HELIUM SPECTRA IN LARGE MAGNETIC FIELDS
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The discovery of circular polarisation in the continuum of the DC white dwarf Grw +70°8247 has been reported by Kemp et al. 7. This polarisation is taken as indicating the presence of a magnetic field of the order of $10^7$ gauss based on the greybody magnetoemission theory of Kemp 7. It is of interest to ask whether or not, and under what conditions a line spectrum would be observable in a star with field of this magnitude and what such a spectrum would look like. The purpose of the present paper is to examine the effects of a field of up to $10^7$ gauss on the spectrum of helium.

The contribution to the Hamiltonian of an electron due to the presence of a macroscopic magnetic field is

$$H_{\mu} = (e/\hbar)A \cdot B + (e^2/2mc^2)A^2.$$

The first term represents the familiar linear Zeeman effect. In terms of the operators $\mathcal{L}$ and $\mathcal{S}$ this part of $H_{\mu}$ becomes

$$H^I = (e/\hbar) \mathcal{L} \cdot (\mathcal{L} \cdot \mathcal{S}).$$

The second term in $H_{\mu}$ gives rise to the quadratic Zeeman effect. For a uniform magnetic field $\mathcal{B} = 1/2(\hat{\mathcal{J}} \times \mathcal{B})$. Hence this term in the Hamiltonian becomes

$$H^{II} = (e^2/8mc^2) \mathcal{B}^2 \cdot \mathcal{J}^2 \sin^2 \theta,$$

where $\theta$ is the angle between $\mathcal{J}$ and the field vector taken to define the $z$ axis. In terms of the spherical harmonic tensor operators $c^r_\ell(k)$ defined by Racah 5 as $(4\pi/2\pi-1)^{\frac{3}{2}}Y_{\ell \ell}$, we can write $H^{II} = (e/12mc^2) \mathcal{B}^2 \cdot \left( C_0^0(\theta)C_0^0(\theta) \right)$. 
Due to the magnitude of the field under consideration, the $H^1$ term gives rise to a complete Paschen-Back effect. We therefore choose our basis functions to be the set $|LM \alpha \beta \gamma \rangle$. In this basis, the matrix elements of $H^1$ are proportional to $M_L + 2M_\alpha$. Note that $H^1$ is also diagonal in $M_L, M_\alpha$. The $H^1$ matrix elements in this basis are then proportional to

$$\langle LM' \mid c(k) \mid LM \rangle$$

where $k = 0, 2$.

The matrix element may now be reduced by applying the Wigner-Eckart Theorem in the form given by Edmonds, which yields

$$\langle LM' \mid c(k) \mid LM \rangle = (-1)^{L+M_L-K+M} \begin{pmatrix} L & K & L' \\ M_L & M & M_L' \end{pmatrix} \langle \alpha L \mid c(k) \mid \alpha' L' \rangle. \quad (1)$$

Now noting that the states of interest in $\text{He I}$ are of the type $2^3S$ or $2^1S$, we can set $L = \lambda$ and $M_L = m$. Since $c(k)$ operates only on the coordinates of the outer electron, we have

$$(0 \alpha L \mid c(k) \mid 0 \alpha' L') = (-1)^{L+M_L+K'} \left[ (2l+1)(2l'+1) \right]^{1/2} \begin{pmatrix} l & 0 & l' \\ \lambda & \alpha' & \lambda' \end{pmatrix} \langle \alpha L \mid c(k) \mid \alpha' L' \rangle \quad (2)$$

for $n$, $n'$ + 1. In the event that the state under consideration is the ground state, we need an additional factor of $2$ in equation (2). Putting (2) into (1) we obtain

$$\langle LM' \mid c(k) \mid LM \rangle = (-1)^{L+M_L+K'+M} \left[ (2l+1)(2l'+1) \right]^{1/2} \begin{pmatrix} l & 0 & l' \\ \lambda & \alpha' & \lambda' \end{pmatrix} \times \begin{pmatrix} l & K & l' \\ \lambda & m & \lambda' \end{pmatrix} \langle \alpha L \mid c(k) \mid \alpha' L' \rangle. \quad (3)$$

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It is now possible to evaluate the $3j$ and $6j$ coefficients for $k = 0, 2$. The reduced matrix elements are given by Racah. The selection rules are that the angular matrix elements vanish unless $l' = l, l \pm 2$. Results for these matrix elements are listed in Table 1.

**Table I**

Angular matrix elements of the Quadratic Zeeman effect for $3nl$ states

<table>
<thead>
<tr>
<th>$l'$</th>
<th>$\langle LM' \mid c(k) \mid LM \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>$\frac{3m^2 - l(l+1)}{(2l+1)(2l+3)(2l'2l'+1)}$</td>
</tr>
<tr>
<td>$l+2$</td>
<td>$\frac{(2l+3)(2l+1)(2l+3)(2l'2l'+1)}{(2l+5)(2l+3)(2l'2l'+1)}$</td>
</tr>
</tbody>
</table>

$m = n, n' + 1$. In this case, $l < l'$.

