15. — STELLAR OSCILLATIONS, STELLAR STABILITY
AND APPLICATION TO VARIABLE STARS *

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ABSTRACT

A general review of current theories of stellar oscillations is presented, including linear, nonlinear, adiabatic, non-adiabatic, radial, and nonradial oscillations. Applications to various known types of intrinsic variable stars are discussed, and some outstanding problems in our current understanding of these stars are summarized.

I. INTRODUCTION

The purpose of the present paper is to provide a general overview of the current status of stellar pulsation theory, its recent successes and failures as applied to actual intrinsic variable stars (i.e., stars whose variations are the result of factors within, or intrinsic to, the star), and some outstanding problems in our present understanding of variable stars. As a detailed review of this subject has recently been published [1], we shall concentrate on recent developments covering, primarily, the past decade or so. Most of this work has been based on conventional, non-relativistic mechanics and Newtonian gravitation theory; so this general basis will be assumed unless specifically stated otherwise. However, considerable work has been done recently on general relativistic pulsation theory, and some of this work will be described briefly. We shall not discuss the subject of secular stability in this paper.

II. THEORETICAL DEVELOPMENTS

In this section we shall summarize what appear to this reviewer to be the most significant recent advances in stellar pulsation theory. In subsection A we consider the oscillations only of spherical, non-rotating stars having no magnetic fields, turbulence, etc., and in subsection B we consider effects on the oscillations of various perturbations, such as rotation, magnetic fields, departures from thermal equilibrium, etc., which may be present in the nonpulsating star. In all cases we ignore molecular and radiative viscosity, whose effects are generally negligible [2], [3].

* The cut-off date for references for this paper is early June, 1974.
The present category includes the bulk of the theoretical work carried out to date on stellar oscillations. We first discuss theories based on the linearized pulsation equations, and shall then consider the non-linear calculations, in both cases restricting attention only to work falling in the present category.

(1) Linear, Adiabatic Radial Oscillations:

It is now somewhat more than half a century since Eddington [4], [5] first gave the theory of small, radial, adiabatic stellar pulsations a firm mathematical foundation. Despite considerable refinements and extensions (see below), this theory still gives a good overall description of many features of pulsating stars. In particular, it yields fairly accurate values of the pulsation periods, through the famous period-mean density relation: \( \Pi \sqrt{\bar{\rho}} = \text{const.} \), where \( \Pi \) denotes the period and \( \bar{\rho} \) the mean density of the star. This relation is usually written in the form

\[
\Pi \sqrt{\bar{\rho}/\bar{\rho}_\odot} = Q,
\]

where \( \bar{\rho}_\odot \) denotes the mean density of the sun. Physically, equation (1) just says that the pulsation period of the star in one of the low modes is of the order of the sound travel time through the diameter of the star. Theory gives, for normal gas characteristics (\( \Gamma_1 = 5/3 \), see below), values of \( Q \) for the fundamental mode (denoted by subscript \( F \)),

\[
Q_F \approx (0.03 - 0.12) \text{ days},
\]

with a typical value for reasonable stellar models of \( Q_F \approx 0.04 \) days. The value of \( Q \) is also found not to depend very sensitively on the details of the stellar model. Its value is, however, fairly sensitive to an appropriately defined average value of the adiabatic exponent \( \Gamma_1 \equiv (d\ln P/d\ln \rho)_{\text{ad}} \) (where \( P \) and \( \rho \) denote, respectively, total pressure and mass density), and approaches infinity, signifying the onset of dynamical instability, as \( \Gamma_1 \) approaches 4/3.

Eddington's theory is based on the assumption that the (Lagrangian) pressure and density variations, \( \delta P \) and \( \delta \rho \), respectively, of a mass element are connected by the relation

\[
\frac{\delta P}{P} = \Gamma_1 \frac{\delta \rho}{\bar{\rho}},
\]

where \( \Gamma_1 \) is the adiabatic exponent defined above. (The validity of equation (3) for actual stars will be discussed below). The (linearized) momentum equation relates the radial variations \( \delta r \) of a mass level and their accelerations \( \partial^2 (\delta r)/\partial \xi^2 \equiv (\delta \rho) \) to the pressure variations and their spatial derivatives \( \partial (\delta P)/\partial m \), where \( m \) denotes the (instantaneous) mass interior to radial distance \( r \). Using equation (3) and the linearized mass equation, the fundamental linear adiabatic wave equation (LAWE) is obtained, which we write in the form

\[
\zeta_r^2 = \zeta_{\frac{4\pi r^2}{d}} \frac{d}{dm} [(3\Gamma_1 - 4)P] + \frac{1}{r} \frac{d}{dm} \left[ 16\pi^2 \Gamma_1Pr^8 \frac{\delta \rho}{\partial m} \right]
\]

\[
\equiv -r \mathbf{L}(\zeta),
\]

which defines the linear operator \( \mathbf{L} \), where \( \zeta \equiv \delta r/r \) and all other quantities not preceded by a \( \delta \) denote values in the equilibrium, non-pulsating, state of the star. Upon assuming that

\[
\zeta(m, t) = \xi(m) e^{i\omega t}
\]

