon}{\partial t} + \vec{v}.\nabla\varepsilon\right) + \vec{\nabla}.(p\vec{v}) = \vec{v}.\vec{F} + Q.$$

Q: Where does expression for ε come from?

~~A: First law of thermodynamics.~~

A: First law of thermodynamics

states that for a given mass element M

$$T\,dS = dE + p\,dV$$

temperature entropy, internal energy, pressure volume

[Q: Λ volume of mass element V or dV?
A: V. dV is change in V from one time to another]

Suppose that

$$E = \beta p V \qquad \text{some constant } \beta.$$

For example:

occup $\ll 1$, so fermi/bose unimportant

non-relativistic ideal Boltzmann gas Q? $\beta = \frac{3}{2}$

relativistic $\beta = 3$,

In the absence of heating or cooling, a mass element evolves adiabatically

ie. Q? $dS = 0$.

$$\begin{aligned} T\,dS &= 0 \\ &= d(\beta p V) + p\,dV \\ &= \beta V\,dp + (1+\beta)\,p\,dV \end{aligned}$$

Divide by pV:

$$\begin{aligned} 0 &= \beta \frac{dp}{p} + (1+\beta)\frac{dV}{V} \\ &= \beta\, d\ln p + (1+\beta)\, d\ln V \\ &= \beta\, d\ln p V^{\frac{(1+\beta)}{\beta}} \end{aligned}$$

ie $pV^{\gamma} = \text{constant}$ where $\gamma \equiv \frac{1+\beta}{\beta}$

$\rho = \frac{M}{V}$

Equivalently, since $V \propto \frac{1}{\rho}$ for a given mass element

$$p\rho^{-\gamma} = \text{constant}.$$

Q: Does this mean $p\rho^{-\gamma}$ = same constant everywhere in fluid?

A: No. It means $p\rho^{-\gamma}$ = constant as a function of time in any mass element.

Q: When does $p\rho^{-\gamma}$ = constant break down in a mass element?

A:
- When fluid is heated eg by radiation
- ——— cools ———
- At shocks, where kinetic energy is converted to internal energy.

β in terms of γ is (Q?)

$$\beta = \frac{1}{1-\gamma}$$

so $$\frac{E}{M} = \beta p \frac{V}{M} = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

is internal energy contribution to ε.

Sound waves

Consider perturbations to a fluid that has uniform initial density $\bar{\rho}$ and pressure $\bar{p}$, and zero velocity $\bar{v}$.

Q: Why might a fluid have a tendency to become uniform?

A: Because it tends to drive towards thermodynamic equilibrium.

Sound waves are small adiabatic perturbations of compression/rarefaction.

Q: Why adiabatic?

A: Because the vibrations are too fast to allow dissipation through energy transfer between neighboring gas elements.

Q: Don't sound waves dissipate?

A: Yes, but slowly compared to frequency.

Write

$$\rho = \underset{\text{mean}}{\bar{\rho}} + \underset{\text{pert}}{\delta\rho} = \bar{\rho}\,(1 + \underbrace{\delta}_{\equiv \frac{\delta\rho}{\rho}})$$

$$p = \underset{\text{mean}}{\bar{p}} + \underset{\text{pert}}{\delta p}$$

Since perturbations are adiabatic

$$p = \text{constant}\ \rho^{\gamma}$$

so $$\frac{\delta p}{\delta \rho} \approx \left.\frac{dp}{d\rho}\right|_s = \frac{\gamma p}{\rho}$$

Define $c_s^2 \equiv \frac{\gamma p}{\rho}$

so $\delta p = c_s^2 \delta\rho = c_s^2 \bar\rho \delta$,

Fluid eqs are

$\rho = \bar\rho(1+\delta)$ $\quad \rho(\vec{v}^{\,0} + \vec{\delta v}) = \bar\rho\,\vec{\delta v}$

Mass: $\frac{\partial \rho}{\partial t} + \vec\nabla \cdot \rho\vec v = 0$

ie $\bar\rho\left(\frac{\partial\delta}{\partial t} + \nabla\cdot\vec{\delta v}\right) = 0$ $\quad$ no external forces

$\vec v + \vec{\delta v}$ $\quad$ $\vec{v}^{\,0} + \vec{\delta v}$ $\quad$ $\bar p + c_s^2 \bar\rho\delta$

since $\vec v = 0$, $\vec{\delta v} = \vec v$. Keep $\vec v$ notation to avoid confusion over meaning of $\vec{\delta v}$.

Momentum $\frac{\partial \vec v}{\partial t} + \underbrace{\vec v \cdot \nabla \vec v}_{\to 0 \text{ to linear order}} + \frac{1}{\rho}\vec\nabla p = 0$

ie $\frac{\partial \vec{\delta v}}{\partial t} + c_s^2 \vec\nabla\delta = 0$

ie $\frac{\partial\delta}{\partial t} + \nabla\cdot\vec v = 0$

$\frac{\partial \vec v}{\partial t} + c_s^2 \vec\nabla\delta = 0$

Eliminate $\vec v$:

$$\frac{\partial^2\delta}{\partial t^2} = -\frac{\partial}{\partial t}\nabla\cdot\vec v = -\vec\nabla\cdot\frac{\partial\vec v}{\partial t} = c_s^2\nabla^2\delta$$

ie $\boxed{\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\delta = 0}$

Q: How to solve in comps?
A: Try $\delta = e^{\lambda t + \mu x}$

Solutions are superposition of sound waves

$$\delta = e^{-i\omega t + i\vec k\cdot\vec x}$$

with $\boxed{\omega^2 = c_s^2 k^2}$

[or equivalently superposition of
$\sin(\omega t - \vec{k}.\vec{x})$ & $\cos(\omega t - \vec{k}.\vec{x})$.]

Note that $-$ve frequency solutions

$$\left[\begin{aligned} \sin(-\omega t - \vec{k}.\vec{x}) &= -\sin(\omega t + \vec{k}.\vec{x}) \\ \cos(-\omega t - \vec{k}.\vec{x}) &= \cos(\omega t + \vec{k}.\vec{x}) \end{aligned}\right]$$

look like $+$ve frequency solutions propagating in the opposite direction.

Elegant and powerful way to obtain sound wave solutions is by Fourier transform

$\vec{\nabla} \to i\vec{k}$

$\frac{\partial}{\partial t} \to -i\omega$ standard physics convention

so

$$\left(-\omega^2 + c_s^2 k^2\right) \delta_{\omega,\vec{k}} = 0 \quad , \quad k \equiv |\vec{k}| .$$

$\Rightarrow$ non-trivial solutions $\delta_{\omega,\vec{k}}$ satisfy

$$\omega = \pm c_s k$$

But it may be convenient to FT space, but not time.

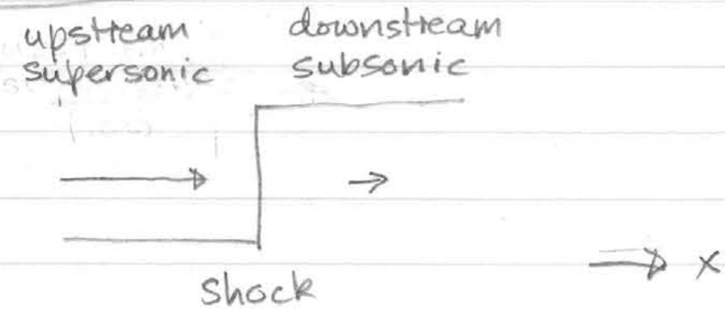
E.g. CMB perturbations.

Example: Project 1

1. Perturbation in Fourier space characterized by superposition of modes $\delta_{\omega,\vec{k}}$ and $\delta_{-\omega,\vec{k}}$ where $\omega \equiv c_s k$.
2. Reality of $\delta(x)$ in real space
$\Rightarrow \delta^*_{\omega,\vec{k}} = \delta_{-\omega,-\vec{k}}$
and $\delta^*_{-\omega,\vec{k}} = \delta_{\omega,-\vec{k}}$
2. In time step Δt
$\delta_{\omega,\vec{k}} \to e^{-i\omega\Delta t}\delta_{\omega,\vec{k}}$
$\delta_{-\omega,\vec{k}} \to e^{i\omega\Delta t}\delta_{-\omega,\vec{k}}$
3. Use FFT to translate between real ($\vec{x}$) and Fourier ($\vec{k}$) space

Sound waves are small (linear) perturbations to a nearly uniform fluid.
But a fluid also supports (non-linear) discontinuities called

Shock waves



For simplicity, work in the frame of the shock front. In this frame the flow is steady, meaning

$$\frac{\partial}{\partial t} = 0.$$

Let the shock be moving along the x-direction, so $\vec{v}$ is along x, and $\vec{\nabla}$ reduces to $\frac{\partial}{\partial x}$.

The fluid eqs are:

Mass: $\frac{\partial}{\partial x}(\rho v) = 0$

Momentum: $\frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0$ assume no external force

Energy: $\frac{\partial}{\partial x}\left(\rho \varepsilon v + p v\right) = 0$ assume no heating cooling

As will be seen, these equations admit shock solutions where ρ, v, p, ε change discontinuously across a shock front.

