SENSITIVITY OF INFERRED SUBPHOTOSPHERIC VELOCITY FIELD TO MODE SELECTION, ANALYSIS TECHNIQUE AND NOISE

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ABSTRACT

The horizontal velocity immediately below the photosphere can be inferred from observations of high-degree solar oscillations by an optimal-averaging inversion technique. We investigate the sensitivity of the results to various details of both the inversion and the determination of the frequencies. The results are shown to be quite stable to the choice of most parameters, suggesting that this procedure produces reliable estimates of the subsurface velocity.

1. INTRODUCTION

Helioseismology has now developed to the point where we are able to infer the physical conditions of the solar interior from observations of the frequencies of the Sun's oscillations. Owing to its comparatively simple effect on the frequencies, the quantity most readily obtained is horizontal velocity. This velocity is primarily differential rotation, though it no doubt contains a contribution from convection cells. Deubner, Ulrich and Rhodes (1979) have attempted to determine the solar rotation from five-minute oscillation frequencies, though their method did not permit them to determine the depth dependence. A similar analysis applied to later observations failed to reproduce the earlier results (Rhodes, Harvey and Duvall 1983).
The prospect of detecting giant cells from their effects on the oscillations is exciting, particularly because earlier searches for surface Doppler velocities with a large spatial scale have failed. This suggests that the photospheric velocity amplitude of giant cells is below the 10 ms\(^{-1}\) sensitivity of the measurements (LaBonte, Howard and Gilman 1981). However, a theoretical model of compressible convection predicts that the horizontal velocity increases with depth (Latour, Toomre and Zahn 1983). We thus might expect to find subsurface flows with amplitudes substantially greater than the limits set by the photospheric observations.

We have sought to infer the horizontal velocity as a function of depth from the positions of the ridges in the k-ω diagram of high-degree (k > 100) five-minute oscillations. Here k is the horizontal wavenumber and ω is the temporal frequency. Figure 1 displays a contour plot of one of our observed k-ω diagrams, showing the ridges corresponding to the f, or fundamental, mode and the p- through p- modes. We have previously reported (Hill, Toomre, and November 1982, 1983) variations in the ridge positions from day to day that were possibly associated with the passage of giant cells across the field of view (e.g. Gough and Toomre 1983).

The possibility of inferring subsurface velocities stems from the fact that wave patterns are advected by a horizontal flow (e.g. Rhodes, Deubner and Ulrich 1979). In the case of high-degree modes, the wavelength of the oscillations is much smaller than the horizontal scale of the flow pattern of interest, namely the giant cells. Then the apparent frequency of a mode with a given (k,ω) is modified by an amount δω = k U. Provided the variation at all detectable depths of the horizontal component of the subphotospheric flow velocity U across the field of view and throughout the observing interval can be ignored, the advection velocity U of the wave pattern is the vertical average of the equatorial component of U weighted by the energy density of the mode (Gough 1978).

We have applied an optimal averaging inversion procedure to perturbations of the ridge position that we believe are caused by large-scale horizontal flows. The inversion procedure is suggested by the work of Backus and Gilbert (1968,1970) and has been most fully developed in the field of geophysics. In our case, the frequency shift δω induced by the flow U is

\[ δ\omega_1 = k_1 \bar{U} = k_1 \int_{\zeta_1}^{\zeta_2} A_1 U H d\zeta, \]

where ζ = log10 p is used as the depth coordinate (p is the pressure in the equilibrium model), H is the pressure scale height and A\(_i\) is proportional to the kinetic energy density of the mode designated by the subscript \(i\). The constant of proportionality is chosen to render H A\(_i\) unimodular; the limits ζ\(_1\) and ζ\(_2\) bound the interval within which H A\(_i\) differs significantly from zero. The inversion procedure involves the construction of linear combinations of equation (1), yielding
Figure 1. Contours of constant-power in the $k-\omega$ plane obtained on day 82 (23 March 1981). The fits resulting from the iterative ridge-fitting procedure are overlaid on the diagram. One of the initial boxes used to isolate the ridges is drawn around the $p_1$ ridge. The cross-sections used to determine the ridge positions are indicated on the $p_3$ ridge. Our ridge classification ignores the possible existence of chromospheric modes.
\[ \int_{\zeta_1}^{\zeta_2} \sum_{i=1}^{N} \alpha_i(\zeta) A_i(\zeta') U(\zeta') \, d\zeta' \equiv \int_{\zeta_1}^{\zeta_2} D(\zeta; \zeta') U(\zeta') \, d\zeta' \]

