SHOCK WAVES IN POPULATION II VARIABLE STARS

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I. HISTORICAL REVIEW

The most conspicuous shock waves occurring in astronomical objects, other than those in the sun, are perhaps the shock waves in Population II variable stars. The symptoms of these shocks are: first, emission in the Balmer lines; second, doubling of the metal lines, that is, the presence at certain phases of two components in each of several metal lines, shifted from each other by some tens of kilometers per second. These phenomena occur during a fraction of the period including the phase of minimum radius. The duration ranges from 2 per cent of the period (e.g. in SW And) to more than 50 per cent (e.g. in W Vir)

The transitory Balmer emission in W Vir was probably first detected by Joy (1937), although at the low dispersion he used he did not observe the associated metal line doubling. The latter was discovered prior to 1952, for in that year Sanford (1952) published velocity measurements based on high-dispersion spectra and called attention to the line doubling phenomenon. In the same year Schwarzschild (1952) proposed the shock wave model as an explanation of the emission and doubling. Sanford's collaborator in this work was Abt, who continued with a thorough study of W Vir (Abt 1954). He used the coarse analysis method to determine the atmospheric parameters, doing this separately for the regions responsible for each component of the metal lines during the double-line phase. However, for lack of a way to find the correct ionization temperature, Abt was led to assume solar metal abundance, thereby overestimating the electron pressure and the density. He concluded that the lines were formed in layers the thickness of
which was only about one per cent of the atmospheric extent. Abt applied the Rankine-Hugoniot relations to the pre-shock (infalling) material, and rejected the shock model because of the very high temperatures implied in the post-shock region. In this conclusion he failed to appreciate the extremely rapid cooling of the post-shock material to roughly the ambient temperature.

This radiative cooling was considered by Whitney (1955) in formulating his W Vir model. In view of the very short typical cooling time, Whitney adopted isothermal conditions for a numerical dynamic calculation of shock generation in the atmosphere of a pulsating star (Whitney 1956a, 1956b). The problems Abt encountered with the shock model were largely solved by Wallerstein (1959) who used a lower $T_{\text{ion}}$, consistent with a lower metal abundance as appropriate to the Population II nature of W Vir stars.

Recently Kraft and his collaborators (Barker et al. 1971) have carefully re-analyzed W Vir, using various independent methods for obtaining $T_{\text{ion}}$, and verify Wallerstein's low metal abundance for this star and that the extent of the atmosphere is compatible with the distance traveled by the shock wave during the time it is visible.

In RR-Lyrae stars the emission is weaker and much more fleeting. It was seen for the first time by Struve (1947) and later by Sanford (1949) in RR-Lyrae itself. These observations, as well as later ones, have shown that only the early Balmer lines and the Ca II K line show doubling, and only the Balmer lines show emission. Evidently the shock is somewhat more superficial in RR Lyrae stars than in W Vir.

Hardie (1955) obtained UBV photometry of RR Lyrae which showed the transitory ultraviolet excess now known to be characteristic of Bailey type a RR Lyrae stars (those with periods near 0.5 and light amplitudes $\Delta V$ near 1 magnitude). Hardie associated this with the Balmer line emission. Abt (1959) corrected the UBV colors for shock emission assuming that the U-B excess was entirely from this source. Oke and Bonsack (1960), however, find that the ultraviolet excess is fully explained by the large increase in
\[ R_{\text{eff}} = g + \hat{R} \] near minimum radius. In fact, it is difficult to imagine a way to separate the effect of the shock on the colors from the pressure effect.

A very significant body of data bearing on shock phenomena in RR Lyrae stars is the single trail spectroscopy of the rising light phase of several stars by Preston (Preston and Paczynski 1964; Preston, Smak and Paczynski 1965; Preston 1965). It is found that only Bailey type a's show \( H \gamma \) emission, and that the emission strength correlates well with Preston's \( \Delta S \)\(^1\) (Preston 1959) and with the period. Fig. 1, from Preston and Paczynski (1964), shows the development with phase of the \( H \gamma \) emission in \( \chi \) Ari and how the maximum emission varies from star to star. The correlation of period, \( \Delta S \) and emission strength is indicated in Fig. 2. Two values of \( \Delta S \) are given for SU Dra, the smaller embodying the view of Oke (1966a) that the metal lines in this star are of comparable strength to those in RR Lyrae.