The radial part of the matrix elements, $\langle n \mid c^2 \mid n' \rangle$, can be calculated in the Coulomb approximation. This method may be expected to give accurate values for the matrix elements since the $c^2$ factor will more heavily weight the outer regions where the Coulomb wave function best represents the true wave function.
These calculations are carried out using the generalized Bates-Dempster approach of Oertel and Shomo. In this treatment the matrix element for He II is given by

$$I_2 = \int \frac{e^2 F(n^*)^2 E (n^*')^2 d\alpha}{(n^*)^2 (n^*')^2 \left(n^* + 1\right) \left(n^* + 2\right) \left(n^* + 3\right) \left(n^* + 1\right)}$$

$$= \sum_{k=0}^{\infty} \frac{\left(n^n + 1\right) \left(n^* + 1\right) \left(n^* + 2\right) \left(n^* + 3\right) \left(n^* + 4\right) \left(n^* + 5\right) \left(n^* + 6\right) \left(n^* + 7\right) \left(n^* + 8\right)}{\left(n^* + 1\right) \left(n^* + 2\right) \left(n^* + 3\right) \left(n^* + 4\right) \left(n^* + 5\right) \left(n^* + 6\right) \left(n^* + 7\right) \left(n^* + 8\right)}$$

where $n^*$ and $n^{**}$ are the effective principal quantum numbers for the states. The recurrence relation for the $a_k$ is given by

$$a_k = (n^n a_{k-1} - 2k) [E_{k+1} - (n^n - k)]$$

with $a_0 = 1$.

The cutoff condition for the sum of $k'k < n^n\cdot n^{**}$.

We now wish to find the eigenvalues of the matrix

$$E_{1,2,3} = \frac{a_0}{(n^n)^3}$$

where the $E_k$ are the unperturbed energies of the He II atom. Since $H^I$ has matrix elements mixing states of all values of $n$, it is necessary to assume a cutoff value of $n$ for the matrix. This value is determined by requiring a certain accuracy for the highest state of interest at the highest field under consideration. This will be an upper limit for the uncertainty of any eigenvalue due to ignoring states of large $n$. In the actual calculation, an accuracy of 5 cm$^{-1}$ was required for the 7p eigenvalue at a field of 10$^7$ gauss. This involves the inclusion of states up to $n = 10$ in the matrix. For each value of $I$ of interest,

states of angular momentum up to $I=2$ were included.

The magnetic Hamiltonian, $H^M$, has no matrix elements between states of opposite spin or parity nor any between states of different $M_z$ and $M_{-z}$. It is then possible to divide the Hamiltonian matrix into submatrices characterized by the different values of these quantum numbers and diagonalize each separately.

The energy eigenvalues were calculated in this way for He I for $n = 1$ to $n = 7$ and for fields from 10$^5$ to 10$^7$ gauss. Typical results are shown from the odd parity singlets and odd parity triplets in Figures 1 and 2 respectively.

Note that in the triplet states there is some degeneracy of components having different values of $M_z$ and $M_{-z}$. This comes about since the $H^I$ matrix elements depend only upon the $M_z$ and those of $H^M$ depend only upon the absolute value of $M_z$. Thus components with these two quantities equal will not be split.

In the lower states the splitting is nearly symmetric as would be expected from the linear Zeeman effect. In the case higher $n$, the quadratic effect becomes more important. It can be shown that for a given value of the field the leading term in the quadratic Zeeman effect goes as $n^n$.

The matrix elements of $H^I$ depend only upon $|M_z|$, as opposed to those of $H^M$ which depend upon the sign of $M_z$. As a result, the points at which the energies of the components with $M_z$ negative reach their minimum and begin to increase indicate approximately the values of the field at which the quadratic effect becomes dominant for each component.
Figure 2:
Energy of $\gamma$P as a function of magnetic field.

The resultant splitting of the spectral lines can be calculated from these eigenvalues. Figure 3 shows the splitting of $\Delta \Omega$ which is the $3\gamma P - 2\gamma P$ transition. The dotted line indicates the splitting caused by the linear Zeeman effect. Note that the effect of the quadratic term is to produce a blue shift of each of the components. Since the transition involves relatively low-lying states, this effect is small compared to the splitting.

Since we have mixing between states with $l$ and $l+2$, one would expect to see the appearance of transitions that are otherwise forbidden. This effect can be calculated from the eigenvectors of our matrix. Figure 4 for example, shows the strength of the forbidden $2s$-nf transitions compared to $2s$-np for $n = 4, 5$. Again we see the enhancement of the effect for larger $n$. In fact, for $n = 5$ and a field of $10^7$ gauss, the two transitions are of almost equal strength.

The components with $\Delta M = \pm 1$ are seen to be much more weakly mixed than those with $\Delta M = 0$. This effect can be understood in terms of the expressions given in Table 1 for the angular matrix elements. For $M_l = 1$ the off-diagonal terms between $p$ and $f$ states are smaller than for $M_l = 0$, while the differences between the corresponding diagonal terms are larger in the first case. Since to first order, the mixing in the wave functions goes as ratio of the off-diagonal term to the difference of the diagonal terms, the $M_l$ dependence of the matrix elements can explain this non-uniform mixing.

It is possible to summarise the effects of a large magnetic field on the He I spectrum as follows. The spectrum
Figure 3: Splitting of $\lambda 5016$ due to large magnetic field 
-- linear Zeeman effect; (-----) linear plus quadratic Zeeman effect.

Figure 4: Comparison of strengths of permitted and forbidden lines due to quadratic Zeeman mixing of singlet F and P states. Only the $|M| = 0$ component is shown for the $4p-2s$ transition ($N = \pm k$ for singlets).

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will consist of a blue shift superimposed on the usual Zeeman splitting. As you go up a given series, the blueshift should increase as approximately $B^4$. At high fields it should also be possible to observe lines which are ordinarily forbidden due to inter-i mixing. Once again these lines should increase in strength as you go up a series. The $\sigma$ components of the forbidden lines should remain weaker than the $\pi$ component, while just the opposite will be true in the case of the allowed lines.

At the present time it is not clear under what conditions these effects would be visible in white dwarf spectra. Future calculations will be directed toward answering this and also toward extending the present calculations to other atoms, including hydrogen.

This work was supported in part by National Science Foundation Grant No. GP-20696.

REFERENCES