\[ \equiv \tilde{U}(\zeta) = \sum_{i=1}^{N} \alpha_i \frac{k_i^{-1}}{\omega_i} \delta_{\omega_i}, \]  

(2)

The coefficients \( \alpha_i \) are chosen to make \( D(\zeta, \zeta') \) resemble a Dirac delta function \( \delta(\zeta-\zeta') \) as closely as possible. Then \( \tilde{U}(\zeta) \) estimates \( U(\zeta) \). The technique of Backus and Gilbert (1970), which takes account of the errors in the data, was used to calculate the coefficients \( \alpha_i \). More details of inversion procedures applied to similar data can be found in Hill, Gough and Toomre (1984), Gough (1984), and Christensen-Dalsgaard and Gough (1984).

The inversion was carried out on \( k-\omega \) diagrams computed from observations of Doppler velocities obtained at Sacramento Peak Observatory (SPO). Data were obtained from the Fe I 5434.5 Å and Mg I 5172.7 Å spectral lines using the diode array situated at the exit of the echelle spectrograph of the vacuum tower telescope. Sequential two-dimensional intensity images in the red and blue wings of both lines were produced by spatially scanning the Sun across the slit of the spectrograph. The scan produced a 256" × 1024" image centered on the solar equator with a nominal resolution of 2". Scans were repeated every 65 or 70 s for observing intervals of 7 to 11 hours.

The data reduction began with the determination of the line-of-sight velocity in the manner described by November et al. (1979). The resulting velocities were averaged perpendicular to the equator to filter out all waves except those with a horizontal wavenumber vector oriented nearly parallel to the equator. The data were projected onto an array with a uniform spacing in longitude of 0.1183°, corresponding to a resolution in \( k \) of \( 8.54 \times 10^{-3} \text{ Mm}^{-1} \). The time series was extended with zeros to obtain a constant resolution of \( 1.5 \times 10^{-4} \text{ s}^{-1} \) in \( \omega \). The resulting data array was then Fourier transformed to produce a \( k-\omega \) diagram such as that in Figure 1. We have considered a total of six different \( k-\omega \) diagrams obtained in February and March of 1981. We shall identify the data by the number of the day in the year when they were obtained. Figure 1 shows the diagram for day 82, which is 23 March 1981.

We have inverted the data to produce an estimate of the horizontal velocity in the 15 Mm immediately below the photosphere. The results are shown in Figure 2, which displays the results for the six different days. There is a general tendency for the horizontal velocity to increase with depth, and there is a variation of the order of 100 ms\(^{-1}\) from day to day in the curves. We suggest that these curves possibly reflect an increasing rotational velocity with depth on which is superimposed a large-scale convective flow of about 100 ms\(^{-1}\). Further discussion of the interpretation of these curves can be found in Hill, Gough and Toomre (1984). What we discuss in this paper is the sensitivity of the results of the inversion to variations of certain details of the data analysis.
Figure 2. Horizontal velocity as a function of depth $\xi (= \log_{10} p)$ inferred from the inversion of data obtained on 6 days in 1981. The curves are identified by the number of the day of the year on which the observations were made. There is a general increase of velocity with depth, with day-to-day variation on the order of 100 ms$^{-1}$. The latter is probably the result of the changing position of giant cells, and is superimposed on a differential rotation velocity that increases with depth.
2. DETERMINATION OF THE FREQUENCIES