The correlation of period with \( \Delta S \) is explainable with the frequently expressed hypothesis that horizontal-branch stars with low metal abundance (hence large \( \Delta S \)) are intrinsically brighter than those with higher metal abundance (low \( \Delta S \)). With reasonable masses for the stars in each group, the effect of the luminosity on the period dominates, and we infer that the low \( \Delta S \) group should have shorter periods than the high \( \Delta S \) group, as they are observed to have. The best temperature determinations for the stars listed in Fig. 3 (Oke and Bonsack 1960; Oke, Giver and Searle 1962; Oke 1966b) suggest that these stars have essentially equal temperatures. They also have very similar velocity amplitudes, so the parameters \( \alpha_a \) and \( \alpha_b \) introduced by Ledoux and

\(^{1}\) \( \Delta S \) is the difference, in spectral sub-classes, between the Balmer line spectral type and the metal line spectral type of a star. It is 0 for metal rich stars and 10 for the most metal poor stars.
Fig. 1 Development of Hγ emission in RR Lyræ stars, from Preston and Paczynski 1964 (reproduced from The Astrophysical Journal by permission of The University of Chicago Press).
<table>
<thead>
<tr>
<th>STAR</th>
<th>PERIOD</th>
<th>ΔS</th>
<th>I(Hγ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SW And</td>
<td>0.442</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>RR Cet</td>
<td>0.553</td>
<td>5</td>
<td>0.7</td>
</tr>
<tr>
<td>TU UMa</td>
<td>0.558</td>
<td>6</td>
<td>0.7</td>
</tr>
<tr>
<td>RR Lyr</td>
<td>0.567</td>
<td>6</td>
<td>0.2-0.6</td>
</tr>
<tr>
<td>RX Eri</td>
<td>0.587</td>
<td>9</td>
<td>&gt;0.4</td>
</tr>
<tr>
<td>X Ari</td>
<td>0.651</td>
<td>10</td>
<td>1.1</td>
</tr>
<tr>
<td>SU Dra</td>
<td>0.660</td>
<td>6,10</td>
<td>≈1</td>
</tr>
</tbody>
</table>

Fig. 2 The correlation of period, metal line strength indicator and Hγ emission strength as inferred from the work of Preston and collaborators.
<table>
<thead>
<tr>
<th>$T$</th>
<th>$\tau_C$</th>
<th>$\tau_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3500 °K</td>
<td>3·10^4 s</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>3·10^4</td>
<td></td>
</tr>
<tr>
<td>4700</td>
<td>1·10^4</td>
<td></td>
</tr>
<tr>
<td>6200</td>
<td>1·10^2 s</td>
<td>1·10^2</td>
</tr>
<tr>
<td>7600</td>
<td>7·10^0</td>
<td></td>
</tr>
<tr>
<td>9000</td>
<td>8·10^-1</td>
<td></td>
</tr>
<tr>
<td>12000</td>
<td>9·10^-2</td>
<td></td>
</tr>
<tr>
<td>15000</td>
<td>2·10^-1</td>
<td></td>
</tr>
</tbody>
</table>

$\rho = 10^{-10}$

Fig. 3 Cooling and heating times ($\tau_C$ and $\tau_W$, respectively) as a function of temperature for a fixed density, 10^{-10} g cm^{-3}.
Whitney (1960) are nearly constant among these stars. \( (a) \) is \( P_g/a \) and \( a_B \) is \( \Delta v/P_g \), where \( P \) is the period, \( g \) the surface gravity, \( \Delta v \) the velocity amplitude, and \( a \) the sound speed. This is because \( P_g \alpha (M/R)^{1/2} \) (Christy 1966b), and is therefore almost constant. These are the only parameters which describe the atmospheric response to the pulsation, and if they are not correlated with shock emission strength then the explanation of variation in that quantity must be sought in the interior.

One parameter which does correlate in the desired sense with emission strength is the combination which describes the motion of the \( H \) ionization zone through mass during the pulsation: \( L P / |\Delta M| \cdot X \), where \( |\Delta M| \) is the mass exterior to the \( H \) ionization zone and \( X \) is the ionization energy per unit mass of hydrogen. If this parameter is large, the motion of the \( H \) zone is more violent, and inversely (Castor 1968). Roughly, it varies as \( g^{-2/3} \) for a given \( T_{\text{eff}} \), and so the low \( \Delta S \), larger \( g \), stars should have more restrained motion of the \( H \) zone, which presumably, in a manner not understood, causes milder shock phenomena.

We would also like to understand the difference between shock behavior in \( W \) Vir, \( RR \) Lyrae and \( \delta \) Cep stars. Shocks have only been detected in the longest period classical cepheids (Kraft 1956, referring to \( X \) Cyg), as doubling of the \( Ha \) line. However, \( Ca II \) emission is seen at minimum radius in several longer period cepheids (Kraft 1957) and it is tempting to identify this with a shock wave. The absence of Balmer line emission can be explained by the smaller velocity amplitudes of most cepheids (typically 40 km s\(^{-1}\)) so that only a small part of the emission is in \( H \) recombination. In any event, the shock in classical cepheids must reach a given strength quite high in the atmosphere in comparison with, say, \( W \) Vir variables. In \( RR \) Lyrae stars the shock evidently forms (i.e. attains a Mach number of about 2 or larger) above the level at which most metal lines originate, but sufficiently low that the emission in the Balmer lines is conspicuous. In \( W \) Vir, the situation must be different. The Balmer line emission is seen first, near minimum light. Later there is a phase during which every line is
double. Evidently the Balmer emission is seen when the shock first emerges, fully developed, at the photosphere, and the shock then traverses the whole atmosphere at its full strength. In order to produce Balmer emission at the photosphere the shock speed (relative to pre-shock material) must be at least 30 or 40 km s\(^{-1}\) even at that depth. This conclusion is important; only in W Vir stars is the shock fully developed below the photosphere. The progression from classical cepheids to RR Lyrae stars to W Vir stars is one of increasing depth of formation of the shock, and only for the last is this depth below the photosphere. This progression is another problem, accompanying the variation among Bailey a's, which can only be solved by understanding the interaction between the interior pulsation and the atmosphere.

