Theory of Bistability and Self-Pulsing in an Optical Ring Circuit Having Saturable Photorefractive Gain, Loss, and Photorefractive Feedback

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ABSTRACT

We present a theory for an optical ring circuit in which gain, loss, and feedback are provided by means of refractive-index gratings in photorefractive crystals. Maxwell's equations and the Kukhtarev charge transport model describe the evolution of the optical fields and the gratings, respectively. Steady-state solutions of the equations exhibit bistability. Dynamic solutions, obtained numerically, exhibit either history-dependent bistability due to the dependence of the feedback coupling element on its past, or self pulsing, depending on the relative speeds of the photorefractive gain and loss.

INTRODUCTION

An optical ring resonator having photorefractive two-beam coupling gain and loss can exhibit bistability [1-3] or self pulsing [3]. We present here the results of a theoretical analysis of a similar system, an optical ring circuit in which feedback is provided by a photorefractive feedback coupling element, which we call a photorefractive mirror. This system too can exhibit bistability or self pulsing, which is qualitatively different than that exhibited by the original system having ordinary feedback because the photorefractive mirror has a memory.

The equations that describe the circuit pictured in Fig. 1 are derived from Maxwell's equations and Kukhtarev's model for charge transport in photorefractive crystals [4]. The steady-state solutions to these equations show that the behavior of the photorefractive mirror is very much like that of an ordinary mirror over a wide range of operating conditions. It should not come as a surprise, then, that the steady-state behavior of the circuit is similar to that of the ring resonator described in [3], i.e., the circuit can exhibit bistability. From the dynamic solutions to the equations, we find that the circuit exhibits one of two types of behavior when the photorefractive mirror is much slower than either the gain or the loss. Depending on the speed of the gain relative to the loss, the circuit can exhibit either bistability whose nature depends on the history of the photorefractive mirror, which we call dynamic bistability, or self pulsing.

THEORETICAL MODEL

The optical ring circuit we are analyzing, pictured in Fig. 1, relies on photorefractive two-beam coupling to provide gain, loss, and feedback. Here we describe the mechanism of photorefractive two-beam coupling, and present the model used to describe the circuit.

When two optical beams interfere in a photorefractive crystal, the resulting interference pattern redistributes mobile charge carriers in the crystal. The new charge distribution produces a space-charge field which induces local changes in the index of refraction through the electrooptic effect, from which a refractive-index grating results. In general, the interference pattern and the grating are not in phase, making energy transfer between the two beams possible, the direction of energy transfer depending on the orientation of the crystal.

The circuit pictured in Fig. 1 contains three active elements, the photorefractive crystals in which two-beam coupling takes place. The gain crystal is oriented so that energy is fed from a gain pump beam into the circuit, the loss crystal so that energy is fed from the circuit into a loss pump beam, and the mirror crystal so that energy is fed from the returning signal beam back into the circuit. The coupling in the gain, loss, and mirror crystals is characterized by the complex grating strengths $G_G$, $G_L$, and $G_M$, which are proportional to the effective electrooptic coefficients and space-charge field amplitudes in their respective crystals. The evolution of each grating is described by the charge transport model of Kukhtarev.
et al. [4], which yields a single equation for each grat- ing strength, and the evolution of the optical fields as they interact with one of the gratings is described by Maxwell’s equations. Together, Maxwell’s equa- tions and Kukhtarev’s model describe the dynamics of photorefractive two-beam coupling in the circuit through the following set of equations:

\[
\frac{\partial G_G}{\partial t} = \gamma_G \left[ -G_G + \frac{\Gamma_G}{2} \frac{E_S E_G^*}{I_G} \right], \quad (1a)
\]

\[
\frac{\partial E_S}{\partial z} = -\frac{\alpha_G}{2} E_S + G_G E_G,
\]

\[
\frac{\partial E_G}{\partial z} = -\frac{\alpha_G}{2} E_G - G_G^* E_S,
\]