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and applying the boundary conditions \( \delta \tau = 0 \) at the center and \( \delta P = 0 \) at the surface, one finds that equation (4) becomes an eigenvalue equation for the various oscillation angular frequencies \( \sigma_k^2 \) and corresponding eigenfunctions \( \xi_k \) for the various modes \( k (k = 0, 1, 2, \ldots) \). Each \( \sigma_k (= 2\pi/\Pi_k) \) obeys the period-mean density relation (1). Physically, the existence of such a relation may be seen simply by a dimensional analysis of equation (4).

It is well known that equations (4) and (5) constitute a well-defined self-adjoint system whose mathematical properties have been extensively discussed (see, e.g., [1] and [2] and references contained therein). We only note here the variational property that is obeyed by the \( \sigma_k^2 \). Define the real number

\[
\Sigma^2 = \frac{1}{J} \int_M u^* L(u) r^2 \, dm,
\]

where

\[
J \equiv \int_M |u|^2 r^2 \, dm.
\]

Here \( u(m) \) is any sufficiently smooth function which obeys the same boundary conditions as do the \( \xi_k(m) \), an asterisk denotes the complex conjugate, and the integrations are carried out over the entire stellar mass \( M \). Now, the variational property states that \( \Sigma^2 \), when regarded as a functional of \( u(m) \), becomes equal to \( \sigma_k^2 \) at the various extrema of the right-hand side of equation (6), which occur when \( u(m) = \xi_k(m) \); and that the smallest value of \( \Sigma^2 \) is equal to \( \sigma_0^2 \), the square of the fundamental eigenfrequency. The quantity \( J \), with \( u \) replaced by \( \xi \), is usually called the oscillatory moment of inertia.

The essential physical content of this variational principle, also sometimes known as Rayleigh's principle, is just that in any vibrating mechanical system the frequency takes on a value such that the average kinetic energy is equal to the average potential energy, both averages being over a time interval equal to an integral number of periods.

The above LAWE has been applied to many stellar models, and analytic solutions are known for several simplified models [6]. Efficient and powerful numerical techniques have recently been developed for its solution [7], [8].

The application of the general theory of relativity to the problem of small, radial, adiabatic oscillations of stars was first effected by Chandrasekhar [9], [10], [11], who showed that a variational principle, analogous to that discussed above, also holds in this more general case. Further discussion of these matters may be found in Thorne [12]; Bardeen, Thorne, and Meltzer [13]; and Misner, Thorne, and Wheeler [14].

(2) Linear, Nonadiabatic Radial Oscillations:

Despite the success of the theory of linear, adiabatic, radial oscillations in predicting accurate pulsation periods for stars, the physical limitations of the «adiabatic» assumption, equation (3), on which this theory is based, are evident as applied to real stars. It is known that transfers of heat, thermonuclear energy generation, etc. actually take place during the pulsations. On general grounds these non-adiabatic effects are expected to be small, however. The reason is that heat gains or losses from the star as a whole generally take place on a «thermal» time scale of the order of the Kelvin time, whereas the pulsation periods are of the order of the «free-fall» time, which is generally many orders of magnitude smaller
than the former time. We may say, crudely, that the star cannot lose (or gain) an appreciable fraction of its internal energy during a period. Nevertheless, these small heat gains and losses will in general result in a slow change in the pulsation amplitude over long periods of time, i.e., in a slow « damping » or « excitation » of the pulsations.

This slow change in the pulsation amplitude is effected by small phase shifts introduced by the above non-adiabatic effects between the pressure and density variations \( \delta P \) and \( \delta \rho \). Because of the presence of these phase shifts, the total work \( \int Pd(1/\rho) \) done by a unit mass on its surroundings around a (closed) cycle will in general be non-vanishing, and the sum over the whole star of all of these elementary work integrals may lead to slow growth or decay of the pulsations. The origin of the small phase shifts themselves arises from the fact that the rate of local heat gains or losses is generally largest in absolute value during instants of maximum compression or rarification. The effects of these heat gains or losses are then felt by the local material most strongly just before or just after maximum compression or rarification; i.e., small phase shifts are introduced. The magnitudes of the variations, are, however, hardly affected by these non-adiabatic effects; this is true in particular of the pressure variations. Consequently, the « restoring forces » on a mass element, which mainly determine the period, are similarly hardly affected. This is the main reason that non-adiabatic effects generally produce only minor changes in the periods of pulsating stars.

