DRIVEN ACOUSTIC OSCILLATIONS WITHIN A VERTICAL MAGNETIC FIELD

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ABSTRACT

We examine the effects of a vertical magnetic field on p-mode frequencies, line widths, and eigenfunctions. We employ a simple solar model consisting of a neutrally stable polytropic interior matched to an isothermal chromosphere. The p-modes are produced by a spatially distributed driver. The atmosphere is threaded by a constant vertical magnetic field.

The frequency shifts due to the vertical magnetic field are much smaller than the shifts caused by horizontal fields of similar strength. A large vertical field of 2000 G produces shifts on the order of a µHz while a weak field of 50 G produces very small shifts of several nHz. We find that the frequency shifts decrease with increasing frequency and increase with field strength.

Coupling of the acoustic fast mode to escaping slow modes is extremely inefficient. Constant vertical magnetic field models are therefore incapable of explaining the high level of absorption observed in sunspots and plage.

Keywords: magnetic fields, p-modes, solar cycle

1. INTRODUCTION

Magnetic fields can affect the propagation of solar acoustic waves in dramatic and detectable ways. For example the observed variation of p-mode frequencies over the solar cycle (Woodard & Libbrecht 1991) are probably caused directly or indirectly by changes in the solar magnetic field. Absorption and scattering of incoming acoustic waves by the magnetic fields in sunspots and plage are established observationally (Braun et al. 1988; Braun et al. 1992; Bogdan et al. 1993). Furthermore, there is an obvious reduction of p-mode surface amplitude in regions of magnetic activity (Lites, White, & Packman 1982; Fernandes et al. 1992; Title et al. 1992). In all of these cases, theorists have been unable to provide firm explanations for the characteristic changes in the acoustic wave field. A substantial reason these explanations have not been found may be due to the lack of knowledge about wave propagation in regions near the surface with predominantly vertical magnetic fields.

MHD problems in gravitationally stratified media have proven difficult to solve except for a limited set of models with special magnetic field configurations. Horizontal magnetic fields without shear have been studied extensively (Yu 1965; Campbell & Roberts 1989; Jan & Roberts 1993), yet seemingly simple field geometries such as a constant vertical magnetic field have defied analysis. The difficulty has arisen because the fast and slow magnetosonic wave modes are inherently coupled.

Goldreich et al. (1991) have stated that vertical magnetic fields cannot change the frequencies significantly because the p-modes are primarily vertical at the surface. This statement is based on the behavior of the acoustic fast mode. However, even for models where the fast and slow modes are locally independent of each other (\(V_A \ll c_s\) or \(V_A \gg c_s\)), both the fast and slow modes are required. The reason is that when a magnetic field is present, additional boundary conditions beyond those required for the nonmagnetic problem must be satisfied. These conditions ensure that the magnetic field and the associated currents are well behaved. In general, both fast and slow modes are needed to meet these additional constraints.

Recent mode calculations in atmospheres with vertical magnetic fields (Spruit & Bogdan 1992; Cally & Bogdan 1993; Cally, Bogdan & Zweibel 1994) have attempted to explain the observed 50% absorption of incoming acoustic wave energy by sunspots. Thus, they have concentrated on field strengths in the kilogauss range. We will examine this problem as well, but we are also interested in understanding how acoustic waves interact with weaker fields. Besides frequency shifts we are also interested in determining the effects of the magnetic field on mode line widths and eigenfunctions. Is the surface velocity reduced for trapped p-modes and increased for high frequency oscillations as is observed in active regions by Brown et al. (1992)?

Additionally, we will examine questions about the driving of p-modes when a magnetic field is present. First of all, we hope to understand whether the characteristics of the solar acoustic source can be determined from observable wave quantities. Kumar & Lu (1991), Kumar (1994), and Kumar et al. (1994) have shown that the high frequency acoustic peaks hold detailed information about the location and spatial extent of the acoustic source. No attention has until now been paid to the effects of a magnetic field. By including a driver in our model we can also study the response of the atmosphere to frequencies not corresponding to a resonant p-mode. The background power may contain more information about the magnetic field than the mode ridges.