Integrating the fluid eqs across the shock gives

Mass: $\left[\rho v\right]_-^+ = 0$ ← + downstream, post-shock; − upstream, pre-shock

Momentum: $\left[\rho v^2 + p\right]_-^+ = 0$

Energy: $\left[\underbrace{(\rho\varepsilon + p)}_{= \frac{1}{2}\rho u^2 + \frac{\gamma}{\gamma-1} p} v\right]_-^+ = 0$

Mass $\Rightarrow$

$$\rho_+ v_+ = \rho_- v_- .$$

Momentum $\Rightarrow$

$$p_+ = p_- + \rho_- v_-^2 - \rho_+ v_+^2$$
$$= p_- + \rho_- v_- (v_- - v_+) .$$

Energy $\Rightarrow$ eliminate ρ_+ with mass; eliminate p_+ with momentum

$$0 = \frac{1}{2}\left(\rho_+ v_+^3 - \rho_- v_-^3\right) + \frac{\gamma}{\gamma-1}\left(p_+ v_+ - p_- v_-\right)$$

$$= \frac{1}{2}\rho_- v_- \left(v_+^2 - v_-^2\right) + \frac{\gamma}{\gamma-1}\left[p_-(v_+ - v_-) + \rho_- v_- (v_- - v_+) v_+\right]$$

$$= (v_+ - v_-)\left\{\rho_- v_- \left[\frac{1}{2}(v_+ + v_-) - \frac{\gamma}{\gamma-1} v_+\right] + \frac{\gamma}{\gamma-1} p_-\right\}$$

$$= \frac{(v_+ - v_-)}{(\gamma-1)} \rho_- v_-^2 \left\{-(\gamma+1)\frac{v_+}{v_-} + (\gamma-1) + \underbrace{\frac{2\gamma p_-}{\rho_- v_-^2}}_{= 2/M^2}\right\}$$

where M is Mach number

$$M \equiv \frac{v_-}{c_{s-}} = \left(\frac{\gamma p_-}{\rho_- v_-^2}\right)^{-\frac{1}{2}} .$$

Thus

either $v_+ = v_-$ ← usual case

or $\boxed{\frac{v_+}{v_-} = \frac{\gamma - 1 + 2/M^2}{\gamma + 1}}$ ← shock solution

Same as

$$\frac{v_+}{v_-} = 1 - \frac{2(M^2-1)}{(\gamma+1)M^2} \quad \text{or} \quad \frac{v_- - v_+}{v_-} = \frac{2(M^2-1)}{(\gamma+1)M^2}$$

Since $\underset{\text{downstream}}{v_+} < \underset{\text{upstream}}{v_-}$

shock solution requires

$$\boxed{M > 1} .$$

What about ρ and p?

$$\frac{\rho_-}{\rho_+} = \frac{v_+}{v_-} = 1 - \frac{2(M^2-1)}{(\gamma+1)M^2}$$

$$\frac{p_+}{p_-} = 1 + \underbrace{\frac{\rho_- v_-^2}{p_-}}_{= \gamma M^2}\left(\frac{v_- - v_+}{v_-}\right)$$

$$= 1 + \frac{2\gamma(M^2-1)}{\gamma+1}$$

Self-similarity

Self-similar solutions have the property that the solution (to the system of PDEs being considered, eg. the fluid eqs) at any one time is related by a scale transformation to the solution at any other.

Usually the scale transformation is a power law $x \propto t^{\alpha}$ ← some constant.

Consequently dimensionless variables describing the shape of the system are functions only of

$$\frac{x}{t^{\alpha}}.$$

Advantage? Self-similarity reduces the dimension of the system of PDEs by one. If the original PDEs depend on time t together with just one spatial dimension, then the system of 2D PDEs reduces to 1D ODEs.

Q: When do PDEs depend on one spatial dim?

A: • Plane symmetry
- Cylindrical symmetry
- Spherical symmetry

Applications of self-similarity

- Supernova/other explosions in uniform/power law external medium (Sedov solution)
- Stellar/galactic winds
- Black holes
- Cosmology

  ⋮

- Riemann problem PS 10.

Reality is usually more complicated than self-sim solutions. But self-sim solns

- Can provide insights into the behavior of highly non-linear systems
  Q: means? A: not small pert
- May actually be a decent approximation to reality.

When is evolution self-similar?

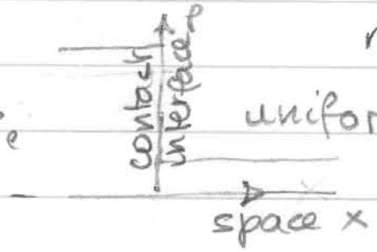
When initial/boundary conditions admit self-similarity.

Riemann problem

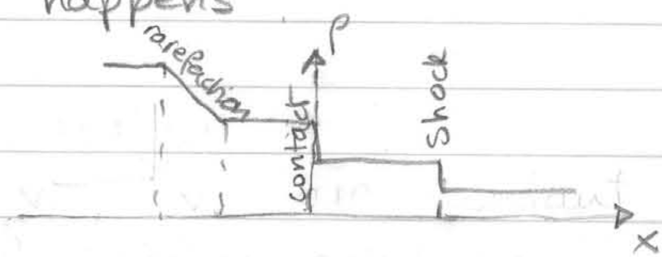
1D "shock tube" problem
for fluid of adiabatic index γ.
Initial ($t=0$) conditions:

left | right

uniform ρ_l, v_l, p_l | uniform ρ_r, v_r, p_r



Go! What happens?



$\left\{\begin{array}{l}\text{Rarefaction} \\ \text{or} \\ \text{Shock}\end{array}\right\}$ waves propagate left and right.

Dimensional analysis:

Initial $\left\{\begin{array}{ll}\text{velocities} & v_l, v_r \\ \text{sound speeds} & c_l, c_r\end{array}\right\}$ constant

$\Rightarrow \exists$ self-similar solutions
expanding at constant velocity
$x \propto t$

$\Rightarrow$ dimensionless quantities depend
only on $\frac{x}{t}$

Can regard $\frac{x}{t}$ as "dimensionless"
hence all velocities as "dimensionless".
Thus $v = v\left(\frac{x}{t}\right)$, $c = c\left(\frac{x}{t}\right)$.

Other self-similar initial conditions?
What could $\rho_\ell, v_\ell, p_\ell$ be
ρ_r, v_r, p_r
and still admit self-similarity?

Could have initially
v_ℓ and c_ℓ $\propto$ $x^{-\lambda}$ ← constant
$\quad{}_r \qquad\quad {}_r$

Dimensional analysis $\Rightarrow$
$$\frac{x}{t} \propto x^{-\lambda} \quad \text{ie} \quad x^{1+\lambda} \propto t$$
ie $x \propto t^{\frac{1}{1+\lambda}}$.

Initial conditions are isentropic either side
if $p \propto \rho^{\gamma}$
ie if $c^2 \equiv \frac{\gamma p}{\rho} \propto \rho^{\gamma-1}$
ie if $\rho \propto c^{\frac{2}{\gamma-1}} \propto x^{-\frac{2\lambda}{\gamma-1}}$.

Could evolution be self-sim
even if not isentropic?
E.g. $\rho \propto x^{-\mu}$ ← constant
Then $p \propto \rho c^2 \propto x^{-\mu-2\lambda}$.
Maybe ...

1D Lagrangian hydrodynamic code

Lagrangian version of fluid eqs

Follow paths of fluid elements

$$\square \xrightarrow{\vec{v} = \frac{d\vec{x}}{dt}}$$

Rate of change $\frac{d}{dt}$ along path of fluid element is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\vec{x}}{dt} \cdot \frac{\partial}{\partial \vec{x}}$$

$$= \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} .$$

Fluid eqs:

Mass: $$\underbrace{\frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho}_{= \frac{d\rho}{dt}} + \rho \vec{\nabla} \cdot \vec{v} = 0$$

ie $$\boxed{\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0}$$

Momentum:

$$\vec{v} \underbrace{\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} \right)}_{= 0} + \rho \underbrace{\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right)}_{= \frac{d\vec{v}}{dt}} + \vec{\nabla} p = \vec{F}$$

ie $$\boxed{\rho \frac{d\vec{v}}{dt} + \vec{\nabla} p = \vec{F}}$$

Energy:

$$\varepsilon \underbrace{\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} \right)}_{0} + \rho \underbrace{\left(\frac{\partial \varepsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \varepsilon \right)}_{= \frac{d\varepsilon}{dt}} + \vec{\nabla} \cdot (p \vec{v}) = \vec{v} \cdot \vec{F} + G$$

ie $$\boxed{\rho \frac{d\varepsilon}{dt} + \vec{\nabla} \cdot (p \vec{v}) = \vec{v} \cdot \vec{F} + G}$$

ID hydro

Assume no sources, $\vec{F} = q = 0$.

Assume 1D.

Then

Mass; $\dfrac{d\rho}{dt} = -\rho \dfrac{\partial v}{\partial x}$ or equivalently $\dfrac{d\frac{1}{\rho}}{dt} = \dfrac{1}{\rho}\dfrac{\partial v}{\partial x}$

Momentum: $\dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

Energy: $\dfrac{d\varepsilon}{dt} = -\dfrac{1}{\rho}\dfrac{\partial (pv)}{\partial x}$

Discretize into zones each of mass m.

$\rho = \dfrac{m}{\Delta x}$ ← mass of zone / ← volume (width) of zone

Mass: $\dfrac{d\Delta x}{dt} = \Delta x \dfrac{\partial v}{\partial x}$

integrates to $\dfrac{dx}{dt} = v$ (yey!)