Apparently the first to attempt to estimate the size of these non-adiabatic effects and their relation to the growth or decay of pulsations was Eddington [15]. He assumed that the pulsation amplitude was proportional to \( e = e^{-i\omega t} \), and he computed \( \zeta \), the stability coefficient, from the formula

\[
-\zeta = \frac{1}{2} \frac{\langle dW/dt \rangle}{\langle \delta \Upsilon \rangle},
\]

(8)

where \( \langle dW/dt \rangle \) and \( \langle \delta \Upsilon \rangle \) are, respectively, the average rate (over a period) at which all mass elements in the star are doing work on their surroundings around a (closed) cycle, and the total average pulsation energy (kinetic plus potential) of the star over a period. The factor 1/2 takes into account the fact that the absolute value of the rate of change of the pulsation amplitude is equal to one-half that of the energy. He computed the numerator of equation (8) by noting that it must be a second-order effect for sinusoidal oscillations and assuming thermodynamically reversible processes, for which \( dq/dt = T \partial s/\partial t \), where \( dq/dt \) is the net rate of gain of heat per unit mass, \( T \) is the local temperature, and \( s \) is the specific entropy. Noting that \( \langle \delta \Upsilon \rangle = \frac{1}{2} \sigma^2 J \) for nearly sinusoidal oscillations, we obtain for Eddington’s expression for the stability coefficient

\[
-\zeta = \frac{1}{\sigma^2 J} \cdot \frac{1}{\Pi} \int dm \int \frac{\delta T}{T} \frac{dq}{dt} dt,
\]

(9)

where \( \Pi \) denotes the period. Writing

\[
\frac{dq}{dt} = \delta \left( \varepsilon - \frac{\partial L_r}{\partial m} \right) = T \frac{\partial (\delta s)}{\partial t},
\]

(10)

where \( \varepsilon \) denotes the rate per unit mass of thermonuclear energy generation (with any neutrino losses regarded as a negative contribution) and \( L_r \) the « interior lumino-
sity », and carrying out the time integration assuming sinusoidal motion, we may write equation (9) in the form

$$\mathcal{C}_r = \frac{C_r}{2\varepsilon^2},$$  \hspace{1cm} (11)

where

$$C \equiv \int_M (\Gamma_3 - 1) \left( \frac{\partial \varphi}{\partial \rho} \right)^* \delta \left( \sigma - \frac{\partial \ln \rho}{\partial m} \right) dm,$$  \hspace{1cm} (12)

subscript r means « real part », subscript s denotes the space part of the total expression for the variation of the indicated quantity, \( \Gamma_3 - 1 \equiv (d \ln T/d \ln \rho)_{\text{ad}} \) is the appropriate adiabatic exponent, and an asterisk denotes the complex conjugate. In deriving equations (11) and (12) we have made use of the thermodynamic identity

$$\frac{\delta T}{T} = (\Gamma_3 - 1) \frac{\delta \rho}{\rho} + \frac{\delta s}{c_v},$$  \hspace{1cm} (13)

where \( c_v \) is the specific heat per unit mass at constant volume; this identity is valid as applied to the Lagrangian variations as long as \( \Gamma_3 \) and \( c_v \) do not depend on time. More formal derivations of equations (11) and (12) have been provided by Ledoux [16] and Cox [3] or Cox and Giuli [17].

Eddington computed the stability coefficient as given essentially by equation (11) by evaluating the quantities appearing therein on the basis of solutions of the LAWE. This procedure has been referred to as the « quasi-adiabatic » approximation, and has been widely used in the study of the vibrational stability of stars. This procedure should give a good approximation to the stability coefficient as long as the outer, « non-adiabatic » region of the star does not make an appreciable contribution to the integral \( C_r \). This is the (outer) portion of the star in which, approximately, the second term on the right-hand side of equation (13) is of the same order of magnitude as the first term. The inner boundary of this portion (not a sharply defined surface) is often called the « transition region », and has been extensively discussed in the literature [3], [1], [17].

It is well known that this last eventuality is indeed realized in the classical Cepheids and in most common types of pulsating stars (see below). Therefore, to obtain a reliable estimate of the stability of these stars, it is necessary to adopt a much more accurate procedure than is provided by the above quasi-adiabatic approximation. It is also well known [18], [19] that the pulsations of these stars are largely confined to the outer regions; the relatively massive stellar core does not, for all practical purposes, participate in the pulsations. Because of this last fact, the total pulsation energy is relatively small. Consequently, the \( e \)-folding times for amplitude growth or decay of common types of pulsating stars are normally found [1] to be several orders of magnitude shorter than the corresponding Kelvin times.

The above considerations led Baker and Kippenhahn [20], [21] and Cox [22] to evaluate the integrals in equation (11) on the basis of a direct solution of the full set of linearized, non-adiabatic pulsation equations for model envelopes of pulsating stars (including Cepheids). This complete set includes, besides the momentum and mass equations referred to above, also the energy conservation and heat transfer equations. Although these techniques effected a large increase in the accuracy of stability calculations, note that these techniques did not yet yield the stability coefficient as a proper mathematical eigenvalue of the full set of linearized equations and boundary conditions.