The discussion given here is a condensation of a more detailed paper which will be submitted to the Astrophysical
Journal. We start this paper by setting up the problem in §2, where we describe the physics included in our wave equation, the solar model employed, and the boundary conditions imposed on the solutions. The numerical method is described in §3. In §4 we present results and §5 provides a summary and conclusions.

2. SETUP OF PROBLEM

2.1 The Equations

In a plane parallel atmosphere with constant vertical gravity and constant vertical magnetic field, the two ideal MHD magnetosonic modes satisfy a set of coupled, second order differential equations (Ferraro & Plumpton 1958; Scheuer & Thomas 1981):

\[ V_A^2 \frac{d^2}{dz^2} + \omega^2 - k^2 (V_A^2 + c_s^2) \xi_z = 0 \] (2.1)

\[ \left( c_s^2 \frac{d^2}{dz^2} - \gamma g \frac{d}{dz} + \omega^2 \right) \xi_z + \left( c_s^2 \frac{d}{dz} - g (\gamma - 1) \right) \xi_x = -g f(z) \] (2.2)

where \( \omega \) is the wave frequency, \( k \) is the horizontal wave number, \( c_s \) is the adiabatic sound speed, \( V_A \) is the Alfvén speed, \( g \) is the gravity, \( \gamma \) is the adiabatic index, and \( \xi_x \) and \( \xi_z \) are the horizontal and vertical components of the fluid displacement. We have placed a driver in the vertical momentum equation. This driver appears as the inhomogeneous term on the right hand side of equation (2.2), where \( f \) is a dimensionless driving profile. For simplicity we have specialized to a driver of the form

\[ f(z) = \exp \left[ -\left( \frac{z - z_d}{\delta} \right)^2 \right] \] (2.3)

where \( \delta \) is the driving width and \( z_d \) is the central driving height.

2.2 The Atmosphere

We employ a simple model designed to simulate several regions of the solar envelope. The lower portion of the atmosphere is a neutrally stratified polytrope, corresponding to the solar convection zone. We truncate the polytrope at a nonzero density and overlay a hot isothermal chromosphere of temperature 6000 K.

Within such an atmosphere the sound and Alfvén speeds are given by

\[ c_s^2 = \begin{cases} -\frac{g z}{m} & \text{for } z < -z_0 \\ \gamma g H_\rho & \text{for } z > -z_0 \end{cases} \]

where

\[ V_A^2 = \begin{cases} \frac{B_0^2}{4\pi \rho_\infty} \left( \frac{-z_0}{z} \right)^m & \text{for } z < -z_0 \\ \frac{B_0^2}{4\pi \rho_\infty} \left( \frac{m + 1}{m} \right) H_\rho e^{-\frac{z_0}{r}} & \text{for } z > -z_0 \end{cases} \]

The height coordinate, \( z \), increases upwards and the polytrope and isothermal atmospheres join at a height of \(-z_0\). The quantity \( z_0 \) is called the truncation depth. The quantity \( \rho_\infty \) is the mass density at the top of the polytrope, \( m \) is the polytropic index, and \( H_\rho \) is the isothermal atmosphere's density scale height. In order for the polytrope to be neutrally stratified the adiabatic exponent must be related to the polytropic index by \( \gamma = 1 + 1/m \). For a \( \gamma = 5/3 \) gas this requires \( m = 3/2 \).

2.3 The Boundary Conditions

We require that the solution has vanishing energy density or represents an outward propagating wave as \( z \to \pm \infty \). Of the four possible solutions of equations (2.1) and (2.2) this condition selects one fast mode and one slow mode. In the asymptotic regime \( z \to \pm \infty \) the well behaved fast mode and slow mode can be generated through series approximations.