Since the individual modes cannot be resolved by these observations, we have estimated the frequencies by a ridge-fitting technique. It appears that this method is useful in overcoming some of the effects of mode beating and atmospheric seeing that are a source of noise. From the size of our resolution bins and the distribution of the solar modes in the k-ω diagram, we estimate that we have about 30 modes in each bin. The beating of these modes produces a jagged mountain-chain appearance to the ridges which changes markedly from day to day. Mode beating can actually place power outside the bin in which the responsible modes reside, though the extent to which this occurs remains to be investigated. In addition, image motion produced by atmospheric seeing can alter the distribution of power along a ridge (Hill 1984). The net result of these processes is to impart a small-scale variation to the location of the ridges which we must try to eliminate in our search for frequency changes due to solar effects. We have thus developed a technique of ridge finding which involves a considerable amount of smoothing.

The ridge-fitting procedure begins with the drawing of a polygonal box around each of the 18 ridges in the +ω and -ω quadrants of the k-ω diagram having k > 0, using the cursor of an interactive graphics device to choose the vertices. The boxes define the limits of slices in the k-ω diagram that are used to calculate initial estimates of the median positions of the ridges. One of these boxes, drawn around the p₁ ridge, is shown in Figure 1. The orientations of the initial slices are vertical (constant k) for the f ridge and the p₁ through p₃ ridges, and horizontal (constant ω) for the p₄ through p₈ ridges. Different orientations are chosen because the higher-order ridges have steeper slopes.

The data along each slice are interpolated onto a grid of 0.1 pixel using a cubic spline. The median of the smoothed ridge cross-section, defined to be the point that divides the integral of the power in half, is computed. The loci of the medians in the k-ω plane are then smoothed with a running mean and fitted with a variable-knot cubic spline. This type of cubic spline allows the specification of the number and the initial positions of knots at which the spline conditions are applied. These positions are then varied to minimize the rms deviation between the fit and the raw medians. In all cases, the initial knots were approximately evenly spaced along the ridges.

Our original data analysis ended at this point, and used the resulting fit computed with 7 knots to provide the frequencies of the modes. We have found, however, that performing the inversion on the frequencies thus obtained gave inconsistent results when we considered different sets of modes that sample similar depths. Further investigation showed that the shape of the boxes drawn about the ridges markedly influenced the frequencies. In addition, the loci of medians computed from cross-sections parallel to the k and ω axes are different. This
arises in part from the curvature of those loci. Thus we have turned to an iterative scheme of ridge finding that calculates cross-sections perpendicular to the ridges. We have also attempted to minimize the influence of noise in the power spectrum by using the positions of adjacent ridges to choose the edges of the boxes.

The orthogonal ridge fitting begins from an initial estimate obtained from the procedure described above, using 3 knots. This is used to estimate the positions of the adjacent ridges which, where necessary, are extrapolated using the functional form deduced by Duvall (1982). At any given k the direction perpendicular to the initial fit is then calculated from the cubic spline coefficients, and the intersection between this line and the adjacent ridges is determined. The values of k at which these perpendicular slices are calculated are evenly spaced in arc length along the ridge.

Next, a small region centered on the ridge at the chosen k is used to provide the coefficients of a two-dimensional cubic spline interpolation. These coefficients determine what we call a slice: the interpolated cross-section of the power along the direction perpendicular to the ridge, with resolution of 0.1 pixel. The slice is then truncated on both sides of the ridge, the points of truncation being determined by a parameter which we call the cut factor; it is defined as a fraction of the distance along the slice to the estimated adjacent ridge. The directions and lengths of the slices for a cut factor of 0.3 are shown superimposed on the p3 ridge in Figure 1.

The medians of the splines along the slices (which we call the raw medians) are determined, and their loci are smoothed, using a 3-knot cubic spline. The latter we refer to as a ridge fit. This provides a first iterate of the position of the ridge. The procedure is repeated once more using the iterate to approximate the adjacent ridge positions and to provide the directions of the cross-sections. The final fit is performed with 7 knots, in order to include intermediate-scale structure along the ridge. Five points are removed from each end of the ridge before the final fit to remove end effects introduced by the running mean.