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The solution of the linear, non-adiabatic problem as applied either to stellar envelope models or to complete stellar models, as a proper eigenvalue problem was effected by Castor [23] and Iben [22]. In this method both the oscillation frequency and the stability coefficient are obtained, for any specified mode, as the real and imaginary parts of a complex eigenvalue whose value is determined, ultimately, by the boundary conditions. The complete set of eigenfunctions of course is also obtained; when used in equation (11), it yields a value of \( \lambda \) which agrees with the value obtained as an eigenvalue, as expected. There have been numerous applications of this kind of calculation, some of which will be discussed in § III.

(3) Linear, Adiabatic Nonradial Oscillations:

In the field of linear, adiabatic, non-radial oscillations of spherical stars, the most significant recent development in basic theory, in this reviewer's opinion (aside possibly from the relativistic extensions to be discussed below), is the explicit demonstration by Chandrasekhar [24] that the eigenfrequencies for this case also obey a variational principle, analogous to that discussed earlier for purely radial oscillations. Such a conclusion is expected on physical grounds in any case, since a system of this kind is a classical, purely conservative mechanical system.

The theory has been extended to include effects of general relativity in a remarkable series of papers by Thorne and collaborators [25], [31]. It is interesting to note that such oscillations, even though strictly adiabatic in the usual thermodynamic sense, are in general not conservative because of the possibility of radiation of gravitational waves. Nevertheless, a variational principle governing such oscillations has recently been discovered (Detweiler and Ipser [32]).

(4) Linear, Nonadiabatic Nonradial Oscillations:

To the best of this reviewer's knowledge, the theory of linear, nonradial oscillations has not yet been extended to include nonadiabatic effects (departures from eq. (3)) fully. In the adiabatic approximation the problem is already of the fourth order in complex variables when the angular dependence of the pulsation variables is expressed in terms of spherical harmonics. (Actually, if no non-adiabatic effects are present, each complex variable may be taken as a real variable, so that the order in this case is only four in real variables.) To include non-adiabatic effects fully would involve increasing the order of the system by two in complex variables. The resulting system, of the sixth order in complex variables, does not seem too formidable in view of the present-day existence and widespread use of fast digital computers.

So far (however, see below), non-adiabatic effects have been taken into account only in the quasi-adiabatic approximation, which gives a formula for the stability coefficient essentially identical to equation (11) (Ledoux and Walraven [2], hereafter referred to as LW, § 81):

\[
- \chi = \frac{1}{2c^2J} \text{Re} \int_V \left( \Gamma_3 - 1 \right) \left( \frac{\delta \rho}{\rho} \right)_s^* \delta \left( \rho - \frac{1}{\rho} \nabla \cdot \mathbf{F} \right)_s \rho d\tau,
\]

(14)

where \( d\tau \) denotes a volume element, \( \mathbf{F} \) denotes the total heat flux, and the integration is extended over the entire volume of the system. Here

\[
J = \int_V |\mathbf{\eta}|^2 \rho d\tau
\]

(15)

is the appropriate oscillatory moment of inertia for the mode under consideration, and \( \mathbf{\eta}(\mathbf{r}) \) denotes the space part of the total (Lagrangian) displacement \( \delta \mathbf{r}(\mathbf{r}, t) \).
Judging from the titles of papers to be presented at this Colloquium, it appears that this is one problem that has already been solved by the time of this writing!

(5) **Nonlinear, Nonadiabatic, Radial Oscillations**

Because of the complexity of the equations for all except essentially trivial, non-realistic stellar models, the calculation of the fully nonlinear oscillations of stars has so far been carried out almost exclusively by purely numerical techniques. Until recently (see below), such techniques for purely radial oscillations have involved conceptually dividing the star into a number (~ 50) of discrete, spherical mass zones or shells, and replacing the nonlinear partial differential equations of hydrodynamics and heat flow by finite difference equations after the manner described by Richtmyer and Morton [53]. The model is then followed step by step in time on a fast computer from certain initial conditions. Published calculations of this kind were pioneered by Christy [34]; Cox, Cox, and Olsen [35]; and Aleshin [36]. More detailed descriptions of such techniques may be found in Christy [37]; Cox, Brownlee, and Eilers [38]; Cox, Olsen, King, and Eilers [39]; and Stobie [40]-[42]. More recent discussions and further developments have been provided, for example, by Fraley [43], Papaloizou [44], and Kutter and Sparks [45]. Some results of applications of these methods will be described in later sections of this paper.