Deep within the polytrope, Cally & Bogdan (1993) have found asymptotic expansions for these two solutions. The well behaved fast mode is refracted by the high internal sound speed and is evanescent. Since the Alfvén speed decreases with depth there is no lower turning point for the slow mode. The slow mode is therefore a propagating wave and we select the wave with downward energy flux.

High in the isothermal atmosphere the solutions can be expressed as power series in the plasma \( \beta \) (Ferraro & Plumpton 1958). We select the fast mode with finite energy density. The propagation properties of the slow modes depend on whether the wave frequency is greater than the acoustic cutoff frequency, \( \omega_{ac} \). If \( \omega > \omega_{ac} \), one slow mode solution is an upward propagating acoustic wave while the second is downward propagating. As before we select the outward radiating solution, i.e. the upward propagating wave. If \( \omega < \omega_{ac} \), one solution is unbounded as \( z \to \infty \) and is thrown away.

Note that in an unmagnetized isothermal atmosphere waves propagate if

\[ \omega^2 > \frac{\omega_{ac}^2 + k^2 c_s^2}{2} + \sqrt{\left( \frac{\omega_{ac}^2 + k^2 c_s^2}{2} \right)} - k^2 c_s^2 N^2 \]

or

\[ \omega^2 < \frac{\omega_{ac}^2 + k^2 c_s^2}{2} - \sqrt{\left( \frac{\omega_{ac}^2 + k^2 c_s^2}{2} \right)} - k^2 c_s^2 N^2 \]

where \( N \) is the buoyancy frequency. With the magnetic field there is no possibility of propagating gravity waves and the lower frequency bound on propagating acoustic waves no longer has \( k \) dependence. This causes waves which would have been trapped by the high sound speed in the outer atmosphere to propagate instead. This hap-
pens because the acoustic waves become ducted along the field lines and there is no horizontal compression of the gas. Therefore the vertical wave number is independent of \(k\).

3. NUMERICAL PROCEDURE

We find the connection coefficients between the asymptotic solutions in the polytrope and the power series solutions in the isothermal chromosphere numerically through a shooting technique. Within each region three different solutions must be computed: the particular solution and a fast and a slow homogeneous solution. The homogeneous solutions are generated by numerically integrating equations (2.1) and (2.2) with \(f = 0\) from starting depths, for which the expansions are valid, to the truncation depth. The particular solution is calculated in exactly the same manner as the homogeneous solutions except that the driving is kept on. For simplicity we have used the fast mode expansion for both the polytropic and isothermal particular solutions.

Physical matching conditions must be met at the truncation depth by the solutions in the polytrope and isothermal atmosphere. The Lagrangian pressure perturbation, the vertical displacement, and both components of the magnetic field perturbation must be continuous across the surface. These four requirements determine the four amplitudes of the homogeneous solutions (two in the polytrope and two in the isothermal atmosphere). Thus the wave function can be determined throughout the atmosphere. This procedure can be performed for any desired driving frequency. Therefore, we can generate power spectra at any height in the atmosphere. We have chosen to calculate velocity spectra at the top of the polytrope.

4. RESULTS

The model atmosphere has six free parameters: \(g\), \(m\), \(H_p\), \(B_0\), \(\rho_c\), and \(z_0\). The driver is characterized by two parameters, \(z_d\) and \(\delta\), and the harmonic degree, \(\ell = k R_0\), can also be varied. We are primarily interested in the effects of the magnetic field. Therefore, we have kept fixed all the atmospheric parameters except the field strength, \(B_0\). These fixed parameters were determined such that the adiabatic index is \(5/3\), temperature minimum occurs at 4200 K, the chromospheric temperature is 6000 K, gravity is \(2.74 \times 10^5\) cm s\(^{-2}\), and the density at the photosphere (i.e., where \(T = 6000\) K) is \(4.8 \times 10^{-7}\) g cm\(^{-3}\). The harmonic degree is fixed at \(\ell = 200\).