3. SENSITIVITY TO ANALYSIS TECHNIQUE

We have varied some of the parameters that enter into our rather complicated ridge-finding procedure to assess the sensitivity of the inversion to the somewhat arbitrary details of the analysis. The parameters that we have varied are: the number of knots in the final fit, the number of knots in the intermediate fits, the number of points in the running mean used to smooth the final fit, the number of points in the running mean used to smooth the intermediate fits, and the cut factor. We have also computed medians using slices truncated symmetrically on both sides of the ridge.
The results are shown in Figures 4 through 9. All of the figures present the horizontal velocity as a function of depth inferred from the inversion of the data obtained on day 82. The curve that we feel is most reliable was produced from the orthogonal ridge-fitting procedure using a 15-point running mean to smooth both the final and intermediate medians: 7 knots for the final fit, 3 knots for the intermediate fits, and a cut factor of 0.3 to each ridge. This curve can be seen in Figure 2.

In Figure 3 we show the results of varying the number of points in the running mean that is used to smooth the final set of medians. A total of 10 different uniformly weighted means were used, ranging from 1 (no running mean at all) to 19 points, with every intervening value being an odd number. All 10 curves are plotted in Figure 3, which shows that the procedure is quite stable to variations of this parameter; the greatest discrepancy between the curves is about 5 ms⁻¹.

Figure 4 shows the result of varying the extent of the running mean applied to the first iterates on the medians; the same selection of widths was used as for the final running mean discussed above. The variation, although still small, is more substantial here, being at most 12 ms⁻¹. This possibly reflects the effect of small differences in the direction of the line along which the cross-section is computed.

Figure 5 shows the effect of varying the number of knots used to define the final fit. A total of six cases was computed, using 3, 5, 7, 10, 15, and 20 knots. Figure 5 shows that the resulting velocity curve is stable to the final number of knots as long as it is above 3. Though the data analysis is designed to suppress small-scale variation along the ridge, we must not obliterate everything. We expect to be able to extract intermediate scales from the noise. It appears that a mere three knots are insufficient for this purpose. The maximum variation for the cases where the number of knots is 5 or greater is about 5 ms⁻¹. On the other hand, the procedure proved to be incapable of successfully finding the complete ridge when an intermediate fit of greater than 3 knots was used. This arises from changes in the ridge slicing direction. When many knots are used to compute the intermediate fit, the small-scale structure results in a rapidly varying slicing direction and renders the procedure unstable.

Figure 6 compares our standard asymmetrical truncation of the slices with a symmetrical truncation. In the latter case the two truncation points were the same distance from the median, and were a constant factor of the distance to the nearest ridge. Thus noise is sampled differently. The distinction between the two methods is not important, however, as the maximum variation between the two curves is only 5 ms⁻¹.

Figure 7 illustrates the result of varying the cut factor. Values of 0.10, 0.15, 0.20, 0.25, 0.30, and 0.35 have been tried. The results indicate that the procedure is rather sensitive to this parameter, for
Figure 3. The horizontal velocity as a function of depth \( \xi \) on day 82 calculated using 10 different widths for the final running mean. The maximum variation between the curves is 5 ms\(^{-1}\), indicating that the inversion procedure is stable to variations of this parameter.

Figure 4. As Figure 3, but for 10 different widths of the intermediate running mean. The variation is greater here being about 12 ms\(^{-1}\). This is due to the effect of small changes in the direction along which the cross-section is computed.
Figure 5. As for Figure 3, but using different numbers of knots in the final fit of the medians. The curve labeled 3 was obtained from 3 knots, and no doubt fails to reflect the structure along the ridge. The results are stable provided the number of knots is no less than 5.