II. TEMPERATURES IN THE DYNAMICAL ATMOSPHERE

With this background I want to discuss some recent work investigating shock waves in model RR Lyrae stars. The majority of it was done by Stephen J. Hill, formerly of the University of Colorado. Before turning to the calculations themselves, it is important to fix in mind the time scales for relaxation of the temperature structure in the atmosphere. We can write the energy equation for the atmospheric material as

\[
C \frac{dT}{dt} + \frac{dW}{dt} = 4\pi \int \kappa_\nu J_\nu d\nu - 4\pi \int \kappa_\nu B_\nu d\nu
\]

using LTE, where \(C\) is an appropriate specific heat of the material, \(dW/dt\) is the rate that the material is doing work per unit mass, and \(J_\nu\) is the mean intensity of the radiation field. Since the atmosphere is optically thin, \(J_\nu\) is fixed by conditions at the photosphere for the most part, and is insensitive to local properties. If the right member of this equation vanishes, the material is in radiative equilibrium. It can be caused to depart from radiative equilibrium by the action of the work term. When the work term is zero, the material tends to cool or heat,
through the two terms in the right member, in order to return to radiative equilibrium. The relevant time scales for cooling, when \( J_\nu \ll B_\nu \), and for warming, when \( J_\nu \gg B_\nu \), are roughly given by

\[
\tau_c = \frac{CT}{4\pi\kappa P} B, \quad \tau_w = \frac{CT}{4\pi\kappa P} J
\]

where \( \kappa \) is the Planck mean opacity. Some typical numbers for a fixed density \( 10^{-10} \text{ g cm}^{-3} \) and a range of temperatures are given in Fig. 3. The value of \( J \) has been set at the Planck function for 6200K. It is of some interest that the cooling times are all quite short, while the warming times can be much longer. This is primarily due to the temperature dependence of the opacity. These times may be compared with the period of an RR Lyrae star, which is about \( 5 \times 10^5 \text{ s} \).

At a fixed temperature, 6200K, the cooling times for various densities are shown in Fig. 4. They are seen to be amply less than the period for essentially all densities of interest.

For assessing the departures from radiative equilibrium, we wish to compare these thermal relaxation times with a typical dynamical time. The logical choice is \( \tau_D = \frac{H}{a} \approx \frac{a}{\gamma}, \) where \( H \) is the atmospheric pressure scale height and \( a \) is the isothermal sound speed. At the photosphere a reasonable estimate of \( H \) is \( 1/\kappa P \), the photon mean free path. Using this, we find for the ratio of the dynamical time to the thermal relaxation time,

\[
\frac{\tau_D}{\tau_c} = \frac{4\pi B}{\rho a CT} \equiv B_0
\]

which is a variant of the Boltzmann number. With conditions appropriate to an RR Lyrae star its value is about \( 10^6 \). Thus we only expect to find departures from radiative equilibrium near the photosphere at the one per cent level. The value of \( \tau_D \) throughout the atmosphere is approximately \( 10^5 \text{ s} \), so we see that it is necessary to go quite high in the atmosphere to
<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\tau_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-9}$ g cm$^{-3}$</td>
<td>$1 \cdot 10^2$ s</td>
</tr>
<tr>
<td>$10^{-10}$</td>
<td>$1 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-11}$</td>
<td>$1 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>$2 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-14}$</td>
<td>$6 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-16}$</td>
<td>$3 \cdot 10^4$</td>
</tr>
</tbody>
</table>

$T = 6200 \degree K$

Fig. 4 Cooling time at 6200$\degree$K as a function of density.
find significant deviations from radiative equilibrium. The same comparison shows that the cooling region behind a shock wave will be very small in extent compared with \( H \), since \( T_C \) is small in comparison with \( T_p \) the time for sound to travel a distance \( H \). Therefore we expect that except for a small region near a shock, the atmosphere is quite close to radiative equilibrium, which in the optically thin atmosphere means isothermal conditions.