All of the above methods, though extraordinarily useful in many ways, possess in common (at least) one severe shortcoming: to study the limiting amplitude, usually periodic, behavior of pulsating stars requires «brute force» integration through a great many pulsation periods and, accordingly, much computer time and possible accumulation of rounding errors. For stars with very slow growth rates or long switching times between modes, the amount of computer time required is so large (reaching astronomical proportions!) as to be totally impracticable. To overcome this difficulty, some recent efforts have been devoted to attempts to obtain more-or-less directly periodic solutions of the nonlinear pulsation equations.

The first such attempts along these lines were due to Baker and von Sengbusch [46] and von Sengbusch [47]. These authors regarded all physical dependent variables for all zones, at each time step in a period, as well as the pulsation period itself, as unknowns, and solved this very large system for all unknowns simultaneously during a period by a sophisticated kind of Newton-Raphson scheme, subject to the periodicity requirement referred to above. Besides the periodic, nonlinear behavior of the model, the method also yielded, as well, information regarding the stability of the nonlinear limit cycles. The method shows much promise; however, the authors report that the calculations are still rather time-consuming, and that certain numerical convergence problems exist in some cases.

More recently, a method has been developed by Stellingwerf [48], [49] which achieves the same goals as the Baker-von Sengbusch method, but by a combination of the two techniques described above. In this method values of all dependent physical variables are guessed at some initial time, as well as the period itself, and the number of time steps in a period is chosen. The model is then integrated forward in time through the specified number of time steps by use of the initial value techniques described above. (However, because the duration of each time step is essentially specified in advance, the integration is most conveniently carried out implicitly rather than explicitly.) The final values of the physical variables at the end of a «period» are then compared with the initial values, and the latter are varied iteratively, as well as the period itself, until strict periodicity is achieved. This method
is found to converge rapidly to high accuracy for realistic stellar models (only a few iterations for each model are required), and appears highly promising. As in the Baker-von Sengbusch method, this method also yields complete stability information for the nonlinear limit cycles for the various modes. Some applications of this method will also be discussed in later sections of this paper.

(6) **Nonlinear, Nonadiabatic Nonradial Oscillations**:

This class includes the most general type of stellar oscillations. The appropriate calculations, however, have not as yet been tackled by anyone, as far as this reviewer is aware.

The only nonlinear, nonradial oscillation calculations attempted so far, to the best of this reviewer's knowledge, are those of Deupree [59], who has restricted himself so far only to the adiabatic approximation and to problems with only two spatial variables (e.g., axial symmetry). Preliminary calculations, which are most conveniently formulated in an Eulerian description, appear very promising. The nonradial supernova calculations of Mahaffy [51] might also be adaptable to the pulsation problem.

**B. Oscillations of Stars When « Perturbations » Are Present**

In this subsection we summarize some important recent work on stellar oscillations when various types of « perturbations » are present in the unperturbed, non-pulsating model. Among the perturbations considered here are rotation (both classical and general relativistic), large-scale magnetic fields, departures from thermal equilibrium (« thermal imbalance »), and convection. Our remarks on these topics will be kept relatively brief.

(1) **Rotating Stars**:

The pulsations of rotating stars have so far been considered only in the adiabatic approximation represented by equation (3). Hence, no results are as yet available (with the exception of Osaki [52]) on the vibrational stability of rotating stars.

One of the most important recent results in this area, in this reviewer’s opinion, is the proof by Clement [53] and Lynden-Bell and Ostriker [54] that the linear, adiabatic oscillations of a uniformly rotating star (rotating as a rigid body) obey a variational principle when the oscillations are dynamically stable, analogous to that described above for small, purely radial, adiabatic oscillations of a non-rotating star. References to other recent work dealing with both linear and nonlinear oscillations of both uniformly and differentially rotating stars, including both « fast » and « slow » rotation, may be found in Cox [1]. Some work which has been published since the above review article appeared are described in the following paragraphs.

In a series of papers by Chandrasekhar and collaborators, the most recent of which is by Wills [55] (references to earlier papers may be found here), the stability of axisymmetric configurations against axisymmetric perturbations in general relativity has been studied.

A case of metastability for slowly, uniformly rotating, supermassive objects against finite-amplitude, isentropic, axisymmetric perturbations in the post-Newtonian theory of general relativity has been discussed by Dedic and Tassoul [56].

The quasi-radial pulsations of rotating relativistic polytropes have been discussed by Papoyan, Sedrakyan, and Chubaryan [57]. The pulsations and stability
of oblate rotating white dwarfs have been studied by Vartanyan [58], and the stability of rotating baryon stars by Arutynyan and Sedrakyan [59]. The relation of adiabatic pulsations to convective instability of uniformly rotating gaseous masses has been investigated by Kochhar and Trehan [60].

We shall conclude this discussion by noting an effect of rotation that has actually been known for some time but not extensively discussed in the literature. This effect refers to the influence of rotation on the (non-axisymmetric) g-modes in a convectively unstable region in a star. These g-modes are known to be dynamically unstable in a non-rotating star, and these unstable motions are presumably a manifestation of the resulting convection. In a rotating star, however, this dynamical instability is transformed into an overstable, i.e. oscillatory motions with increasing amplitude (e.g., LW, § 82). This fact has been made use of by Osaki [62] in his recent theory of the excitation mechanism in the β Cephei stars (see § III H).