Momentum: $\dfrac{dv}{dt} = -\dfrac{\Delta x}{m}\dfrac{\partial p}{\partial x} = -\dfrac{1}{m}\Delta p$

Energy: $\dfrac{d\varepsilon}{dt} = -\dfrac{\Delta x}{m}\dfrac{\partial (pv)}{\partial x} = -\dfrac{1}{m}\Delta (pv)$

Label zones like this:

$x_{i-\frac{1}{2}}$	x_i	$x_{i+\frac{1}{2}}$
left edge	zone	right edge

Integrate over timestep Δt:

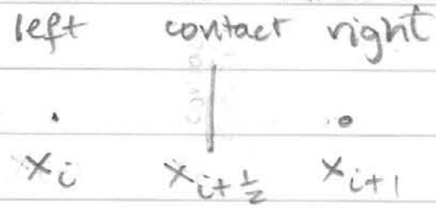
$$x_{i+\frac{1}{2}} \mathrel{+}= \Delta t \, v_{i+\frac{1}{2}}$$

$$v_i \mathrel{+}= \frac{-\Delta t}{m}\left(p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}\right)$$

$$\varepsilon_i \mathrel{+}= \frac{-\Delta t}{m}\left(p_{i+\frac{1}{2}} v_{i+\frac{1}{2}} - p_{i-\frac{1}{2}} v_{i-\frac{1}{2}}\right)$$

Key issue:
How to estimate values
$v_{i+\frac{1}{2}}$ and $p_{i+\frac{1}{2}}$ at zone edges
from value v_i, p_i at zone centers?

Godunov approach:
Use solution to Riemann problem



Treat ρ, v, p as uniform within each zone. Solve for $v_{i+\frac{1}{2}}$ and $p_{i+\frac{1}{2}}$ at contact interface.

"Riemann solvers" are at heart of modern hydrodynamic "shock-capturing" codes.
Advantages:
+ resolve shocks in few zones
+ conserve entropy outside of shocks, preventing undesirable dissipation.

Disadvantages:

- Involves iterative solution of implicit equation, which is numerically expensive.

In PS 10, found that implicit equation is

$$p_c = \boxed{p_\ell f\left(\frac{v_\ell - v_c}{c_\ell}\right) = p_r f\left(\frac{v_c - v_r}{c_r}\right)}$$

where $f(z) = \begin{cases} \left(1 + \frac{\gamma-1}{2} z\right)^{\frac{2\gamma}{\gamma-1}} & z \le 0 \quad \text{rarefaction} \\ 1 + \gamma z M & z \ge 0 \quad \text{shock} \end{cases}$

where M is mach number

$$M = \frac{\gamma+1}{4} z + \sqrt{\left(\frac{\gamma+1}{4} z\right)^2 + 1} \, .$$

Ways codes improve on Godunov's approach:

- model ρ, v, p in each zone as polynomial instead of constant
- find faster approximate solution to Riemann problem.

<u>Courant condition on timestep Δt</u>

Timestep Δt must be small enough that information propagates across less than 1 zone.

In Lagrangian code,

$$\Delta t \leq \frac{\Delta x}{c} \quad \text{sound-crossing time}$$

In Eulerian code, where fluid moves through zone at velocity v,

$$\Delta t \leq \frac{\Delta x}{|v| + c} .$$

For PS11, take

$$\Delta t = \underset{\uparrow \atop \text{Courant}}{\ell} \min_{\text{zones}} \left(\frac{\Delta x}{c} \right)$$

Take $\ell < 1$ with a safety factor.

eg. $\ell = 0.7$.

Solution of PDEs by characteristics

1st order linear PDE

Consider 1st order linear PDE

$$a(t,x)\frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} = f(t,x,u)$$

Compare to identity

$$dt\,\frac{\partial u}{\partial t} + dx\,\frac{\partial u}{\partial x} = du$$

Define characteristic curves by

$$\frac{dx}{dt} = \frac{b}{a}$$ Q: Is this an ODE? A: Yes. 1st order ODE.

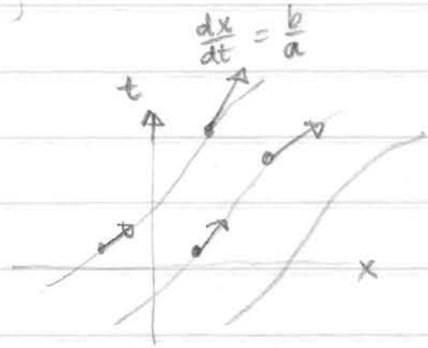
Along characteristic curve

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{b}{a}\frac{\partial u}{\partial x} = \frac{f}{a}.$$ Q: ODE? A: Yes

Summary: Solution of 1st order linear PDE in 2 variable reduces to solution of 2 1st order ODEs

$$\boxed{\begin{aligned}\frac{dx}{dt} &= \frac{b}{a}\\ \frac{du}{dt} &= \frac{f}{a}\end{aligned}}$$

Compactly: $\frac{dt}{a} = \frac{dx}{b} = \frac{du}{f}$.



Solutions of

$$\frac{dx}{dt} = \frac{b}{a}$$

define family of curves

$$\xi(t,x) = \text{constant}.$$

Called "characteristic curves", or "characteristics".

What do characteristics mean physically?

Characteristic equation

$$\frac{dx}{dt} = \frac{b}{a}$$

defines a flow pattern, moving with velocity b/a at each point.

Q; Do flow lines intersect? A: No.

Typically the velocity has a physical meaning, eg.

- flow velocity
- wave velocity (eg. sound speed).

<u>Homogeneous case $f \equiv 0$</u>

Then

$$\left.\frac{du}{dt}\right|_{\xi} = \frac{f}{a} = 0 \quad \text{along char curve } \xi = \text{const}$$

$\Rightarrow$ u = constant along characteristic ξ
(but can be a different constant for each ξ)

$\Rightarrow$ $\boxed{u_{hom} = g(\xi)}$ = arbitrary function of ξ.

Fix $g(\xi)$ by boundary conditions.
Need exactly one b.c. for each ξ.

<u>Inhomogeneous case $f \neq 0$</u>

Then

$$\left.\frac{du}{dt}\right|_{\xi} = \frac{f}{a}$$

Integrate along characteristic ξ = const to find particular solution u_{part}.

General solution is sum of particular and homogeneous solns

$$\boxed{u = u_{part} + g(\xi)}$$

Example:

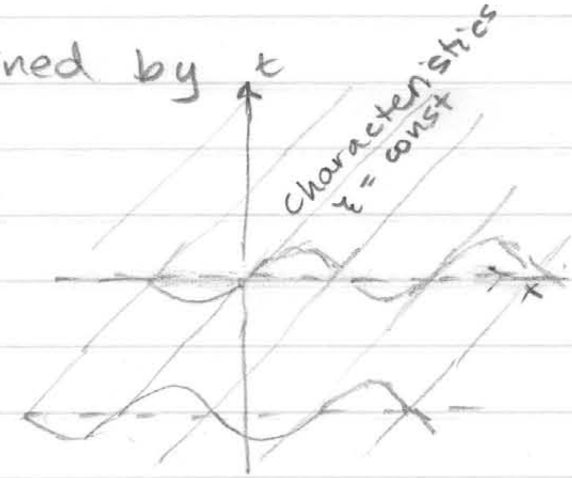
1st order wave or advection equation

$$\frac{\partial u}{\partial t} + c_s \frac{\partial u}{\partial x} = f(t, x, u)$$

constant wave or advection speed

Then $a = 1$, $b = c_s$.

Characteristic curves defined by



$$\frac{dx}{dt} = \frac{b}{a} = c_s$$

$\Rightarrow$ $x = c_s t$ + constant

Choose

$$\xi = x - c_s t = \text{constant}$$

Homogeneous solution $f = 0$:

$$u_{hom} = g(\xi) = g(x - c_s t) = \text{arb func of } x - c_s t.$$

Inhomogeneous solution $f \neq 0$:

$$\left.\frac{du}{dt}\right|_{\xi} = \frac{f}{a} = f(t, x, u)$$

a = 1 here

Along ξ = constant, $x = c_s t + \xi$, so

$$\frac{du}{dt} = f(t, c_s t + \xi, u) \qquad \text{1st order ODE.}$$

If $f = f(t, x)$ only, indept of u, then

$$u_{part} = \int^t f(t, c_s t + \xi)\, dt$$

A more realistic, and complicated example:

Characteristics of 1D fluid eqs

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial(\rho v^2 + p)}{\partial x} = 0$$

$$\frac{\partial \rho \varepsilon}{\partial t} + \frac{\partial(\rho \varepsilon v + p v)}{\partial x} = 0$$

Three 1st order nonlinear PDEs.

Eqs can be written

$$\frac{\partial q_i}{\partial t} + \frac{\partial f_i}{\partial x} = 0$$

$$q_i \equiv \{\rho, \rho v, \rho \varepsilon)$$
$$f_i \equiv \{\rho v, \rho v^2 + p, \rho \varepsilon v + p v\}.$$

Fluxes f_i are functions only of variables q_i, not of t or x explicitly.

So can write fluid eqs as

$$\frac{\partial q_i}{\partial t} + \frac{\partial f_i}{\partial q_j}\frac{\partial q_j}{\partial x} = 0.$$

Diagonalize $\partial f_i / \partial q_j$: hydro_characteristics.nb

$$\frac{\partial f}{\partial q} = P \underset{\substack{\uparrow \\ \text{matrix of} \\ \text{eigenvalues}}}{\Lambda} P^{-1}$$

P = matrix of eigenvectors

so

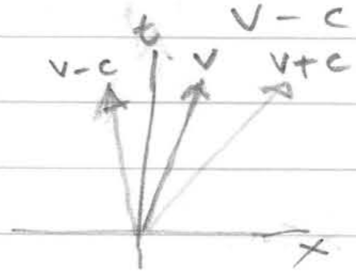
$$P^{-1}\frac{\partial q}{\partial t} + \Lambda P^{-1}\frac{\partial q}{\partial x} = 0$$

Matrix Λ of eigenvalues proves to be

$$\Lambda = \begin{pmatrix} v & & \\ & v+c & \\ & & v-c \end{pmatrix}$$

Thus fluid eqs define 3 characteristic curves,

- one moving at the fluid velocity v;
- other two moving along the sound cone, with velocity $v+c$ to right, $v-c$ to left



Along characteristic ξ with eigvalue λ

$$\left.\frac{d}{dt}\right|_{\xi_\lambda} = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x} .$$

Fluid eqs may thus be written

$$P^{-1} \left.\frac{dq}{dt}\right|_{\xi} = 0$$

These are 3 coupled ODEs

BUT they are to be integrated along 3 different characteristic curves.