Figure 6. As for Figure 3, but comparing the effects of symmetrically truncated slices whose lengths are determined by the distance to the nearest ridge (curve A) and asymmetrical truncations determined by the positions of both ridges (curve B).
the variation of the inferred velocity can be as much as 50 ms\(^{-1}\). The sensitivity is least for cut factors in the narrow range 0.25 - 0.35, where the maximum variation is 7 ms\(^{-1}\). Cut factors smaller than 0.25 result in the loss of the edges of the ridges and do not provide an accurate median. The ridge finding procedure is incapable of finding the complete ridges when the cut factor is greater than 0.35, because larger cross sections encompass power from adjacent ridges in some locations, which pulls the fit away from the desired ridge in the iteration.

4. SENSITIVITY TO MODE SELECTION AND NOISE

The inversion technique results in the construction of a set of optimal averaging kernels \(D(\zeta, \zeta')\). As is evident from equation (2), each optimal kernel provides a weight function to average the flow velocity \(U\), the result of which is a weighted average of the data. Because the original kernels \(A_1\) effectively span a space of dimension less than the number of data, there can be redundancy in the data, and the combinations that yield \(D\) reduce the effects of noise. However, there is a tradeoff between resolution and error. A narrower kernel \(D\) requires coefficients \(a_1\) with greater magnitudes, resulting in a greater contribution from uncorrelated noise to the inferred velocity. Thus it is expedient to sacrifice some resolution.

The inversion technique takes errors into account by means of a parameter \(\theta\) whose value can vary from 0 to \(\pi/2\) (Backus and Gilbert 1970; see also Gough 1984). When \(\theta = 0\), errors in the data are ignored and the widths of the optimal kernels are minimized to provide the highest resolution. Random errors are greatly magnified, and yield an unrealistically rapidly varying velocity. On the other hand, when \(\theta = \pi/2\), the kernels \(A_1\) are ignored entirely when determining \(a_1\), and the velocity curve that is produced has unacceptably poor resolution. An intermediate value of \(\theta\) must be chosen by performing the inversion for a number of different choices for \(\theta\) and selecting a value at which the shape of the velocity curve is stable. If the inversion is nowhere stable, the data are too heavily contaminated by error to permit an inference of the flow velocity, unless one is prepared to assume some constraint on \(U\).

The procedure is illustrated in Figure 8, which shows the result of the inversion with \(\theta = 0.001, 0.01, 0.1, 0.5, 0.7,\) and 1.0. The first three values produce widely varying velocities characteristic of inversions that ignore errors. The curves stabilize at \(\theta = 0.5\), and the smaller-scale structure begins to be smoothed out as \(\theta\) approaches \(\pi/2\). This can be seen more clearly in Figure 8b, which shows the curves for \(\theta = 0.5, 0.7,\) and 1.0 on an expanded scale. Figure 8b also shows that the depths at which the optimal kernels are centered changes as \(\theta\) is varied. We have chosen \(\theta = 0.5\).

The effectiveness of the inversion procedure in reducing errors is shown by comparing the noise in the raw data with the errors in the inferred velocity. We have defined the noise in the data to be the rms

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Figure 7. As for Figure 3, but comparing the results from different lengths of the cross-section. The curves are labeled with the cut factor used to compute them. The cut factor is the fraction of the distance to the next ridge that is covered by the cross-section. The results vary considerably, except in the narrow range 0.25 - 0.35. We have used a cut factor of 0.30 to compute the curves in Figure 2.
difference in pixels between the final ridge fit and the final raw medians. This error varies between 0.1 and 0.6 pixels, being higher for the higher-order and lower-degree modes. To this is added an error that is linear in frequency and is taken from our estimates of residual scale uncertainties (Hill, Toomre and November 1983). When these errors are translated to velocities, they range from 5 ms\(^{-1}\) to nearly 600 ms\(^{-1}\). The noise in the inferred velocity is found by combining the raw errors using the same coefficients \(a_i\) that give the velocity curve. For our standard case, with \(\theta = 0.5\), the resulting error in the velocity ranges from 7 to 34 ms\(^{-1}\). The error is higher at greater depths, which are sampled by the higher-order and lower-degree modes. Table 1 summarizes the range of errors in the inferred velocity for the various values of \(\theta\).