### III. ISOTHERMAL SHOCK DYNAMICS

If we isolate that part of the atmosphere near a shock (call it a black box) in which there are significant departures from isothermality, we can assume that the flow is steady through the box because it is so small. In a coordinate frame moving with the shock, we see the material enter the box with density \( \rho_0 \), velocity \( v_0 \), and temperature \( T_0 \), and exit from it with density \( \rho_1 \), velocity \( v_1 \), and temperature again \( T_0 \). Mass and momentum, but not mechanical energy, are conserved in passing through the box, so we have:

\[
\rho_0 v_0 = \rho_1 v_1
\]

and

\[
p_0 + \rho_0 v_0^2 = p_1 + \rho_1 v_1^2.
\]

With a perfect gas equation of state the isothermal condition results in:

\[
\frac{p_0}{\rho_0} = \frac{p_1}{\rho_1} = a^2,
\]

where \( a \) is the isothermal sound speed. With the Mach number defined by

\[
M = \frac{v_0}{a} > 1,
\]
the solution of these equations gives:

\[
\frac{v}{a} = \frac{1}{M}, \quad \frac{\rho_1}{\rho_0} = \frac{p_1}{p_0} = M^2.
\]

We see that the isothermal shock produces indefinitely large compression as the Mach number becomes large, in contrast to the adiabatic shock for which the compression is limited to \((\gamma + 1)/(\gamma - 1)\).

I will describe a simple calculation of a piston driven atmosphere with the temperature held constant throughout the atmosphere at all times, which is a reasonable approximation in view of the preceding section. This is very similar to Whitney's (1956a and 1956b) calculation with these differences: (1) pseudoviscosity method rather than characteristics and shock fitting, (2) atmospheric extent -- 25 scale heights in exterior mass, (3) piston motion -- given by the equation of a non-linear oscillator which resembles the actual photospheric motion of a pulsating star, with the correct period and amplitude, (4) duration of the calculation -- it is run to near-periodicity.

The principal results are shown in Fig. 5. The upper section indicates the trajectories in exterior mass per unit area, \(m\), versus time for the principal shock (solid line) and the subsidiary shocks (dashed lines). The piston is always located at \(m = 1\). The lower section shows the pre-shock, post-shock and piston velocities on the same time scale as the upper section. The interesting thing is that from about the phase of maximum outward velocity on, the Mach number is constant, and in fact, both the pre-shock and post-shock velocities are constant. This is only a property of the periodic case -- it does not occur in the first period of the calculation, for example. This phase is initiated when the main shock overtakes a weaker shock formed earlier quite high in the atmosphere. The region above this secondary shock behaves much as in Whitney's calculation. The effective gravity \(g_{\text{eff}}\).
Fig. 5 Upper: mass-time trajectories of the principal shock (solid line) and subsidiary shocks (dashed lines) in a periodically pulsating isothermal atmosphere. Lower: velocity curve for the base of the atmosphere (piston), shown as a continuous curve, and the pre-shock (outside) and post-shock velocities. The pre-shock velocity is uppermost in the figure. Positive values of the ordinate indicate infalling material.
defined by $g_{\text{eff}} = p/m$, is constant with depth and its reciprocal increases linearly with time since the velocity has a constant gradient with $\xi n(1/m)$ given by $d\xi /d\xi n(1/m) = a$. The picture this forms is that each mass element follows a parabolic arc, accelerating inward at $g$, receiving an impulse once each period from the shock. The motion of the upper layers is retarded in phase relative to the lower layers by the shock travel time between them.

The kinematics of this requires that the shock Mach number be constant at the value $M = PG/a = a\lambda$. Further, the effective gravity varies between a minimum of $g/M$ and a maximum of $g \cdot M$.

The secondary shock results indirectly from the pressure gradient set up when the main shock increases rapidly in strength after maximum compression. This pressure gradient causes that material to over-expand, and subsequently it accelerates inward under gravity more rapidly than the rest, forming the early shock. The main shock is formed by non-uniform piston acceleration somewhat later. A second subsidiary shock is also formed by non-uniform piston acceleration after the main shock, and very quickly overtakes it. It is interesting to note that the main shock and the second subsidiary shock form about 3 to 4 scale heights in exterior mass above the piston. The early shock forms about 10 scale heights above the piston.