(2) Magnetic Stars :

The theory of the oscillations of magnetic stars is in even a more primitive state than the corresponding theory for rotating stars (see above). Some references to recent work in this area have been given by Cox [1].

Perhaps the most important of this recent work is the demonstration by Kovetz [64] that a variational principle exists for the linear, adiabatic oscillations of a star possessing a magnetic field. A brief survey of work on oscillations of magnetic stars, through late 1973, may be found in Cox [1].

Recent studies of the effects of magnetic fields on the adiabatic pulsations of polytropes have been carried out by Miketinac [62] and Billings, Singh, and Trehan [63].

To conclude this discussion, we emphasize that neither non-adiabatic effects (which may include effects of finite electrical conductivity) nor nonlinear effects have as yet been considered in the oscillations of magnetic stars. Hence, essentially nothing is now known about the vibrational stability of these objects. A perhaps even more basic difficulty emerges from the recent work by Tayler and collaborators [64]-[66] on the stability of certain idealized magnetic field configurations which are usually assumed in calculations of properties of magnetic stars. This work seems to suggest that many (if not most) of these configurations are unstable and so could presumably not exist in real stars.

(3) Stars in Thermal Imbalance :

Stars in thermal imbalance are stars which in their non-pulsating, hydrostatic equilibrium states are not in thermal equilibrium, i.e., not in energy balance. In such stars the rates per unit mass of thermonuclear energy generation ε (with any neutrino losses regarded as a negative contribution) and heat losses (1/ρ) ∇·F (F = total heat flux, ρ = density) are not balanced. These stars will therefore evolve on a time scale governed in part by the size of the difference ε — (∇·F)/ρ. This « slow », or secular, time tε is often of the order of the Kelvin time and is usually much longer than the « free-fall » time tff.

The effects of these departures from thermal equilibrium on stellar vibrational stability, usually ignored in ordinary pulsation theory, have recently been worked out, to first order in tff/tε, in a series of papers by Cox and collaborators [67]-[72] and by Demaret [73] (certain previous discussions have been found to be inadequate or incorrect). Since these effects are discussed in detail elsewhere in this Colloquium,
our remarks here will be kept relatively brief. The main result of these studies is that, for purely radial oscillations, the contribution of these thermal imbalance effects to the stability coefficient, and hence to the vibrational stability (here we mean stability with respect to the \emph{absolute} variations $\delta T$, rather than to the \emph{relative} variations $\delta T / T$) consists mainly of two terms, one arising from purely «dynamical» effects (analogous to a slow strengthening or weakening of the spring constant in a simple mechanical system), the other from small «non-adiabatic» effects which are always present whenever a star is not in thermal equilibrium. The dynamical contribution is relatively insensitive to the internal structure of the star and is proportional to the fractional rate of change of the period arising from the slow evolution (\emph{stabilizing} for a \emph{decrease} in the period, normally associated with \emph{contraction}). The non-adiabatic contribution depends much more strongly on the internal structure of the star and is usually opposite in sign to the dynamical contribution (thus, \emph{contraction} usually leads to \emph{destabilization} in terms of the non-adiabatic contribution).

It has been shown [72] that the expression referred to above, as well as the above conceptual separation, also apply \emph{exactly} to the \emph{nonradial} oscillations of spherical, non-rotating stars in thermal imbalance. Applications to the nonradial oscillations of models of $\beta$ Cephei stars are now in progress [74] (it has been shown by Davey [75] that these thermal imbalance effects are too small to excite \emph{radial} oscillations).

Numerical calculations only recently completed [74] of the nonradial oscillations of a 10 M$_{\odot}$ star model in the evolutionary phases immediately following core hydrogen exhaustion show that these thermal imbalance effects, while destabilizing on the overall contraction phase, are much too small, compared with the «ordinary» radiative damping effects, to produce instability.

(4) \emph{Effects of Convection}:

The effects of convection on theories of pulsating stars are important in two main respects: First, convection, which represents the bodily (essentially random) motion of fluid elements («eddies»), contributes extra terms to the momentum and energy equations, since these bodily motions represent extra stresses (e.g., turbulent pressure) and extra (kinetic) energy. Second, convection represents an additional mechanism for heat transfer besides those normally considered (radiation and conduction). It is obvious that a \emph{time-dependent} theory of convection is essential if the important parameters in some part of the physical system are changing on time scales of the same order as or smaller than characteristic «convective», «eddytturnover», times. This is precisely the situation which normally seems to prevail in pulsating stars. Hence, effects of convection on pulsation properties of stars are only rather poorly known at present.