for the gain crystal,

\[
\frac{\partial G_L}{\partial t} = \gamma_L \left[ -G_L + \frac{\Gamma_L}{2} \frac{E_S^* E_L}{I_L} \right], \quad (1d)
\]

\[
\frac{\partial E_S}{\partial z} = -\frac{\alpha_L}{2} E_S - G_L^* E_L,
\]

\[
\frac{\partial E_L}{\partial z} = -\frac{\alpha_L}{2} E_L + G_L E_S,
\]

for the loss crystal, and

\[
\frac{\partial G_M}{\partial t} = \gamma_M \left[ -G_M + \frac{\Gamma_M}{2} \frac{E_S E_M^*}{I_M^*} \right], \quad (1g)
\]

\[
\frac{\partial E_S}{\partial z} = -\frac{\alpha_M}{2} E_S + G_M E_M,
\]

\[
\frac{\partial E_M}{\partial z} = -\frac{\alpha_M}{2} E_M - G_M^* E_S,
\]

for the mirror crystal. The complex field amplitudes \(E_S, E_G, E_L,\) and \(E_M\) are the slowly-varying amplitudes of the signal, gain pump, loss pump, and mirror pump beams, respectively. The total intensity in each crystal is

\[
I_{j0}(z,t) = |E_S|^2 + |E_j|^2
\]

\[
= I_S(z,t) + I_j(z,t), \quad j = G, L, M, \quad (2)
\]

and \(\alpha_j\) is the corresponding absorption coefficient. The coupling constants \(\Gamma_G, \Gamma_L,\) and \(\Gamma_M\) for each crystal depend on the crystal’s orientation with respect to the two interacting beams and are proportional to the crystal’s effective electrooptic coefficient. The decay rates \(\gamma_G, \gamma_L,\) and \(\gamma_M\) determine the speed with which each crystal responds to changes in the local fields, and are proportional to the total intensity:

\[
\gamma_j = \gamma_{0j} I_{j0}(z,t), \quad j = G, L, M, \quad (3)
\]

where \(\gamma_{0j}\) is a material dependent rate constant.

The circuit’s description is completed by specifying the boundary conditions. The gain and loss pump amplitudes satisfy

\[
E_G(z_G, t) = I_G^{1/2}, \quad E_L(z_L, t) = I_L^{1/2}, \quad (4a)
\]

where \(z_G (z_L)\) is the distance from the mirror crystal input (\(z_M = 0\)) to the gain (loss) crystal input, and \(I_G\) and \(I_L\) are the incident gain and loss pump intensities, respectively. Both pump beams are assumed to oscillate at the same frequency \(\omega_p\).

There are two boundary conditions to specify at \(z = 0\), one for the signal beam and one for the mirror pump beam. For the signal beam,

\[
E_S(0,t) = \sqrt{I_{inj}(t)}, \quad (4b)
\]

where \(I_{inj}(t)\) is the intensity of the injected signal whose frequency is also \(\omega_p\). The mirror pump beam is simply an attenuated and phase-shifted version of the loss crystal’s output. An interesting property of photorefractive two-beam coupling is that the amplitudes of the output beams are independent of the relative phase of the input beams, as long as it is constant. It follows that the output of the mirror crystal, \(E_M(l_M, t)\), where \(l_M\) is the length of the mirror crystal, is independent of any constant phase shift experienced by the pump beam; as a result there is no round-trip phase condition to satisfy, and we have for the mirror pump beam

\[
E_M(0,t) = \sqrt{R} E_S(z_M + l_M, t). \quad (4c)
\]

By the same reasoning we can ignore all constant phase shifts due to propagation, and specify the boundary conditions for the signal at the inputs to the gain and loss crystals as

\[
E_S(z_G, t) = E_S(l_M, t), \quad E_S(z_L, t) = E_S(z_G + l_G, t). \quad (4d)
\]

This property of two-beam coupling holds approximately even if the relative phase is time varying, as long as it varies slowly enough that the mirror

\[\text{Figure 1. A unidirectional ring circuit with photorefractive gain loss, and feedback.}\]
grating can follow the resulting changes in the interference pattern. If the mirror crystal is fast enough to follow such changes, then it can compensate for slow variations in the relative phase, brought about by changes in the round-trip path length, for example.