Figs. 6 and 7 show the pressure distribution with mass for a sequence of times in a calculation of shock propagation into material which is in hydrostatic equilibrium. In Fig. 6 the slow increase of Mach number with height and the large pressure ratio across the shock are evident. In Fig. 7 we see how after the shock reaches the outer boundary of the calculation region ($t_0$) and after a subsequent phase during which a rarefaction wave is visible propagating inward ($t_0$) the atmosphere adopts a condition with $g_{\text{eff}}$ constant in depth and decreasing in time ($t_6$ and $t_7$). The events associated with $t_0$ and $t_5$ occur very quickly and the motion in the post-shock-escape phase is predominantly of the type at $t_5$ and $t_7$. At the same time as that pressure distribution is being established, the velocity distribution is becoming of the form $U = c-g\cdot t-a\cdot \xi n(m)$. 
Fig. 6 Pressure profiles at successive times in the calculation of the propagation of an isothermal shock wave into an isothermal hydrostatic equilibrium atmosphere. The line labelled $t_0$ is the undisturbed atmosphere. Depth increases toward the right.
Fig. 7 The same as Fig. 6 for times after the shock was allowed to escape at the surface-most mass point. Of special note is the return of the pressure structure to a scaled hydrostatic equilibrium state.
It should be emphasized that the growth of the Mach number with height is very much slower for the isothermal shock than for an adiabatic one. A comparison of two identical calculations except for the replacement of isothermality with adiabacity is shown in Fig. 8. The growth of Mach number with height in the adiabatic case is in excellent agreement with the similarity solution for strong shocks (Zel'dovich and Raiser 1967): 
\[ M \propto \frac{1}{m} \approx 0.204. \]
The growth of Mach number in the isothermal case does not admit a similarity solution, but applying Whitham's (1958) approximate method yields 
\[ M = C - \frac{1}{\ell} n(m), \]
in fair agreement with the numerical results.

IV. RADIATIVE SHOCK DYNAMICS

Stephen Hill has made a study of the development of a shock in an RR Lyrae model using the numerical nonlinear initial value method. The techniques are quite standard (Christy 1964; Cox, Brownlee and Eilers 1966) except for the treatment of radiative transfer and the boundary conditions, and so perhaps some comment on these aspects is in order.

Letting \( J_\nu \), \( B_\nu \) and \( K_\nu \) be the usual first three moments of the radiation field with respect to angle, the first two moment equations in the LTE approximation become

\[
\frac{dH_\nu}{dm} = \kappa_\nu (J_\nu - B_\nu)
\]

and

\[
\frac{1}{\kappa_\nu + \sigma_\nu} \frac{d\kappa_\nu}{dm} = B_\nu
\]

in which \( \kappa_\nu \) is the mass absorption coefficient and \( \sigma_\nu \) is the mass scattering coefficient. By integrating each equation over \( \nu \) and inserting the Planck and
Fig. 8 Mach number of a shock wave propagating into a hydrostatic equilibrium isothermal atmosphere as a function of the mass depth of the shock front. For the curve labelled \( \gamma = 5/3 \) the motion was assumed to be adiabatic. For the other, isothermality of the motion was assumed.
Rosseland mean opacities as implied by the way each equation has been written, we obtain

\[
\frac{dl}{dm} = \kappa_p (J-B) \quad \text{and} \quad \frac{1}{(\kappa + \sigma) R} \frac{dK}{dn} = H
\]

which become, in the Eddington approximation \( K = (1/3) J \),

\[
\frac{1}{3} \frac{d}{dn} \left( \frac{1}{(\kappa + \sigma) R} \frac{dl}{dm} \right) = \kappa_p (J-B)
\]

This is the equation of transfer Hill has adopted. The corresponding form for the (Lagrangian) energy equation is

\[
\frac{\partial E}{\partial t} + p \frac{\partial v}{\partial t} = 4\pi \kappa_p (J-B).
\]

In the numerical scheme the transfer equation is written, in finite difference form, in terms of quantities entirely at one instant, say \( t^{n+1} \). The right member of the energy equation in difference form is an average of values at \( t^n \) and \( t^{n+1} \), so \( T \) and \( J \) at \( t^{n+1} \) must be solved for simultaneously. The Newton-Raphson procedure is used, and because the energy equation is a local equation, the correction to \( T \) in a zone is easily expressed in terms of the correction to \( J \) in that zone.

The calculation uses plane-parallel symmetry. The boundary conditions at the base of the atmosphere are

1) radiative flux is prescribed at \( \tau = 4 \) as a function of time;

2) velocity of the fluid is prescribed at \( \tau = 4 \) as a function of time.

These boundary conditions are applied at \( \tau = 4 \), even though this location varies in mass, since zones are added or subtracted at the bottom as necessary.
The boundary conditions at the top of the atmosphere are

3) \( H = J/\sqrt{3} \)

4) pressure vanishes.

Boundary condition 4) insures that small amplitude waves are completely reflected at the surface; however, a special treatment is used for shock waves approaching the surface --- these are allowed to escape without reflection, which is realistic and also expedites the calculation.

The flux and velocity at \( T = 4 \) are taken as a function of phase from the model generated by Castor (1966). This is quite similar to Christy's (1966a) model designated 5gP, except that much finer zoning was used in the \( H \) ionization region, and a form of radiation transport replaced the diffusion approximation.

This model has the parameters

\[ T_{\text{eff}} = 6500^\circ \text{K} \]

\[ M_{\text{bol}} = 0.82 \]

\[ M = 0.58 M_\odot \]

\[ Y = 0.30 \]

The results verify that, indeed, material which is heated by the passage of the shock very quickly cools to radiative equilibrium. Usually \( T \) returns to approximately 5000°K within about 2 zones behind the shock front, a distance short enough to not affect the dynamics. A very interesting result is that when \( T \) is lowered below 5000°K by adiabatic expansion, the whole period may not be long enough for \( T \) to return to 5000°K. This occurs as shown in the m-t diagram (Fig. 9).