For these models the field was varied from as low as 50 G to as high as 3000 G. The height of the surface where \(V_A = c_s\) is 1027 km above the photosphere for the 50 G model and is 140 km below the photosphere for the 3000 G model. For a mode with a cyclic frequency of 3 mHz the upper turning point is 319 km below the photosphere. Therefore, for all these models the magnetic field is a perturbation because \(V_A \ll c_s\) within the acoustic cavity. For these calculations the driving depth was 498 km, corresponding to 142 km below the photosphere, and the width was 100 km.

4.1 Spectra

Figures 1a and 1b show the square of the vertical velocity at the top of the polytrope. Figure 1a corresponds to a model with no magnetic field, while Figure 1b shows the results for a model with a 2000 G magnetic field. The Lamb and acoustic cutoff frequencies of the upper isothermal atmosphere are shown as vertical lines. The \(p\)-modes are trapped only for frequencies between these two critical frequencies. At \(\ell = 200\) only 8 trapped modes exist. The acoustic cutoff frequency, \(\nu_{ac} = 4.5\) mHz, is less than the solar global cutoff, \(\nu_{ac} \sim 5.2\) mHz, because the chromosphere in our model is hot. The \(f\)-mode and modes \(p1\) through \(p6\) have shapes similar to those published recently by Roxburgh & Vorontsov (1985) who consider driven acoustic oscillations without a magnetic field. The highest frequency mode, \(p7\), is coincident with the cutoff and has a very asymmetric profile. Besides the \(f\) and \(p\)-modes there exist smaller and broader peaks for frequencies above the acoustic cutoff. These are due to the constructive interference effect noted by Kumar & Lu (1991). Similarly there are also destructive interference valleys, where the direct and reflected waves interfere, creating no response above the driver. Such destructive interference valleys occur within both the \(p\)-mode and high frequency bands. They are possible because our driver is thin compared to a \(p\)-mode wave length; therefore, the direct and reflected waves have well-defined phases.

![Figure 1. Square of the vertical velocity at the top of the polytrope as a function of cyclic frequency for (a) \(B_0 = 0\) and (b) \(B_0 = 3000\) G. The vertical lines indicate the Lamb and the acoustic cutoff frequencies of the isothermal chromosphere.](image_url)
The destructive interference valleys produce a structured background of power. This background is orders of magnitude smaller than the power in the lines. Therefore, the lines themselves can be fit extremely well with Lorentzian profiles. We have performed such a fitting, producing central frequencies, line widths, and mode amplitudes. From the central frequencies, we have calculated frequency shifts between models with different field strengths. Each of these three quantities will be discussed separately in the following sections.

4.2 Frequency Shifts

The frequency dependence of the frequency shifts produced by this model do not agree with observations. Between the years 1986 and 1988, Woodard & Libbrecht (1991) observe p-mode frequency shifts which are essentially zero at low frequency, increase to a maximum value of roughly 400 nHz at a frequency of 3.8 mHz, and then decrease rapidly. Over the period of these observations, Goldreich et al. (1991) estimate from observations that the rms field strength due to intense fibrils changed from 120 G to 100 G. Figure 2 shows the change in frequency between a model with a field strength of 200 G and one with a 100 G field. The frequency shift is largest at low frequencies and has a maximum shift of only 41.6 nHz. Jain & Roberts (1993) find that increases in field strength for horizontal chromospheric fields produce shifts which increase with frequency. In order to explain the decreasing frequency shift for frequencies above 3.8 mHz, their models rely on an increase of the chromospheric temperature. This study shows that the frequency dependence of magnetic frequency shifts are strongly controlled by the magnetic geometry.

![Figure 2](image)

Figure 2. Frequency shift as a function of frequency between a 200 G model and a 100 G model. The vertical line is the acoustic cutoff frequency of the isothermal chromosphere.