The 1st of the 3 eqs, along v characteristic, is

$$\boxed{\left.\frac{ds}{dt}\right|_{\xi_v} = 0} \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$$

which is conservation of entropy / mass s

$$s \equiv \frac{1}{\gamma-1} \ln\left(\frac{p}{\rho^{\gamma}}\right) = \ln\left(\frac{c^{\frac{2}{\gamma-1}}}{\rho}\right) .$$

The other 2 eqs, along $v \pm c$ characteristics, are

$$\boxed{\left.\frac{d}{dt}\right|_{\xi_{v\pm c}} \underbrace{\left(v \pm \frac{2}{\gamma-1} c\right)}_{\text{"Riemann invariants"}} \mp \frac{c}{\gamma} \left.\frac{ds}{dt}\right|_{\xi_{v\pm c}} = 0} .$$

Q: Was that useful to know?

A: Yes and no.

No: Does not yield 3 coupled ODEs that can be solved in usual for ODEs way, because characteristic curves are along 3 different directions.

Yes: + In special cases, system of ODEs can be solved.

+ Yields insight into solution (maybe?)

+ May motivate a strategy for solution.

<u>Changing variables to characteristics</u>

Characteristic curves τ, ξ, η satisfy

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\tau = 0 \quad \Rightarrow \quad \tau_t + v\tau_x = 0$$

$$\left(\frac{\partial}{\partial t} + (v+c)\frac{\partial}{\partial x}\right)\xi = 0 \quad \Rightarrow \quad \xi_t + (v+c)\xi_x = 0$$

$$\left(\frac{\partial}{\partial t} + (v-c)\frac{\partial}{\partial x}\right)\eta = 0 \quad \Rightarrow \quad \eta_t + (v-c)\eta_x = 0.$$

$$\frac{\partial}{\partial t} + \frac{v\partial}{\partial x} = (\tau_t + v\tau_x)\frac{\partial}{\partial \tau} + (\xi_t + v\xi_x)\frac{\partial}{\partial \xi} + (\eta_t + v\eta_x)\frac{\partial}{\partial \eta}$$

$$= c\left(-\xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial t} + (v+c)\frac{\partial}{\partial x} = c\left(\tau_x\frac{\partial}{\partial \tau} + 2\eta_x\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial t} + (v-c)\frac{\partial}{\partial x} = c\left(-\tau_x\frac{\partial}{\partial \tau} - 2\xi_x\frac{\partial}{\partial \xi}\right)$$

Does changing variables to characteristics help?

Maybe.

- Lagrangian code follows v-characteristic. (goes with the flow)
  v-characteristic curve is just position
  $$x_{Lag}(t, x)$$
  of Lagrangian mass element.

  Lagrangian (t, x_{Lag}) as opposed to Eulerian (t, x) codes are fairly common.
  + Potentially more accurate, eg. larger timestep.
  - In $\geq 2D$, Lagrangian mesh gets quite distorted.

  But be careful!
  $$\frac{ds}{dt} = 0 \quad \text{(Lagrangian)}$$
  fails at shocks.

- Sometimes see codes (more commonly in general relativity) in which variables are $v \pm c$ - characteristics.
  Eqs wrt $v \pm c$ - characteristics remain PDEs, not ODEs, so not usually much advantage.

Solution of PDEs by characteristics (cont)

Recall that for 1st order linear PDEs, considered

$$a u_t + b u_x = f$$

combined with the identity

$$dt\, u_t + dx\, u_x = du.$$

Pair of eqs can be written compactly

$$\begin{pmatrix} a & b \\ dt & dx \end{pmatrix} \begin{pmatrix} u_t \\ u_x \end{pmatrix} = \begin{pmatrix} f \\ du \end{pmatrix}.$$

Characteristics are defined by

$$\frac{dx}{dt} = \frac{b}{a}$$

or equivalently

$$\begin{vmatrix} a & b \\ dt & dx \end{vmatrix} = 0.$$

Convenient abbreviation

$$u_t \equiv \frac{\partial u}{\partial t}, \qquad u_x \equiv \frac{\partial}{\partial x}$$

2nd order linear PDE

Consider

$$a(t,x)\, u_{tt} + 2b(t,x)\, u_{tx} + c(t,x)\, u_{xx}$$

$$= f(t, x, u, u_t, u_x).$$

Also have identities

$$dt\, u_{tt} + dx\, u_{tx} = du_t$$

$$dt\, u_{tx} + dx\, u_{xx} = du_x$$

i.e.

$$\begin{pmatrix} a & 2b & c \\ dt & dx & 0 \\ 0 & dt & dx \end{pmatrix} \begin{pmatrix} u_{tt} \\ u_{tx} \\ u_{xx} \end{pmatrix} = \begin{pmatrix} f \\ du_t \\ du_x \end{pmatrix}$$

Characteristic curves are defined by

$$\begin{vmatrix} a & 2b & c \\ dt & dx & 0 \\ 0 & dt & dx \end{vmatrix} = 0$$

Q: Why?
A: Because, as we'll see, transformation to characteristic variables kills 2nd derivs...

ie

note - sign came in

$$a\,dx^2 - 2b\,dt\,dx + c\,dt^2 = 0$$

ie $$a\left(\frac{dx}{dt}\right)^2 - 2b\frac{dx}{dt} + c = 0$$

ie $$\boxed{\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$ characteristic eqn.

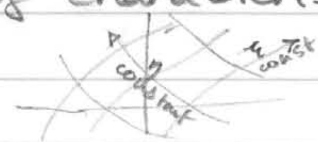
Q: ODE?

A: Yes, if a, b, c funcs of t, x (not u).

Characteristic eqn has 2 solutions provided that $b^2 - ac \neq 0$:

$$\xi(t,x) = \text{constant} \quad \text{and} \quad \eta(t,x) = \text{constant}$$

which constitute 2 families of characteristic curves.



Along characteristic ξ = constant, have

$$d\xi = 0 = \xi_t \, dt + \xi_x \, dx$$

ie $$\frac{dx}{dt} = -\frac{\xi_t}{\xi_x}$$ back to + sign

ie $$a\,\xi_t^2 + 2b\,\xi_t \xi_x + c\,\xi_x^2 = 0$$

Likewise

$$a\,\eta_t^2 + 2b\,\eta_t \eta_x + c\,\eta_x^2 = 0.$$

Now look at what happens when you transform original PDE to new variables

$$t, x \longrightarrow \xi, \eta$$ (provided that $b^2 - ac \neq 0$, to ensure 2 characteristics)

Then

$$a u_{tt} + 2b u_{tx} + c u_x^2 = f$$

transforms to

$$A u_{\xi\xi} + 2B u_{\xi\eta} + C u_\eta^2 = f - D u_\xi - E u_\eta$$
$$\equiv F$$

where (straightforward but laborious application of the chain rule)

$$A = a\xi_t^2 + 2b\,\xi_t\xi_x + c\,\xi_x^2 \qquad = 0$$
$$B = a\,\xi_t\eta_t + b(\xi_t\eta_x + \xi_x\eta_t) + c\,\xi_x\eta_x$$
$$C = a\,\eta_t^2 + 2b\,\eta_t\eta_x + c\,\eta_x^2 \qquad = 0$$
$$D = a\,\xi_{tt} + 2b\,\xi_{tx} + c\,\xi_{xx}$$
$$E = a\,\eta_{tt} + 2b\,\eta_{tx} + c\,\eta_{xx}\,.$$

Along characteristics (provided that $b^2 - ac \neq 0$, to ensure 2 characteristics)

$$A = C = 0$$

so PDE reduces to

$$2B\,u_{\xi\eta} = F$$

or

$$\boxed{u_{\xi\eta} = \frac{F}{2B}}$$

Note that

$$B^2 - AC = (b^2 - ac)(\xi_t\eta_x - \xi_x\eta_t)^2$$

so (provided that everything is real) sign of discriminant $b^2 - ac$ is preserved under change of variables $t, x \to \xi, \eta$.

Q: Really? Aren't characteristics imaginary?

A: Well, I'm taking ξ η here to be general variables, not nec. characteristics.

Traditional classification of 2nd order linear PDEs

Assume a, b, c real.

(1) Hyperbolic $\quad b^2 > ac \quad$ wave equation
(2) Parabolic $\quad b^2 = ac \quad$ diffusion equation
(3) Elliptic $\quad b^2 < ac \quad$ Poisson equation.

(1) Hyperbolic $b^2 > ac$.

Characteristic eqn

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

defines 2 families of real characteristic curves, propagating along real paths from past into future.

Equations can be solved starting from prescribed initial conditions.

Sets of PDEs all of whose characteristics are real, and therefore can be solved by integrating forward from prescribed initial conditions are commonly called hyperbolic.