The depth in the atmosphere where the over expansion caused a shock in the isothermal model is taken by a cold region (\( T < 1000^\circ \text{K} \)) in the radiative model. A shock caused by material falling back upon the piston occurs lower in the atmosphere. The low temperature region and the main shock are conspicuous in Fig. 10. The main shock follows the piston very closely until about the phase of maximum outward
Fig. 9 Mass versus time array of significant phenomena occurring in S. J. Hill's radiative transfer RR Lyrae atmosphere pulsation calculation. The ordinate is linear in the logarithm of the exterior mass, with the base of the atmosphere at the bottom. Approximately nine decades in exterior mass are represented. The curves with arrows attached are the discernible shocks which occurred in the calculation. The hatched regions are cooler than 3000⁰K, with increasing numbers of strokes indicating lower temperatures in the sequence 3000, 1000, 10⁰K. Thus the triple cross-hatching indicates temperatures less than 10⁰K.
Fig. 10 Height versus time trajectories of individual mass zones. The zero-point of the height scale is arbitrary. The densely grouped zones which are ascending are located immediately interior to the principal shock. This shock formed in the atmosphere at the times approximately 7, 12, and 17 \cdot 10^4 sec. The densely grouped descending zones lie in the low temperature region indicated in Fig. 9.
piston velocity, which is also when it passes through the low temperature region, after which it moves rapidly away from the piston.

The lower section of Fig. 11 shows the pre-shock, post-shock and piston velocities, as for the isothermal case in Fig. 5. The piston velocity is the continuous periodic curve, the pre-shock velocity is the upper curve, and the post-shock velocity is coincident with the piston velocity until \( t = 26 \times 10^3 \) s, after which time it is shown by the lowest curve. That the pre-shock and post-shock velocities are not constant, though their difference is, after the shock has passed the low temperature region is due to lack of periodicity of the calculation as a whole.

When the material flows through an isothermal shock (as the shocks in this calculation very nearly are) the enthalpy returns after the shock to approximately its pre-shock value, but the kinetic energy per unit mass changes by approximately \( \frac{1}{2} \dot{v}_0^2 \), therefore the energy per unit area per unit time which must be radiated by the material in passing through the shock is \( \frac{1}{2} \dot{v}_0^2 \) (for a strong shock). This quantity, which we may call \( P_S \), is compared with the total flux from the star, \( F \), in the upper section of Fig. 11. We see that \( P_S / F \) twice attains the value 10 per cent, each time after merging with one of the subsidiary shocks. The second maximum corresponds well with minimum radius, while the first maximum is at about the phase of the bump in the velocity curve discussed by Christy (1966a). We see that \( P_S \) drops sharply after maximum light.

We show the velocity jump across the shock as a function of the height of the shock in Fig. 12. This shows the constancy of shock strength above about \( m = 10^{-2} \) g cm\(^{-2}\) when the shock has emerged from the cold material.

We can calculate the total shock dissipation for one passage through the atmosphere. It is (for the third period of calculation)

\[
\text{dissipation} = 1 \times 10^{38} \text{ erg},
\]

which, in terms of \( K \), the total pulsation energy, is

\[
\text{dissipation} = 0.004 \text{ } K.
\]
Fig. 11 Upper: The energy flux radiated by the shock expressed as a fraction of the total flux of the star as a function of time. Lower: The piston or photospheric velocity and the pre- and post-shock velocities as a function of time, as in Fig. 5.
Fig. 12  Velocity jump across the shock as a function of its mass depth, analogous to Fig. 8.
Since the total driving of the pulsation is 0.035 K, this extra dissipation is quite significant. This dissipation is also about 12 per cent of the work done by one side of the $T = 1$ interface upon the other side during a half-period. That is, this interface absorbs about 12 per cent of the mechanical energy incident on it. This must be contrasted with the assumption often made that this interface provides complete reflection of the mechanical energy flux.

V. SHOCK STRUCTURE

What about the internal details of the shock? This was considered by Ledoux and Whitney (1960) and by Whitney and Skalafuris (1963). The subject flourished for several years in the 1960's and studies were made by Heaslet and Baldwin (1963) for the gray LTE case, and for a resonance continuum non-LTE model by Clarke and Ferrari (1965). Surprisingly, little seems to have been done on this problem for the last few years. The book by Zel'dovich and Raizer (1967) has a very useful discussion, as does Part I.E of Thomas (1967).