4.3 Line Widths

Figure 3 shows the line width as a function of mode frequency for several field strengths. The line widths are extremely small. At 100 G the f-mode has a line width of 3.9 nHz while at 2000 G it has a width of 1.3 µHz. Only the line widths for the largest field strengths approach the values of observed p-mode line widths; Duvall et al. (1991) find mode line widths that range from 1 µHz at frequency of 2 mHz to 100 µHz at a mode frequency of 5 mHz. As expected the conversion of p-modes into downward propagating slow modes is unimportant in terms of the global damping. Furthermore, these line widths are also too small to explain absorption of p-modes by active regions and sunspots. Even models with a field of a few kilogauss have line widths of only a few nHz. The attenuation length of the wave should be approximately the phase speed multiplied by the mode lifetime, ωτ/k. For a mode with a cyclic frequency of 3 mHz and a harmonic degree of 200 the attenuation length is on the order of 10 Ṙ. No significant absorption can take place over an active region scale, much less over a sunspot. If fast mode conversion to slow modes is responsible for the observed absorption of p-modes by sunspots, the horizontal variation of the field must be crucial.

![Figure 3](image)

Figure 3. Line widths as a function of frequency for models with different field strengths.

4.4 Mode Amplitudes

A direct examination of the spectra reveals a drastic reduction of mode power with field strength. Between B0 = 50 G and B0 = 2000 G the power in the f-mode changes by more than a factor of 10^5. This reduction is caused by our assumption that the only damping mechanism is the radiation of slow magnetoacoustic waves.

Without the magnetic field, resonant oscillations have a formally infinite amplitude because there is no energy dissipation mechanism to eliminate the energy input by the driver. As the field strength increases the conversion of fast modes to radiating slow modes becomes more efficient. Therefore, if the rate of energy input were to remain constant a steady state would be attained with a lower mode amplitude. The energy input rate is a function of mode amplitude, however:

\[
\frac{dE}{dt} = - \int_{-\infty}^{\infty} g p_0 u_f dz. \tag{4.1}
\]

Since the mode amplitude decreases as the field strength increases, the energy input also decreases. This tends to decrease the mode amplitude more. The mode amplitudes are therefore sensitive functions of the field strength.
Mode amplitudes in the Sun are determined by other processes, however. The observed line widths are much larger than the line widths produced by the conversion of fast modes into radiating slow modes. Therefore, the dissipation of wave energy is dominated by damping mechanisms not provided for in this calculation. The dissipation of solar acoustic waves is only weakly affected by the magnetic field and the mode amplitudes will therefore undergo only modest changes in the presence of the magnetic field.

4.5 Eigenfunctions

Figures 4a and 4b display the horizontal and vertical velocity components of the $f$-mode as a function of height. The solid curve is for a magnetized model with a field strength of 3000 G and the dashed curve is for a nonmagnetized model. The vertical lines are located at two special locations. The deepest corresponds to the $V_A = c_s$ surface while the highest is the truncation height. To display both modes on the same scale the nonmagnetized $f$-mode has been normalized to have the same vertical velocity as the magnetic solution at the top of the polytrope. The vertical velocity is dominated by the fast mode even for this extremely large value of the field strength. The slow mode appears as a short wave length perturbation to the fast mode oscillation. The fast mode is essentially stationary, however, while the slow mode is propagating. The horizontal velocity has roughly equal contributions from the fast mode and slow mode.

For smaller field strengths the height where $V_A = c_s$ can be quite high in the atmosphere. For a field strength of 50 G this height occurs 1027 km above the photosphere. The slow mode is not reflected at the top of the acoustic cavity. It undergoes a reflection at the height where $V_A = c_s$. For low field strengths this allows the slow mode to propagate fairly high into the chromosphere.

One observable consequence of the slow modes would be an enhancement of power at higher altitudes. Furthermore the motions of the slow mode are not primarily vertical as the fast mode is. This enhancement of power above the upper turning point should be observable in the horizontal velocity as well.