The first of the above two categories of effects of convection, which we shall refer to here as the «dynamical» effects, has usually been either ignored or treated in the manner following from the discussion by Ledoux (LW). In this treatment the convective elements, or eddies, are regarded essentially as «molecules» in a gas, and gas-kinetic theory is applied to them. However, such a treatment may imply the existence of a «turbulent viscosity» which could conceivably be orders of magnitude larger than ordinary molecular or radiative viscosity and which could, accordingly, have far-reaching effects on stellar vibrational stability. As far as this reviewer is aware, the question of the physical reality of such a turbulent viscosity is still unresolved.
The second of the above categories, the direct effects of convective heat transfer on stellar pulsation properties, we shall here call the "thermal" effects. In the static case (time scales for the physical system much larger than convective turnover times) conventional mixing length theory (Vitense [76]; Bohm-Vitense [77]; Cox and Giuli [78]) is normally used. In applications to pulsating stars, these effects of a time-varying convective flux are usually either ignored, i.e., variations in convective flux are set equal to zero; or else simple modifications (e.g., ref. 38) are introduced into the mixing-length theory to take partial account of these effects. In both cases, however, effects of convection on the structure of the static, non-pulsating state of the star are often taken into account [21], [33] (however, see King, Cox, Eilers, and Davey [79]; Stellingwerf [48], [80]; and Spangenberg [81]).

It is certainly true that effects of convection are among the least understood of all physical phenomena bearing on stellar pulsations.

III. RECENT APPLICATIONS OF PULSATION THEORY TO STARS

In this section we shall summarize some recent applications of pulsation theory to actual pulsating stars. We shall see that, overall, the broad features of stellar

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**Fig. 1.** Location of various types of intrinsic variables on the Hertzsprung-Russell (H-R) diagram (from Cox [1], © The Institute of Physics).
pulsation as observed in nature are reasonably well accounted for by current pulsation theory. There are, however, severe problems in a number of respects, and these problems will be discussed mainly in the next section.

For orientation, we illustrate in Figure 1 the locations of the principal types of pulsating stars on a Hertzsprung-Russell (H-R) diagram. Note that most of the best-known kinds of pulsating stars (the classical Cepheids, RR Lyrae variables, W Virginis variables, and the δ Scuti variables and dwarf Cepheids) all lie in the long, narrow, almost vertical, cigar-shaped region indicated by dashed lines. This region is sometimes referred to as the «instability region», the «Cepheid region», etc.

A. Classical Cepheids

We begin our discussion with the detailed application in the early 1960's of linear, non-adiabatic, radial pulsation theory to models of classical Cepheids by Baker and Kippenhahn [20] and Cox [22]. (A discussion of earlier searches for the destabilizing mechanism has been provided by King and Cox [82]). These calculations definitively established singly ionized helium (He+) in the envelope as an effective destabilizing agent which is probably the main cause of pulsational instability in the classical Cepheids and in other stars lying in the general Cepheid region of the H-R diagram, as had first been suggested by Zhevakin [83]-[85]. The detailed physical mechanism of operation of this ionization region (and of other important ionization regions as well) has been explained in a number of review articles (e.g., Cox [3], King and Cox [82], Cox and King [89], Cox [1]). Briefly, the enhanced ionization of the abundant element during compression renders the stellar material in this region relatively opaque and keeps it relatively cool. Hence, the large amount of energy impinging on this region from the interior at this instant is trapped, thus rendering the pressure in this region larger during the expansion phase than during the contraction phase. The net effect is a strong «driving» of pulsations in the vicinity of the ionization region; this driving can more than compensate for the interior damping, and hence can produce pulsational instability, when the parameters of the star form a favorable combination. This mechanism has been found to give a good general account of many of the qualitative features, particularly of the blue, or hot, side, of the Cepheid region. The red, or cool, edge of the region, however, has not yet been satisfactorily accounted for in detail theoretically.

The difficulty with the red edge was first explicitly emphasized in the calculations of Baker and Kippenhahn [21], who studied the pulsational stability of realistic Cepheid envelope models which included convection according to conventional mixing-length theory in the static models. The authors ignored the variations in the convective flux, and found that the models tended toward neutral stability (but never actually became stable) as the red edge was approached from the blue. The authors suggested that convection was probably the physical agent that actually terminated instability on the red side of the region. This is probably the correct explanation, but theoretical calculations have yet to account for this phenomenon in detail (see also Tuggle and Iben [87]).

Meanwhile, Christy [34] emphasized the importance of the hydrogen (H) ionization region in the general problem of Cepheid instability. Subsequent calculations (see below) have shown that Christy was qualitatively right, but that the effect was actually smaller than his results suggested.