We can use the approximation of steady flow, justified by the small scale of the shock structure, to construct conservation equations for mass, momentum, and energy which are:

$$
\rho v = c_1
$$

$$
p + \rho v^2 = c_2
$$

$$
\rho v(E + p/\rho + \frac{1}{2}v^2) - F = c_3.
$$

While $c_1$ and $c_2$ follow from the known upstream conditions; $c_3$ is not known a priori since it is related to the solution of the radiative transfer problem. These equations may be rewritten (using $V = 1/\rho$) as
\[ v = C_1 v \]
\[ p = C_2 - C_1^2 v \]
\[ E + V(C_2 - \frac{1}{2} C_1^2 v) = \frac{F + C_3}{C_1} \]

By using the second equation the temperature may be found in terms of \( V \), which allows us to find the left member of the third equation also. Thus both \( T \) and \( F + C_3 \) may be regarded as functions of \( V \).

These relations are sketched in Fig. 13.

For strong shocks, in which the upstream temperature is negligible compared with temperatures within the shock, the material passes along the bold sections of the curves in Fig. 13. Material far upstream of the shock is represented by point A. Nearing the shock, it is heated by the shock radiation so that immediately in front of the discontinuity itself it has reached conditions B. Passing through the discontinuity brings it to point C, and the flow through the cooling region takes it on to point D, back at the low ambient temperature. Since the flux must be continuous across the hydrodynamic discontinuity, the points B and C are joined by a horizontal line on the flux curve. In moving from point C to point D, the cooling material radiates an amount of energy equivalent to a flux increment \( \Delta F_1 + \Delta F_2 \).

The amount \( \Delta F_1 \) is just what is absorbed by the upstream material, while \( \Delta F_2 \) escapes. That is, \( \Delta F_1 \) is the energy radiated in the upstream direction to which the upstream material is opaque; this might be the Lyman continuum radiation. The remainder of the radiation, \( \Delta F_2 \), is in frequencies to which the upstream material is transparent, such as the Balmer or Paschen continua. If the upstream material is opaque to all radiation, then \( \Delta F_2 \) vanishes and point D is replaced by point \( D' \). We notice that the Rankine-Hugoniot relations apply between points A and \( D' \) and between B and C.
Fig. 13 Schematic curves showing the relations of temperature and energy flux to specific volume implied by the conservation equations, for flow in the vicinity of the shock. The points A, B, C, D', and D are discussed in the text.
That Rankine-Hugoniot conditions do not apply between the distant upstream and downstream conditions, i.e. between points A and D, is due to the existence of spectral windows which allow the flux $\Delta P_2$ to escape; isothermal, rather than R-H, conditions are the result.

An approximate solution for the shock structure is possible using Zel'dovich and Raizer's method of approximating the sections A-B and C-D of the curves in Fig. 13 by the linear relation

$$F + C_3 = C_1 CT$$

The appropriate value of $C$ would be $C_1$ at point A, and $C_P$ at point D. With the gray Eddington model we have

$$\frac{d^2 F}{dT^2} + 3F = -4\pi \frac{dB}{dT}$$

so

$$\frac{d^2 F}{dT^2} + K \frac{dF}{dT} - 3F = 0$$

where

$$K = \frac{4\pi}{C_1 C} \frac{dB}{dT} = B_0$$

is the local Boltzmann number. Treating $K$ as if it were constant, we find solutions $F \propto e^{3 \tau/K}$, and $F \propto e^{-KT}$. The first is applicable upstream, while the second applies downstream. Since $K$ is usually quite large ($10^3$ is typical), the length scale is very much longer in the upstream material (precursor) than in the downstream material (tail). The typical temperature profile of the shock is sketched in Fig. 14. Since in the gray model there is no transparent window in the spectrum, the conditions far in the tail are those of point D' in Fig. 13. An interesting feature of
Fig. 14  The schematic temperature structure of a gray, LTE shock. $K$ is the Boltzmann number for this problem.
this temperature profile is that most of the cooling in the tail occurs in a very optically thin temperature spike, which therefore contributes little to the mean intensity at the shock face (only an amount of order 1/K). That mean intensity is therefore fixed by the temperatures at points D' and B. The long scale of the precursor implies that it is very close to radiative equilibrium, so J nearly equals B at point B, consequently the temperatures at points D' and B must be nearly equal.

For the case of two frequency bands with constant opacity in each, a similar simple discussion has been given by Olfe and Cavalliri (1967), also assuming LTE. If the two bands are called "1" and "2", with $\kappa_1 >> \kappa_2$, and if it assumed that band 1 is always optically thin, there are three relevant scales (in mass depth m):

\[
\begin{align*}
\text{precursor:} & \quad 1/(\sqrt{3}\kappa_2) \\
\text{tail:} & \quad 1/(K\kappa_p) \quad \text{and} \quad (1/\sqrt{3}\kappa_2)
\end{align*}
\]

where $\kappa_p$ is the Planck mean of $\kappa_1$ and $\kappa_2$. The result for the case that $K$ is larger than $\kappa_2/\kappa_1$ is the temperature structure illustrated in Fig. 15. In this case the precursor and the extended part of the tail have the same characteristic length, which is one mean free path of the more opaque frequency band. In fact, the structure is quite symmetrical about the temperature spike. The spike is optically thin to all radiation.