Assuming all the fast mode energy which is converted into slow modes propagates upwards allows an estimate to be made of the energy flux available for atmospheric heating. The slow mode energy flux is

$$Q = \frac{\gamma EN}{4\pi R_o^2}$$ \hspace{1cm} (4.2)

where $\gamma$ is a typical line width, $E$ is the energy per mode, and $N$ is the number of excited $p$-modes. Using a line width of 5 nHz, representative of the 100 G model, an energy per mode of $10^{28}$ ergs (Libbrecht & Woodard 1991), and $10^7$ modes, the heating rate is on the order of $10^5$ ergs s$^{-1}$ cm$^{-2}$. This is about 3 orders of magnitude smaller than the flux necessary to maintain chromospheric temperatures above active regions.

5. CONCLUSIONS

We have solved the linearized MHD equations in an atmosphere consisting of a truncated neutrally stratified polytrope overlaid by a hot isothermal chromosphere and threaded by a constant vertical magnetic field. Our goal was to illustrate the general effects of the coupling between the acoustic waves and magnetic slow modes. Therefore, we have made many simplifying assumptions. We have ignored sphericity, perturbations to the gravitational potential, nonadiabatic effects, and variations in the adiabatic exponent. More importantly, we have restricted our model to a very simple magnetic field configuration. Solar active regions and sunspots have a very complicated geometry, however. It is unknown which aspects of the active region magnetic field are important for the observed changes in acoustic oscillations. Thus, it is important to examine simpler field geometries in order to determine which structures are necessary to explain the observations.

With these limitations in mind, we can draw several conclusions about the effects of vertical magnetic field on solar acoustic oscillations.

5.1 Frequency Shifts

The frequency shifts caused by the magnetic field for field strengths of several hundred Gauss are extremely small (several nHz). The shift is largest at low frequency and decreases with increasing frequency. The vertical field model is not successful in explaining the observations of
p-mode frequency changes over the solar cycle. The size of the shift is too small by an order of magnitude and the frequency dependence is not the same. The conjecture of Goldreich et al. (1991), that vertical magnetic fields are negligible for frequency shifts because the motions of the p-modes are primarily along the field at depths where the field is important, appears to be substantiated by our results. The frequency shift caused by coupling to slow modes is much too small to explain observed frequency shifts.

5.2 Line Widths

The conversion of acoustic energy into downward propagating slow modes appears to be an inefficient process. The line widths caused by the escape of slow mode waves are several orders of magnitude smaller than observed, except for models with kiloGauss magnetic fields. Our model cannot explain the 50% absorption of acoustic energy observed for sunspots or the slightly weaker absorption for plage. It is possible that horizontal variation could be important in this process, allowing Alfvén resonances to exist (Hollweg 1988; Sakurai et al. 1991).

5.3 Eigenfunctions

A vertical magnetic field changes p-mode surface velocities only slightly except inside sunspot umbra where the field strength is several kilogauss. The presence of the slow modes does not greatly affect the vertical velocity of a p-mode. The fast mode is mostly vertical at the surface and the slow mode is primarily transverse. Therefore, the slow mode only appears as a small perturbation even for the largest field strengths. For identical reasons the horizontal velocity is dominated by the slow mode. However, the vertical velocity remains much larger than the horizontal velocity for observations near the top of the acoustic cavity. The slow modes can extend much higher in the atmosphere than the fast mode because the slow mode is reflected at the \( V_A = c_s \) height. Above the photosphere the ratio of the vertical to horizontal velocity should decrease with height. Observations made away from disk center toward the limb might be able to detect this enhanced horizontal component.

5.4 High Frequency Oscillations and Upper Atmosphere Reflectivity

The presence of the vertical magnetic field has very little effect on the high frequency spectrum. Any changes that do occur cannot be distinguished from a slightly different driving profile. The reflection properties of the upper atmosphere are drastically altered by the vertical magnetic field. For \( V_A > c_s \) the lower frequency bound for propagating acoustic waves no longer has \( k \) dependence. This means that the high temperature of the corona cannot reflect high frequency waves. The high frequency acoustic oscillation model of Balmforth & Gough (1990) does not work in the presence of a vertical magnetic field. Furthermore, the gravity wave propagation band does not exist when \( V_A > c_s \). All gravity waves are reflected near the \( V_A = c_s \) surface for all frequencies.

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