"Normal form" of hyperbolic 2nd order linear PDE is

$$u_{\xi\eta} = \frac{F}{2B}.$$

Example of hyperbolic 2nd order linear PDE:

Wave (Helmholtz) equation

$$u_{tt} - c_s^2 u_{xx} = \underbrace{f(t,x)}_{\text{source}}$$

Q: If you met this in comps, how would you solve it?
A: Try $e^{\lambda t + \mu x}$.

c_s = wave speed (not to be confused with c in a, b, c).

Here $a = 1$, $b = 0$, $c = -c_s^2$.

$b^2 - ac = c_s^2 > 0 \Rightarrow$ hyperbolic.

Characteristic eqn

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm c_s$$

has solutions

$$x = \pm c_s t + \text{constant}.$$

Characteristic curves are

$$\xi = x - c_s t$$
$$\eta = x + c_s t$$

Here $\xi_t = -c_s$, $\xi_x = 1$, $\xi_{tt} = \xi_{tx} = \xi_{xx} = 0$,

$\eta_t = +c_s$, $\eta_x = 1$, $\eta_{tt} = \eta_{tx} = \eta_{xx} = 0$

$\Rightarrow$ $A = C = 0$

$B = -2c_s^2$

$D = E = 0$

$F = f$

so

$$u_{\xi\eta} = \frac{-f}{4c_s^2}$$

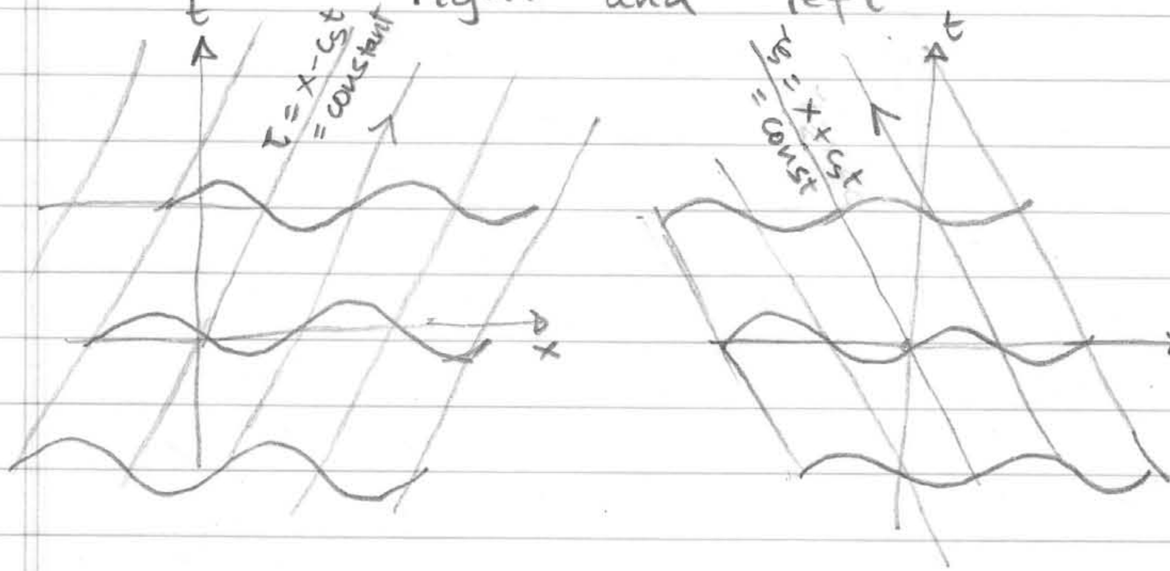
Homogeneous solution $f=0$:

$$u_{\xi\eta} = 0$$

$$\Rightarrow u_{hom} = g(\xi) + h(\eta) \qquad g, h \text{ arbitrary funcs}$$

$$= g(x - c_s t) + h(x + c_s t).$$

Physically, these represent waves travelling right and left



$f(t,x)$ represents source of waves injected at time t, position x.

(2) Parabolic $b^2 = ac$

Char equ

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{b}{a}$$

defines only one set of characteristic curves.

Say $\xi(t,x)$ = constant are char curves

Transform PDE

$$t, x \longrightarrow \xi, x$$

Then

$$A = a\xi_t^2 + 2b\xi_t\xi_x + c\xi_x^2 = 0$$

and

$$B^2 - AC = (b^2 - ac)\xi_t^2 = 0$$

$$\Rightarrow B = 0$$

$$C = a x_t^2 + 2b x_t x_x + c x_x^2$$
$$= c$$

$$D = a\xi_{tt} + 2b\xi_{tx} + c\xi_{xx}$$
$$= \text{something } (\neq 0)$$

$$E = a x_{tt} + 2b x_{tx} + c x_{xx}$$
$$= 0$$

So PDE reduces to

$$c u_{xx} = F \equiv f - D u_\xi - E u_x$$
$$= f - D u_\xi$$

ie $\boxed{u_{xx} + \frac{D}{c} u_\xi = f}$ "normal form" of parabolic eq

Example of parabolic 2nd order linear PDE

<u>Diffusion equation</u>

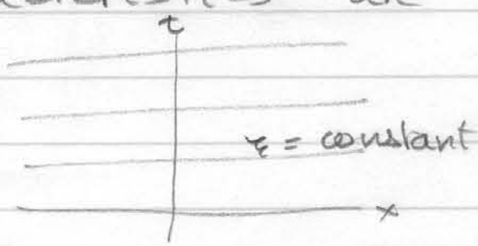
$$u_t - \kappa u_{xx} = f(t,x)$$

κ = diffusivity = const

$$a = b = 0, \quad c = -\kappa$$

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm\sqrt{\frac{-c}{a}} = \pm\sqrt{\frac{\kappa}{0}} = \infty.$$

Here characteristics are horizontal lines.



Characteristic curves can be taken to be

$\xi = t$.

Q: How to solve hom eq?

A: Try

$$u = e^{\lambda t + \mu x}$$

Then

$$u_t - \kappa u_{xx} = \lambda - \kappa\mu^2 = 0$$

ie $\lambda = \kappa\mu^2$.

Physical solutions have $\lambda < 0$

$\Rightarrow \mu^2 < 0$

Write $\mu = ik$.

$\Rightarrow$ Solns are superpos of

$$e^{-\kappa k^2 t + ikx}$$

25.10.

More formally, scale x so $\kappa = 1$.

GF of diffusion eq

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = \delta_D(t-t_0)\,\delta_D(x-x_0).$$

FT wrt x: $\int_{-\infty}^{\infty} e^{-ikx}/\sqrt{2\pi}\,dx$

$$\frac{\partial \tilde{u}}{\partial t} + k^2\tilde{u} = \delta_D(t-t_0)\,\frac{e^{-ikx_0}}{\sqrt{2\pi}}$$

Homog solution is Q?

$$\hat{u}_k \propto e^{-k^2 t}$$

To find const

$$\int_{t_0-}^{t_0+}\left(\frac{\partial \hat{u}}{\partial t} + k^2\hat{u}\right)dt = \int_{t_0-}^{t_0+}\delta_D(t-t_0)\,\frac{e^{-ikx_0}}{\sqrt{2\pi}}\,dt$$

$$= \left[\tilde{u}\right]_{t_0-}^{t_0+} = \frac{e^{-ikx_0}}{\sqrt{2\pi}}$$

With retarded b.c.s,

$$\Rightarrow \hat{u}_k = \begin{cases} 0 & t < t_0 \\ \dfrac{e^{-k^2(t-t_0)-ikx_0}}{\sqrt{2\pi}} & t \geq t_0 \end{cases}$$

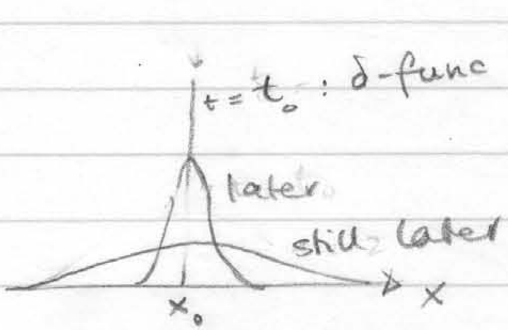
FT back into x space

$$\int_{-\infty}^{\infty}\hat{u}_k\,\frac{e^{ikx}}{\sqrt{2\pi}}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2(t-t_0)+ik(x-x_0)}\,dk$$

$$= \frac{1}{\sqrt{4\pi(t-t_0)}}\exp\left[-\frac{(x-x_0)^2}{4(t-t_0)}\right]$$

Looks like a gaussian that spreads out with width $\propto \sqrt{t-t_0}$



Diffusion!

Q: source is, eg.? A: Particles, heat, ...

25.11

Q: How to do that $\int$?

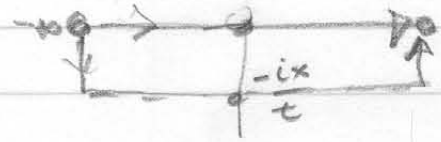
$$\int_{-\infty}^{\infty} e^{-k^2 t + ikx}\, dk$$

$$= \int_{-\infty}^{\infty} e^{-t\left(k - \frac{ix}{t}\right)^2 - \frac{x^2}{4t}}\, dk$$

$$= e^{-x^2/4t} \int_{-\infty}^{\infty} e^{-z^2} \frac{dz}{\sqrt{t}}$$

$$= \sqrt{\pi}\, \frac{e^{-x^2/4t}}{\sqrt{t}}$$

Shift integration path in complex plane



$z \equiv \sqrt{t}\left(k - \frac{ix}{t}\right)$

Q: Can parabolic PDE be solved, like hyperbolic, by integrating forward from prescribed initial conditions?