RESULTS

Here we discuss the results obtained from the model described in the previous section. We show that the steady-state solutions to Eqs. (1) and Eqs. (4) exhibit bistability. We also show that dynamic solutions to the same equations, obtained numerically, exhibit either history-dependent bistability or self pulsing, depending on the relative speeds of the gain and loss crystals.

The steady-state solutions to the equations for two-beam coupling in a single photorefractive crystal are well known (see [4], for example). From these solutions, we define open-loop gain $G > 0$ and loss $L > 0$:

\[
G \equiv \log_{10} \left[ \frac{I_S(z_G + l_G)}{I_S(z_G) \exp(-\alpha_G l_G)} \right] = \log_{10} \left[ \frac{\tau_G + 1}{\tau_G + \exp(-\Gamma_G l_G)} \right], \tag{5a}
\]

\[
L \equiv -\log_{10} \left[ \frac{I_S(z_L + l_L)}{I_S(z_L) \exp(-\alpha_L l_L)} \right] = -\log_{10} \left[ \frac{\tau_L + 1}{\tau_L + \exp(\Gamma_L l_L)} \right], \tag{5b}
\]

where $\tau_G = I_S(z_G)/I_G$, and $\tau_L = I_S(z_L)/I_L$. The coupling constants $\Gamma_G$ and $\Gamma_L$ determine the small-signal ($\tau_G \to 0$, $\tau_L \to 0$) gain $G_0 = \Gamma_G l_G \log_{10} e$ and loss $L_0 = \Gamma_L l_L \log_{10} e$, and the incident gain and loss pump intensities $I_G$ and $I_L$ determine the saturation intensities for $G$ and $L$, respectively. The open-loop gain $G$, loss $L$, and combined gain $G - L$ characterize the combined effect of the gain and loss on the signal field in steady state. The effect of the photorefractive mirror on the signal field in steady state is characterized by the mirror reflectivity $R_M = I_S/l_M/[l_M(0) \exp(-\alpha_M l_M)]$:

\[
R_M = \frac{1 + \tau_M}{1 + \tau_M^{-1} \exp(-\Gamma_M l_M)}. \tag{6}
\]

For $\exp(-\Gamma_M l_M) \ll \tau_M \ll 1$, $R_M \approx 1$, i.e., the photorefractive mirror acts like an ordinary mirror over this operating range; the gain $G$ and loss $L$ then determine the circuit’s steady-state behavior over this range.

We choose the pump intensities so that $I_L \ll I_G$, so that the loss saturates at a much lower intensity than the gain. For signal intensities $I_S \equiv I_S(l_M) \ll I_G$, the gain is linear, but the loss is like that due to a saturable absorber. One expects bistability in such a system, an expectation confirmed by Fig. 2, which is a graphic representation of a relation between $I_S$ and $I_{in}$ obtained from Eqs. (4b), (5), and (6). One sees here an S-curve indicating bistability.

![Figure 2. Steady-state transmission diagram indicating bistability. For all figures $G_0 = 0.5$, $\Gamma_M l_M = 6.9$, $l_G = l_l = l_M = 3.0$ mm, $\alpha_G = \alpha_l = \alpha_M = 0.15$ mm$^{-1}$, $R = 0.10$, $I_G = 50$ mW/mm$^2$, and $I_l/l_M = 0.02$.](image)

Information concerning the dynamic behavior of the circuit is obtained by numerically integrating Eqs. (1) in conjunction with the boundary conditions Eqs. (4). The results show that the circuit can exhibit two types of behavior when $\gamma_M \ll \gamma_G, \gamma_L$, i.e., when the photorefractive mirror is much slower than either the gain or the loss. First, if the gain is not too fast, the circuit can exhibit a form of bistability that depends on the history of the circuit, and in particular on the history of the photorefractive mirror; we call this type of behavior dynamic bistability. Second, if the gain is much faster than the loss, the circuit can self pulse in the absence of an injected field, with an amplitude that initially increases in time.