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>Error (ms(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5 - 600</td>
</tr>
<tr>
<td>0.001</td>
<td>76 - 166</td>
</tr>
<tr>
<td>0.01</td>
<td>40 - 81</td>
</tr>
<tr>
<td>0.1</td>
<td>17 - 47</td>
</tr>
<tr>
<td>0.5</td>
<td>7 - 34</td>
</tr>
<tr>
<td>0.7</td>
<td>6 - 30</td>
</tr>
<tr>
<td>1.0</td>
<td>5 - 26</td>
</tr>
</tbody>
</table>

The choice of modes used in the inversion can also affect the resolution and accuracy of the inferred velocity. The modes, of course, must sample all the depths of interest, and increasing their number improves the resolution and decreases the final error. Figure 9 shows the effects of changing the size of the mode set. The full set is the one we have used throughout this study, and consists of 196 modes: they are approximately evenly spaced in \(f\) on the \(f\) and the \(p_1\) through \(p_8\) ridges, except that they avoid the frequency band of the chromospheric modes. The half mode set consists of 98 modes and was created by eliminating every other mode in the list of the full mode set. The quarter mode set totals 51 modes and was constructed from the half mode set in a similar manner. Figure 9 shows inversions for the three sets, and illustrates the degradation of resolution that results from the use of a lower number of modes. The gross features of the curve remain in all three cases. Again we see changes in the depths at which the optimal kernels are centered as fewer modes are used. This change is similar to that observed in Figure 8b for larger values of \(\theta\). Of course, one would like to include all possible observed modes, but constraints of computer resources prevent it.

That the cancellation properties of the inversion procedure are effective in dealing with noisy data can also be demonstrated by examining the velocity curves resulting from applying the inversion to the final raw medians rather than the fitted data. This is shown in Figure 10, which compares our standard inversion obtained from the ridge fits
Figure 8. The effect of varying the parameter \( \theta \) in the inversion procedure. This parameter controls the amount of weight placed on the errors in the data: \( \theta = 0 \) implies no weight on the errors and \( \theta = \pi/2 \) implies no weight on the data. Panel (a) shows the curves for \( \theta = 0.001, 0.01, 0.1 \) and 0.5. A wide variation results when \( \theta \) is small. Panel (b) shows the curves for \( \theta = 0.5, 0.7 \) and 1.0. We have used \( \theta = 0.5 \) elsewhere in our analysis.
Figure 9. The effect of restricting the number of modes used in the inversion. Curve F was determined using the full 196 mode set. Curve H was computed using only 98 modes, selected by rejecting every other mode from the full set. Curve Q was calculated using a set chosen by rejecting every other mode from H.

Figure 10. Illustration of the ability of the inversion procedure to cope with noisy data. Curve A was computed using the final fitted medians from the iterative orthogonal scheme, and curve B was calculated using the unsmoothed medians.
with that obtained from the unsmoothed medians. The curves are quite
similar in shape, with that deduced from the unsmoothed medians being
some 10 to 20 ms⁻¹ higher in magnitude. The inversion is thus quite
effective in handling noisy data. We must point out, however, that
these final unsmoothed medians do not have as noisy a character as the
medians resulting from simply cutting the ridge in the vertical
direction.

5. CONCLUSIONS

We have shown that our iterative ridge-fitting procedure is stable
for certain choices of parameters. Moreover, the inversion procedure is
quite effective in reducing errors, even when dealing with the noisy
data considered here.

Of course, a reduction in the noise in the observations would be of
great benefit in increasing the accuracy of the results. The increased
stabilization of future instruments for measuring Doppler shifts will be
of considerable advantage. Another obvious gain would come if one were
to observe from space, where the data would be free from atmospheric
seeing. A certain degree of improvement could also be obtained from the
longer intervals of continuous observation that would be obtained,
either from a spacecraft in a continuously sunlit orbit or from a
ground-based network, but that improvement might only be moderate
because the observing interval would still be limited by the evolution
timescale of the giant cells.

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