VI. COMPUTED SHOCK STRUCTURES

Hill has made numerical calculations of radiating shock structures using this model:

1) LTE;
2) gray in each of two frequency bands, one above and one below the H I Lyman edge, with Planck and Rosseland mean opacities in each;
3) radiation pressure is neglected;
4) Population I composition;
Fig. 15. The schematic temperature structure for a two-frequency model with \( \frac{k_2}{k_1} \).
5) boundary conditions for the steady-flow equations taken from the dynamical calculation described in Section IV.

The numerical method is analogous to that of Auer and Mihalas (1968). Four models were constructed, labelled (a) to (d). Model (a) refers to a phase quite soon after shock formation. Model (b) is at the phase of the first maximum in \( F_\text{B} \). Model (c) is at the phase of the second maximum in \( F_\text{B} \), i.e., essentially at minimum radius. Model (d) is typical of the maximum strength shock, and is taken at about the phase of maximum outward piston velocity. The parameters for the four shock models are given in Fig. 16.

The principal result of each calculation is the temperature distribution. These are shown in Figs. 17-20. Shock (a) has no precursor, and only raises \( T \) to 8000⁰K, giving rise to mostly \( \text{H}^{-} \) recombination radiation. Shock (b), with its speed of 27 \( \text{km} \ \text{s}^{-1} \), can not quite ionize \( \text{H} \), so it probably divides its radiation between the \( \text{H}^{-} \) bound-free and Balmer continua.

There is a slight precursor. Shock (c), which has the largest value of \( F_\text{B} \), is able to ionize \( \text{H} \) easily \( (v_0 \text{ is } 71 \ \text{km} \ \text{g}^{-1}) \). The precursor is visible extending 3.10⁻⁵ g cm⁻² ahead of the shock. The cooling is gradual, beginning only 10⁻⁸ g cm⁻² behind the shock and reaching a plateau at the precursor temperature (about 13000⁰K) and then finally dropping near 3.10⁻⁵ g cm⁻² from the shock face. This suggests the symmetry of precursor and tail indicated in the simple two-band picture. Shock (d) shows substantially the same effects, but in this case a temperature plateau also occurs at the \( \text{HeII} \) recombination temperature 40000⁰K. Calculations are under way of the \( \text{H}_\gamma \) emission from these shocks. These indicate that the strongest emission is produced by shock (c), coinciding with minimum radius, as observed.
<table>
<thead>
<tr>
<th>SHOCK</th>
<th>$\rho_0$</th>
<th>$v_0$</th>
<th>$T_0$</th>
<th>$m_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$4.2 \times 10^{-9}$</td>
<td>13.6</td>
<td>5240</td>
<td>39.5</td>
</tr>
<tr>
<td>b</td>
<td>$8.5 \times 10^{-10}$</td>
<td>27.2</td>
<td>5040</td>
<td>8.7</td>
</tr>
<tr>
<td>c</td>
<td>$6.7 \times 10^{-11}$</td>
<td>71.3</td>
<td>4640</td>
<td>1.18</td>
</tr>
<tr>
<td>d</td>
<td>$1.0 \times 10^{-11}$</td>
<td>145.9</td>
<td>2500</td>
<td>0.148</td>
</tr>
</tbody>
</table>

Fig. 16. The physical parameters of four shock models intended to represent successive stages in the outward propagation of a shock in the RR Lyrae atmosphere. $\rho_0$ and $v_0$ are the pre-shock density in g cm$^{-3}$ and the speed of the pre-shock material in km sec$^{-1}$ relative to the shock face, respectively. $T_0$ is the pre-shock temperature. $m_s$ is the mass depth of the shock front in g cm$^{-2}$. 
Fig. 17. The temperature profile of shock model $a$. The left panel shows pre-shock material, and the right panel shows post-shock material. The abscissas in both cases is the logarithm of the displacement in mass depth from the shock front.
Fig. 19: The same as Fig. 17 for shock model c.
Fig. 20  The same as Fig. 17 for shock model d.
VII. CONCLUSION

What can we learn from these simple models and the numerical calculations? We now are able to make some definite statements about the H\gamma emission in RR Lyrae stars. We are able to predict emission at the phase when it is actually observed. We can conclude from the constant Mach number of the shock at great height in the atmosphere and the lack of any tendency for long-term heating of the atmosphere, that the shock wave should not cause more than a negligible amount of mass loss. We can say that shock dissipation is sufficiently large to substantially affect the pulsation of the star as a whole, although it has been omitted from all calculations to date by the use of a truncated atmosphere. The need is clear for dynamical full envelope calculations with a much more extensive atmosphere to clarify the relationship between the shock and the envelope pulsation, and also to explain the differences in shock strength among RR Lyrae stars and between W Vir, RR Lyrae and \delta Cep stars.

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