Once the parameters relevant to the circuit's steady-state behavior have been chosen ($G_0$, $L_0$, $R$, $\Gamma_M l_M$, $I_G$, and $I_L$), it is the decay rates of the gain, loss, and mirror crystals that determine the dynamic behavior of the circuit. We now choose the photorefractive rate constants $\gamma_G$ and $\gamma_L$ so that the decay rates $\gamma_G$ and $\gamma_L$ are of the same order of magnitude. This choice is mandated by the stability conditions for a ring resonator with photorefractive gain and loss and an ordinary mirror [2,3]:

\[
\frac{\gamma_L}{\gamma_G} > \frac{G_0 - C}{L_0 + C} \tag{7a}
\]

and

\[
G_0 - L_0 < C. \tag{7b}
\]

Here $C$ represents the passive losses in the absence of a photorefractive mirror. The off state $E_S(z,t) = 0$ is stable when the loss is fast enough to prevent the gain from turning the resonator
on [Eq. (7a)] and when the small-signal combined gain is insufficient to overcome the passive losses [Eq. (7b)]. The off state is unstable if either condition is violated. Because the photorefractive mirror responds so slowly to changes in the field in comparison to the gain and the loss, it acts like an ordinary mirror when no injected signal is present; it has a dynamic reflectivity

\[ R_M(t) = \frac{I_S(l_M, t)}{I_M(0, t) \exp(-\alpha_M l_M)}. \]  

(8)

We can apply Eqs. (7) to an optical circuit with a photorefractive mirror by replacing the passive loss \( C \) by the total dynamic loss \( C_T(t) \):

\[ C_T(t) = C + C_M(t) = C - \log_{10}[R_M(t)], \]  

(9)

where \( C \) is given by

\[ C = -\log_{10}\left\{ Re^{-\left(\gamma_G l_G + \alpha L + \alpha_M l_M\right)} \right\}. \]  

(10)

A stable off state is required if the system is to exhibit bistability; our choices of \( \gamma_G \) and \( \gamma_L \) ensure the stability of the off state for reasonable values of \( C_T(t) \). As we shall see below, the off state is unstable when \( \gamma_G \ll \gamma_L \) and self pulsing can occur.

Figure 3 shows the results of a numerical integration of Eqs. (1). A writing beam of intensity \( I_W \) and duration \( t_W \) is injected into the circuit until it approaches steady state; we judge it to have done so when the average mirror grating strength

\[ \bar{G}_M(t) = \frac{1}{I_M} \int_0^{\pm} G_M(z, t) \, dz, \]  

(11)

shown in Fig. 3a, is nearly constant. After the writing beam is terminated, \( \bar{G}_M(t) \) initially increases in time, due to the interference of the incident beam \( I_M(0, t) \) with light scattered from the mirror grating, which rewrites the grating stronger on average than during the writing phase.

The quantity \( \bar{G}_M(t) \) is significant because it determines the fraction of the returning signal intensity that is fed back into the circuit. The evolution of \( R_M(t) \), and hence \( C_T(t) \), follows that of \( \bar{G}_M(t) \). Because the mirror grating evolves so slowly, it depends not only on the current state of the circuit, but also on past states; that is, the photorefractive mirror has a memory. As a result, the reflectivity \( R_M(t) \) and the total loss \( C_T(t) \) are history dependent.