A: Yes.

Another example of parabolic 2nd order linear PDE

<u>Schrödinger's equation</u>

$$u_t - \underset{\substack{\uparrow \\ i=\sqrt{-1}}}{i}\, u_{xx} = \underset{\text{source}}{f(t,x)}$$

Characteristic are same as diffusion eq, namely horizontal lines (lines of constant t).

Q: Solve hom eq?

A: Try $u = e^{\lambda t + \mu x}$

Then

$$u_t - i u_{xx} = \lambda - i\mu^2 = 0$$

ie $\lambda = i\mu^2$

Try Fourier expansion w.r.t x, ie take $\mu^2 < 0$. Write $\mu = ik$, so $\lambda = -ik^2$.

Solutions are waves

$$e^{-ik^2 t + ikx}$$

$$= e^{-ik(kt - x)}$$

with wavespeed k (= faster for shorter wavelengths).

Very short wavenumbers, $k \to 0$, propagate at ∞ velocity!

Makes sense? Schrödinger eq is valid only at non-relativistic velocities. Replaced by Dirac eqn at relativistic velocities.

(3) Elliptic $b^2 < ac$

Char eqn

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

has 2 complex conjugate solutions

$$\xi(t,x) = \mu(t,x) + i\,\nu(t,x)$$
$$\eta(t,x) = \mu(t,x) - i\,\nu(t,x)$$

with μ, ν real.

Earlier eqn

$$u_{\xi\eta} = \frac{F}{2B}$$

is still valid. But since ξ, η complex, it's more natural to transform

$t, x \to \mu, \nu$ instead of ξ, η.

Result is

$$\frac{B}{2}\left(u_{\mu\mu} + u_{\nu\nu}\right) = F$$

with B & F as before

$$B = a\,\xi_t \eta_t + b(\xi_t \eta_x + \xi_x \eta_t) + c\,\xi_x \eta_x$$

etc.

$$\boxed{u_{\mu\mu} + u_{\nu\nu} = \frac{2F}{B}}$$

is "normal form" of elliptic eqn.

Check transformation

$$\xi, \eta \to \mu, \nu$$

$$\xi = \mu + i\nu$$

$$\eta = \mu - i\nu$$

$$\Rightarrow \mu = (\xi + \eta)/2$$

$$\nu = (\xi - \eta)/2i$$

$$\left.\frac{\partial}{\partial \xi}\right|_{\eta} = \left.\frac{\partial \mu}{\partial \xi}\right|_{\eta} \left.\frac{\partial}{\partial \mu}\right|_{\nu} + \left.\frac{\partial \nu}{\partial \xi}\right|_{\eta} \left.\frac{\partial}{\partial \nu}\right|_{\mu} = \frac{1}{2}\frac{\partial}{\partial \mu} + \frac{1}{2i}\frac{\partial}{\partial \nu}$$

$$\left.\frac{\partial}{\partial \eta}\right|_{\xi} = \left.\frac{\partial \mu}{\partial \eta}\right|_{\xi} \left.\frac{\partial}{\partial \mu}\right|_{\nu} + \left.\frac{\partial \nu}{\partial \eta}\right|_{\xi} \left.\frac{\partial}{\partial \nu}\right|_{\mu} = \frac{1}{2}\frac{\partial}{\partial \mu} - \frac{1}{2i}\frac{\partial}{\partial \nu}$$

So

$$\frac{\partial^2}{\partial \xi \partial \eta} = \left(\frac{1}{2}\frac{\partial}{\partial \mu} + \frac{1}{2i}\frac{\partial}{\partial \nu}\right)\left(\frac{1}{2}\frac{\partial}{\partial \mu} - \frac{1}{2i}\frac{\partial}{\partial \nu}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2}\right)$$

Example of elliptic 2nd order PDE:

Poisson equation — insert $4\pi G$ for gravity in 3 spatial dimensions

$$\nabla^2 \Phi = \underbrace{\rho}_{\text{source}}$$

in 2D is

(25.17) $$\Phi_{xx} + \Phi_{yy} = \rho$$

Hom solutions?

(1) In 2D (as here),
the Re and Im parts of any analytic complex func $f(z)$

$$f(z) = g(z) + i h(z)$$

satisfy

$$\nabla^2 g = \nabla^2 h = 0 .$$

cut

Proof

These come from Cauchy-Riemann conditions

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

is same regardless of direction of Δz.

Take $\Delta z = \Delta x$ along x

$$\frac{df}{dz} = \lim \left[\frac{g(x+\Delta x, y) - g(x,y)}{\Delta x} + i \frac{h(x+\Delta x, y) - h(x,y)}{\Delta x} \right]$$

$$= \frac{\partial g}{\partial x} + i \frac{\partial h}{\partial x}$$

Take $\Delta z = i\Delta y$ along y

$$\frac{df}{dz} = \lim \left[\frac{g(x, y+\Delta y) - g(x,y)}{i\Delta y} + i \frac{h(x, y+\Delta y) - h(x,y)}{i\Delta y} \right]$$

$$= \frac{1}{i} \left(\frac{\partial g}{\partial y} + i \frac{\partial h}{\partial y} \right)$$

25.16

Equate Re & Im parts:

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \quad \text{and} \quad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$$

Hence

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 h}{\partial x \partial y} = -\frac{\partial^2 g}{\partial y^2}$$

and

$$\frac{\partial^2 h}{\partial x^2} = -\frac{\partial^2 g}{\partial x \partial y} = -\frac{\partial^2 h}{\partial y^2}$$

ie $\nabla^2 g = \nabla^2 h = 0$

as claimed QED.

$a = c = 1 \quad , \quad b = 0$

Char eqn

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm\sqrt{-1} = \pm i .$$

Solutions

$$x = \pm it + \text{const}$$

Characteristic curves are

$$\xi = x - it$$
$$\eta = x + it$$
$$\mu \ \left[= \tfrac{1}{2}(\xi + \eta) \right] = x$$
$$\nu \ \left[= \tfrac{1}{2i}(\xi - \eta) \right] = t$$

Soln of

$$u_{\xi\eta} = 0$$

is

$$u_{\text{hom}} = f(\xi) + g(\eta)$$
$$= f(x - it) + g(x + it) .$$

Consistent with fact that any analytic function $f(z)$ satisfies

$$z = x + iy$$

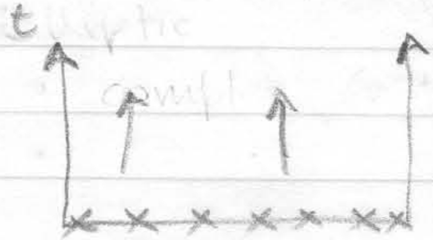
$$\nabla^2 \, \text{Re} f = \nabla^2 \, \text{Im} f = 0$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

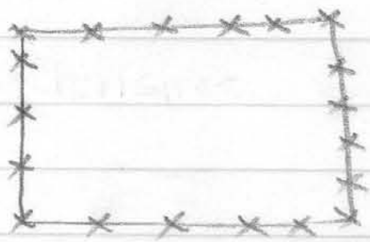
Boundary conditions for PDEs

Hyperbolic/parabolic:

- real characteristics
- solve from given initial conditions



hyperbolic/parabolic



elliptic

Elliptic:

- complex (not real) characteristics
- boundary conditions must be specified on whole domain
- harder to solve than hyp/par.

Realistic systems of PDEs are often mix of hyperbolic and elliptic
(i.e. some characteristics real, others not).
Can sometimes converted mixed system to fully hyperbolic system.

GF of Poisson's eqn

$$\nabla^2 \phi = \underbrace{\delta_D(\vec{r} - \vec{r}_0)}_{\text{point source at } \vec{r}_0}$$

Q: Is 1D source (line) in 3D equivalent to pt source in 2D?
A: Yes. If source indept of z, then $\partial/\partial z = 0$.

line $\uparrow z$

Eqn is spherically symmetric about $\vec{r}_0$ so convert ∇^2 to spherical coordinates.

In n-dimensions

$$\nabla^2 = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r}$$

$$\frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} \phi = \delta_D^{(n)}(\vec{r})$$

$$\Rightarrow r^{n-1} \frac{\partial}{\partial r} \phi = \int \delta_D^{(n)}(\vec{r}) \underbrace{r^{n-1} dr}_{= \frac{dV}{A_n}}$$

$$A_n = \begin{cases} 2\pi & n = 2 \\ 4\pi & n = 3 \end{cases}$$

surface area of unit sphere in n dimensions.

$$= \frac{1}{A_n}$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{1}{A_n r^{n-1}} \quad \text{gravity}$$

$$\Rightarrow \phi = \frac{1}{A_n} \int^r \frac{dr}{r^{n-1}}$$

$$= \frac{r^{2-n}}{(2-n) A_n} \quad \text{or} \quad \frac{\ln r}{A_2} \quad \text{if } n = 2.$$

up to arbitrary constant.

For $n > 2$ can chose $\phi = 0$ at ∞, but for $n \leq 2$ ϕ diverges at ∞.