After the writing beam is terminated, \( \bar{G}_M(t) \) changes. Suppose we fix the mirror grating, and thus \( \bar{G}_M(t) \), at some time \( t > t_W \). We can then find the steady-state output intensity \( I_S \) as a function of an input intensity \( I_{inj} \), which is done in Fig. 3b for \( C_T(t) \) corresponding to \( \bar{G}_M(t) \) at the times marked by dots in Fig. 3a, and we obtain S-curves indicating bistability. At time \( t_1 \), just after termination of the writing beam, one obtains an S-curve similar to the steady-state S-curve in Fig. 2. As \( \bar{G}_M(t) \) increases initially, \( C_T(t) \) decreases, reaching a minimum at \( t_2 \) when \( \bar{G}_M(t) \) is maximum. During this time the loss decreases until there are two intensities \( I_S \) at which oscillation can occur with \( I_{inj} = 0 \). As \( C_T(t) \) decreases, \( I_S \) increases and \( I_S \) decreases until the separation \( I_S - I_S \) becomes maximum at \( t_3 \). The process partially reverses itself for \( t > t_3 \) as \( C_T(t) \) increases, but stops

Figure 3. (a) Average grating strength \( \bar{G}_M(t) \). \( I_W/I_G = 0.01, t_W = 5.0 \) min, and \( \gamma_G : \gamma_L : \gamma_M = 3 : 21 : 1 \). Points a to e correspond to those with the same labels in Fig. 3b. (b) Dynamic bistability. Curves a–e are obtained by fixing the mirror grating at the times marked by dots in Fig. 3b, \( t_1 = 5.0 \) min., \( t_2 = 6.5 \) min., \( t_3 = 9.3 \) min., \( t_4 = 12.5 \) min., and \( t_5 = 17.0 \) min. The corresponding reflectivities \( R_M(t) \) are 0.664, 0.734, 0.787, 0.746, and 0.693, respectively. (c) Hysteresis.
when $C_T(t)$ is just large enough to shut the system off; at this point, the corresponding S-curve (curve e in Fig. 3b) is just tangent to the vertical axis. At any time $t > t_W$, the state of the mirror grating depends on the history of the circuit, and determines the characteristics of the S-curve obtained by fixing the mirror grating and thus $C_T(t)$; we call this type of history-dependent behavior dynamic bistability.

Bistability also manifests itself as hysteresis, as shown in Fig. 3c. Figure 3c shows the results of a numerical integration of Eqs. (1) in which a low-intensity injected signal is slowly ramped up and then down beginning at some time $t > t_L$ after the circuit has shut off.

We now choose the photorefractive rate constants so that $\gamma_c > \gamma_L$, and find that the circuit self pulses after an initial writing beam is terminated at time $t = t_W$. The self-pulsing mechanism is described in [3]. Figure 4 shows the results of numerical integrations of Eqs. (1) for three different writing beam intensities. Only self pulsing occurs in Fig. 4a. The circuit exhibits somewhat different behavior for a lower writing beam intensity, as shown in Fig. 4b. The circuit self pulsates for a time after $t = t_W$ before starting to oscillate steadily. Steady oscillation begins at $t = t_W$ in Fig. 4c. The initial growth of the oscillation and self-pulsing amplitudes seen in Fig. 4 is due to the initial growth of the average mirror grating strength $G_M(t)$, which grows and decays in a manner similar to that in Fig. 3a.

The different kinds of behavior seen in Fig. 4 can be explained using the slowly-varying total loss $C_T(t)$. The writing beam intensity in Fig. 4a is such that the average mirror grating strength at $t = t_W$ is relatively small; as $G_M(t)$ grows for $t > t_W$, $C_T(t)$ decreases, but because $C_T(t)$ was high to begin with, it cannot decrease to the level at which the circuit can oscillate steadily. In this case, the circuit can only self pulse. The writing beam intensity in Fig. 4b is such that $G_M(t)$ is larger at $t = t_W$ than in Fig. 4a. In fact, $G_M(t)$ is large enough that at some point during the mirror grating's initial growth phase, $C_T(t)$ drops below the level at which steady oscillation can occur, and self pulsing ceases as steady oscillation begins. The oscillation amplitude increases as $C_T(t)$ continues to decrease. After $G_M(t)$ starts to decrease, the mirror grating eventually decays to the point at which steady oscillation can no longer occur, and self pulsing begins anew. In Fig. 4c, the writing beam writes a larger grating than in either Figs. 4a or 4b, large enough that steady oscillation begins $t = t_W$.  

**REFERENCES**