In $n = 3D$, $\phi = \frac{1}{4\pi r}$

Numerical solution of PDEs (Ch3 of Trefthen PDE notes)

Basic issues same as ODEs:

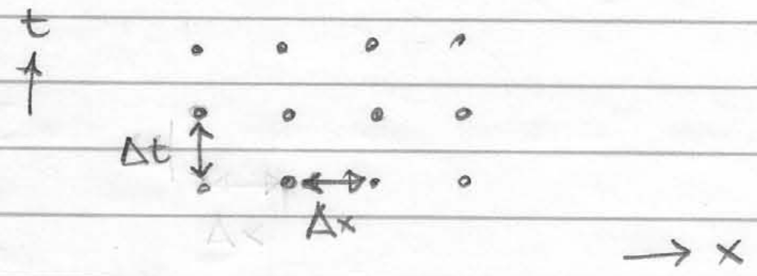
- Accuracy
- Stability

Here deal only with initial value PDEs, ie, hyperbolic/parabolic systems.

Essential starting point:

Discretization

Prototypical approach is to discretize on to a uniform grid



Prototypical equations:

$u_t = u_x$ hyperbolic 1D wave/advection

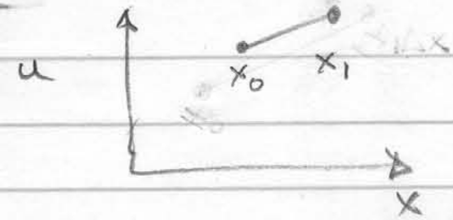
$u_t = u_{xx}$ parabolic diffusion

Q: How to estimate derivatives u_t, u_x, u_{xx} from values of u on grid?

A: Standard (but not unique) answer is to fit a polynomial

Simplest polynomial:

Linear



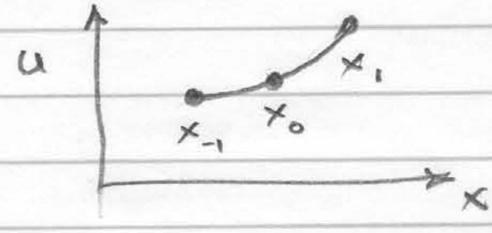
Fit linear polynomial through 2 points $u_0 = u(x_0)$, $u_1 = u(x_1)$

$$u(x) = u_0 + \left(\frac{x - x_0}{x_1 - x_0}\right)(u_1 - u_0)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u_1 - u_0}{\underbrace{x_1 - x_0}_{= \Delta x}} = \frac{u_1 - u_0}{\Delta x}$$

But linear yields only 1st derivative, not 2nd. Lowest order polynomial yielding 2nd derivative is:

Quadratic



Quadratic polynomial through $u_i \equiv u(x_i)$, $i = -1, 0, 1$, is

$$u(x) = u_0 + \frac{(x - x_0)}{2\Delta x}(u_1 - u_{-1}) + \frac{(x - x_0)^2}{2\Delta x^2}(u_{-1} - 2u_0 + u_1)$$

giving

$$\left.\frac{\partial u}{\partial x}\right|_{x = x_0} = \frac{u_1 - u_{-1}}{2\Delta x}, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x = x_0} = \frac{u_{-1} - 2u_0 + u_1}{\Delta x^2}$$

Convenient notation

Define centered difference operator δ_0 by

$$\delta_0 u \equiv \frac{u_1 - u_{-1}}{2\Delta x}$$

Define 2nd difference operator $\delta_\wedge$ by

$$\delta_\wedge u \equiv \frac{u_{-1} - 2u_0 + u_1}{\Delta x^2}$$

So to quadratic order

$$\frac{\partial u}{\partial x} \approx \delta_0 u \quad , \quad \frac{\partial^2 u}{\partial x^2} \approx \delta_\wedge u \quad .$$

Same notation but with raised indices for time t:

$$\frac{\partial u}{\partial t} \approx \delta^0 u \equiv \frac{u(t+\Delta t, x) - u(t-\Delta t, x)}{2\Delta t}$$

$$\frac{\partial^2 u}{\partial t^2} \approx \delta^\wedge u \equiv \frac{u(t-\Delta t, x) - 2u(t,x) + u(t+\Delta t, x)}{\Delta t^2}$$

Also define forward difference δ_+

$$\delta_+ u \equiv \frac{u_1 - u_0}{\Delta x}$$

and backward difference δ_-

$$\delta_- u \equiv \frac{u_0 - u_{-1}}{\Delta x} \quad .$$

26.4

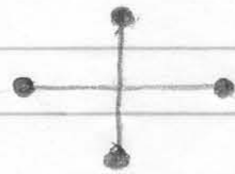
1D wave/advection eqn — Leap Frog

LF approximates

$$u_t = u_x$$

by

$$\delta^0 u = \delta_0 u$$



ie, explicitly

$$\frac{u(t+\Delta t, x) - u(t-\Delta t, x)}{2\Delta t} = \frac{u(t, x+\Delta x) - u(t, x-\Delta x)}{2\Delta x}$$

LF is 2nd order accurate in t and x, explicit, time symmetric, and, as we will see later, stable.

But suppose for some reason you want to integrate forward each time step without leap-frogging. The 2nd order method that achieves this is

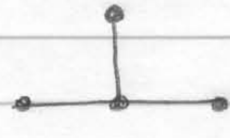
Lax - Wendroff

LW approximates

$$u_t = u_x$$

by

$$\delta^+ u = \delta_0 u + \frac{\Delta t}{2} \delta_{\wedge} u .$$



Derivation:

Taylor expand to 2nd order:

$$u(t+\Delta t, x) = u + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}$$

But $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$

and $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

So $u(t+\Delta t, x) = u + \Delta t\, u_x + \frac{\Delta t^2}{2} u_{xx}$

ie $\frac{u(t+\Delta t, x) - u}{\Delta t} = u_x + \frac{\Delta t}{2} u_{xx}$

ie $\delta^+ u = \delta_0 u + \frac{\Delta t}{2} \delta_\wedge x$

to quadratic order.

LW is 2nd order in t & x,

explicit, and stable but dissipative.

see later.

Is dissipation good or bad?

Good if you want it — shocks.

Bad if you don't want it — other than shocks.

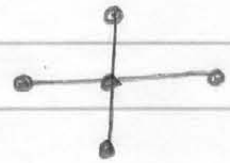
Diffusion eqn — Leap Frog

LF approximates

$$u_t = u_{xx}$$

by

$$\overset{\circ}{\delta} u = \delta_\wedge u$$



ie. explicitly

$$\frac{u(t+\Delta t, x) - u(t-\Delta t, x)}{2\Delta t} = \frac{u(t, x-\Delta x) - 2u(t,x) + u(t, x+\Delta x)}{\Delta x^2}$$

LF is 2nd order in t and x, explicit, but unstable, ie there is undesired growing solution as well as desired decaying solution $\Rightarrow$ no good.

not Nicholson

Instability is cured by an implicit method

Crank - Nicolson

CN approximates

$$u_t = u_{xx}$$

by

$$\delta^+ u = \frac{1}{2}\left[\delta_\wedge u(t) + \delta_\wedge u(t+\Delta t)\right]$$

ie. explicitly

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = \frac{1}{2\Delta x^2}\left[\begin{array}{l} u(t, x-\Delta x) - 2u(t,x) + u(t, x+\Delta x) \\ + u(t+\Delta t, u-\Delta x) - 2u(t+\Delta t, x) + u(t+\Delta t, x+\Delta x)\end{array}\right]$$

CN is 2nd order in t & x, implicit, stable.

Show differenceoperators.nb

Derivatives from higher order polynomial fits

There is a simple recursive algorithm (Fornberg 1988 Math Comp 51, 699) to generate coefficients of arbitrary order polynomial fit through arbitrarily spaced points.

For important case of equally spaced points, fit by polynomial of even order $2p$ gives

$$\frac{\partial}{\partial x} = \sum_{j=1}^{p} \alpha_j \, \delta_0(j\Delta x)$$

signifies δ_0 on interval $j\Delta x$

$$\frac{\partial^2}{\partial x^2} = \sum_{j=1}^{p} \alpha_j \, \delta_\wedge(j\Delta x)$$

where $\alpha_j = (-)^{j+1} \dfrac{2\,(p!)^2}{(p-j)!\,(p+j)!}$

Quadratic $(p=1)$:

$$\frac{\partial}{\partial x} = \delta_0(\Delta x) \qquad \frac{\partial^2}{\partial x^2} = \delta_\wedge(\Delta x)$$

Quartic $(p=2)$:

$$\frac{\partial}{\partial x} = \frac{4}{3}\delta_0(\Delta x) - \frac{1}{3}\delta_0(2\Delta x)$$

$$\frac{\partial^2}{\partial x^2} = \frac{4}{3}\delta_\wedge(\Delta x) - \frac{1}{3}\delta_\wedge(2\Delta x)$$

Infinite order $(p=\infty)$

$$\frac{\partial}{\partial x} = 2\delta_0(\Delta x) - 2\delta_0(2\Delta x) + 2\delta_0(3\Delta x) - \ldots$$

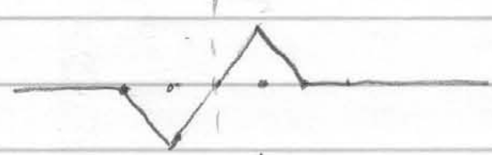
$$\frac{\partial^2}{\partial x^2} = 2\delta_\wedge(\Delta x) - 2\delta_\wedge(2\Delta x) + 2\delta_\wedge(3\Delta x) - \ldots$$

Difference operators as convolution

The difference operators are all convolution operators

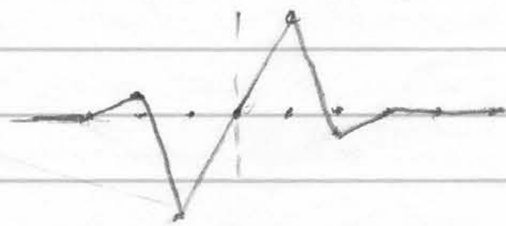
$\frac{\partial}{\partial x}$:

Quadratic:



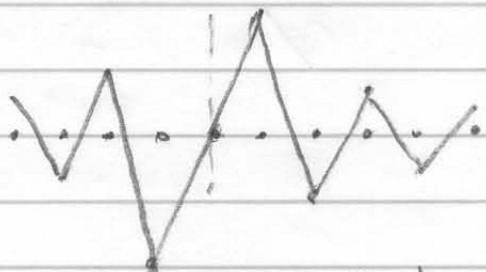
$-\frac{1}{2} \quad 0 \quad \frac{1}{2}$

Quartic:



$\frac{1}{12} \quad -\frac{2}{3} \quad 0 \quad \frac{2}{3} \quad -\frac{1}{12}$

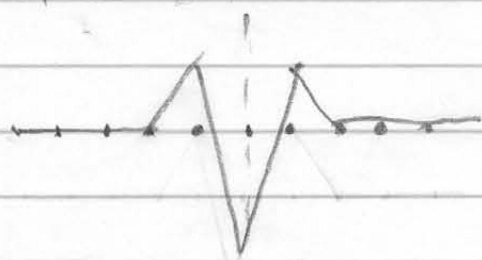
Infinite



$\frac{1}{4} \quad -\frac{1}{3} \quad \frac{1}{2} \quad -1 \quad 0 \quad 1 \quad -\frac{1}{2} \quad \frac{1}{3} \quad -\frac{1}{4}$

$\frac{\partial^2}{\partial x^2}$:

Quadratic:



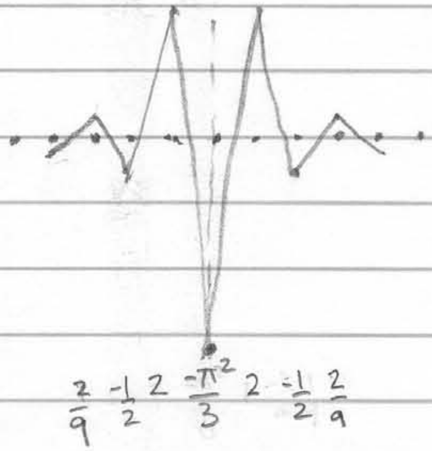
$1 \quad -2 \quad 1$

Quartic:

$-\frac{1}{12} \quad \frac{4}{3} \quad -\frac{5}{2} \quad \frac{4}{3} \quad -\frac{1}{12}$

26.9

Infinite :



Integration as convolution

Difference operators are convolutions,
$\Rightarrow$ all explicit integration methods considered
(LF, LW applied to $u_t = u_x$ or u_{xx})
are convolutions, taking form

$$u(t+\Delta t, x) = a \circ u$$

convolution function — signifies convolution

Since convolution $\equiv$ multiplication in Fourier space,
it's natural to FT.

Since x runs over discrete grid

$$x_n = n\Delta x \quad , \quad n \text{ integer}$$

the FT is a series type

$$u_k = \sum_n u(x_n)\, e^{-ikx_n}/\sqrt{2\pi}$$

If n extended over all integers,
then k would be periodic over $\left[\frac{-\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]$.
and

$$u(x) = \int_{-\pi/\Delta x}^{\pi/\Delta x} u_k \frac{e^{ikx}}{\sqrt{2\pi}}\, dk .$$

Nyquist frequency.

Or, if x runs over a finite periodic grid, then FT is discrete.

The details of the FT are unimportant.
What matters is that, in Fourier space

$$u_k(t+\Delta t) = a_k\, u_k$$

FT of convolution operator

ie stepping forward Δt = mult in Fourier space.

→ More correctly, if the fastest growing solution is the solution you want.

26.11

Stepping forward n steps gives

$$u_k(t+\Delta t) = a_k^n \, u_k$$

n'th power of a_k.

Method is stable if $|a_k| \leq 1$.
Notice that stability depends on wavenumber k.

Instead of going via FT, it is more straightforward to

<u>Test stability</u>

by trying

$$u(t + n\Delta t, x + m\Delta x) = g^n e^{ikm\Delta x}$$

g = growth factor

and seeing what happens.
Q: At $n = m = 0$? A: $u = 1$.

<u>Stability of LF, 1D wave</u>

LF approximates

$$u_t = u_x$$

by

$$\overset{\circ}{\delta} u = \delta_\circ u$$

ie

$$\frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

ie $$\frac{g - g^{-1}}{2\Delta t} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}$$

$$= \frac{i \sin k\Delta x}{\Delta x}$$

ie $g^2 - 2i\dfrac{\Delta t}{\Delta x}\sin k\Delta x \; g - 1 = 0$

ie $g = ib \pm \sqrt{-b^2+1}$

where

$$b \equiv \frac{\Delta t}{\Delta x}\sin k\Delta x .$$

Q: Ny freq?

A Highest freq measurable from grid spaced at Δx.

Note $k_{Ny}\Delta x = \pi$ at Nyquist freq $k_{Ny} = \dfrac{\pi}{\Delta x}$.

if $\Delta t \le \Delta x$, then $b \le 1 \;\forall k$.

Write $b = \sin(\omega\Delta t)$ then

$$g = i\sin(\omega\Delta t) \pm \cos(\omega\Delta t)$$
$$= \pm e^{\pm i\omega\Delta t}.$$

Thus solutions have neutral stability $\forall k$, provided that $\Delta t \le \Delta x$.

For long wavelengths $k\Delta x \ll 1$,

$$b \approx k\Delta t$$

so, provided that $g = e^{i\omega\Delta t}$,

$$\omega \approx k$$

which is as it should be.

The $g^{\pm}$ is not a problem, because LF advances each quantity by 2 time steps at a time, and $(\pm 1)^2 = 1$.

Stability of LW, 1D wave

LW takes

$$\delta^+ u = \delta_0 u + \frac{\Delta t}{2} \delta_\wedge u$$

Recall that, for trial $u = g^n e^{ikm\Delta x}$,

$$\delta_0 u = i \frac{\sin k\Delta x}{\Delta x}$$

Likewise

$$\delta_\wedge u = \frac{u(x-\Delta x) - 2u(x) + u(x+\Delta x)}{\Delta x^2}$$

$$= \frac{e^{-ik\Delta x} - 2 + e^{ik\Delta x}}{\Delta x^2}$$

$$= \frac{\left(e^{-ik\Delta x/2} - e^{ik\Delta x/2}\right)\left(e^{-ik\Delta x/2} - e^{ik\Delta x/2}\right)}{\Delta x^2}$$

$$= -\left(\frac{2\sin(k\Delta x/2)}{\Delta x}\right)^2$$

So LW is

$$\frac{g-1}{\Delta t} = i \frac{\sin k\Delta x}{\Delta x} - \frac{\Delta t}{2}\left(\frac{\sin(k\Delta x/2)}{\Delta x}\right)^2$$

ie

$$g = 1 + i \frac{\Delta t}{\Delta x} \sin k\Delta x - \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2\left(\frac{k\Delta x}{2}\right).$$

The abs value squared of this is

$$|g|^2 = 1 - 4\left(\frac{\Delta t}{\Delta x}\right)^2\left(1 - \frac{\Delta t^2}{\Delta x^2}\right)\sin^4\left(\frac{k\Delta x}{2}\right)$$

which is $\leq 1 \quad \forall k$

provided that $\Delta t < \Delta x$

Thus LW is stable for $\Delta t < \Delta x$, but dissipative. Dissipation is greatest at smallest scales, near Nyquist frequency

$$k\Delta x \approx \pi$$

Stability of LF, diffusion eqn

LF approximates

$$u_t = u_{xx}$$

by

$$\delta^0 u = \delta_\wedge u .$$

With trial $u = g^n e^{ikm\Delta x}$,

$$\frac{g - g^{-1}}{2\Delta t} = -\left(\frac{2\sin(k\Delta x/2)}{\Delta x}\right)^2$$

$$= -\mu^2 \quad \text{say (defining } \mu \geq 0)$$

Then

$$g^2 + 2\Delta t \mu^2 g - 1 = 0$$

ie

$$g = -\Delta t \mu^2 \pm \sqrt{\Delta t^2 \mu^2 + 1}$$

The desired solution is the decaying solution

$$g = \sqrt{1 + \Delta t^2 \mu^2} - \Delta t \mu$$

$$= \frac{1}{\sqrt{1 + \Delta t^2 \mu^2} + \Delta t \mu}$$

$$\leq 1 \quad \forall \Delta t$$

But there's also an undesired growing solution

$$g = -\sqrt{1 + \Delta t^2 \mu^2} - \Delta t \mu$$

$$< -1 \quad \forall \Delta t \quad \Rightarrow \text{no good.}$$

Stability of CN, diffusion eqn

CN approximates

$$u_t = u_{xx}$$

by implicit

$$\delta^+ u = \frac{1}{2}\left[\delta_\wedge u(t) + \delta_\wedge u(t+\Delta t) \right]$$

For trial $u = g^n e^{ikm\Delta x}$

$$\frac{g-1}{\Delta t} = -\left(\frac{2\sin(k\Delta x/2)}{\Delta x}\right)^2 \frac{g+1}{2}$$

ie $g - 1 = -(g+1)\nu$

where $\nu = \frac{\Delta t}{2}\left(\frac{2\sin(k\Delta x/2)}{\Delta x}\right)^2 \geq 0$

ie $g(1+\nu) = 1 - \nu$

ie $g = \frac{1-\nu}{1+\nu}$

So $|g| \leq 1$ for all k

$\Rightarrow$ stable

Good.