These nonlinear calculations of Christy [34] were the first to demonstrate expli-
citly self-excitation of pulsations in a stellar model due to H ionization. The calculations of Cox, Cox, and Olsen [36] were the first to demonstrate self-excitation as a result of He\(^+\) ionization and the attainment of a stable limiting amplitude having limit cycle characteristics. Further details and extensions of these latter calculations were published by Cox, Cox, Olsen, King, and Eilers [38] and King, Cox, and Eilers [88].

The classical «phase lag» of maximum luminosity behind minimum radius was first computed by Christy [37]. Its physical explanation was provided by Castor [89], and subsequently confirmed in detail by Keller and Mutschleener [90] and others (see below). Physically, the phase lag owes its origin to the sweeping through mass of the thin H ionization zone, with its large heat capacity, during the pulsations. Subsequent calculations, both linear and nonlinear (e.g., Stobie [40]-[42], Castor [8], Iben and Tuggle [91], [92], and King, Cox, Eilers, and Davey [79]), have consistently revealed the proper phase relationships provided that the outermost regions of the model are suitably zoned and that the transfer is assumed to be predominantly radiative in these regions. This last proviso suggests that convection, at least in these regions, may not be as important as conventional mixing length theory suggests (Castor [9]). Also, the phase lag is definitely a linear phenomenon, affected, but not caused, by nonlinear effects.

The by now classical «Cepheid mass discrepancy» first originated with attempts by Christy [93] to reproduce with his nonlinear pulsation calculations the observed progression of «bumps» in the velocity curves with periods of Cepheids; this progression is usually referred to as the «Hertzsprung relation». Christy found that the Hertzsprung relation could be reproduced only if the assumption was made that Cepheid masses were only about one-half the values expected on the basis of conventional evolution calculations without mass loss (e.g., Hofmeister, Kippenhahn, and Weigert [94], [95]; Kippenhahn, Thomas, and Weigert [96]; Meyer-Hofmeister [97]; Iben [98]; Paczynski [99], [100]). Stobie's [40]-[42] calculations led to a similar conclusion. A qualitatively similar (but less extreme) conclusion was reached by Cogar [101] on the basis of the period-mean density relation as obtained from linearized calculations and also by Rodgers [102]. This mass discrepancy has since then generated a good deal of discussion (e.g., Tayler [103]; Fricke, Stobie, and Strittmatter [104], [105]; Cox, King, and Stellingwerf [106]; Iben and Tuggle [91], [92]; Schmidt [107]; Schmidt, Rosendhal, and Jewsbury [108]; King, Cox, Eilers, and Davey [79]; and Takeuti [109]), and it is not yet entirely clear whether a real discrepancy exists (e.g., King, Hansen, Ross, and Cox [110]). However, masses as small as one-half the «evolutionary» masses can probably be ruled out on the basis of evolutionary calculations: a star in the Cepheid mass range will apparently not execute a «loop» into the Cepheid region during core helium burning if the star loses even 10 per cent of its mass during the pre-Cepheid, «red giant» phases of its evolution (Lauterborn, Refsdal, and Weigert [111]; Lauterborn and Siquig [112]).

Iben and Tuggle [91] have suggested that one way of resolving this mass discrepancy is to assume, for the most part, that the present distance scale is slightly incorrect and that the Cepheids are actually some 0.93 more luminous than heretofore believed. This resolution has been criticized by King, Hansen, Ross, and Cox [110], who point out that a more likely resolution is provided by assuming that the most fashionable color-effective temperature transformation relations now in use are yielding effective temperatures a few hundred degrees too hot, for given color, for these stars. The relation these authors propose differs only slightly from the above relations and is not outside the range of others which are presently available.

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B. Double Mode Cepheids

The recent discussion by Petersen [113] of the eight or so known "double mode Cepheids" has generated considerable interest. The pulsations of these stars appear to consist of a more-or-less permanent mixture of two modes, usually assumed to be the fundamental and first harmonic. The longer of the two periods generally lies between about 2 and 4 days. Since both periods are determined almost solely by the stellar mass M and radius R (see, e.g., Cox, King, and Stellingwerf [106]), then, given the two periods, M and R can be determined. The sorting out of the two periods appears to be a straightforward and reasonably reliable procedure; however, as pointed out by King, Hansen, Ross, and Cox [110], slight errors in the periods are magnified quite considerably in terms of M and R. Nevertheless, both Petersen [113] and the above authors come to the conclusion that these stars are probably low mass (1 to 2 solar masses), highly luminous (several hundred solar luminosities) stars which constitute a heretofore unrecognized class of stellar object. Preliminary non-linear calculations by Stellingwerf (see the paper in this Colloquium by King, Hansen, Ross, Stellingwerf, and Cox), show that models with the required masses and radii, and having sufficiently low effective temperatures, do indeed exhibit the double mode behavior.

These double mode Cepheids point up another difficulty about stars such as classical Cepheids whose pulsations are driven by "envelope ionization" mechanisms located in the outer stellar layers. This difficulty concerns the large- (or limiting-) amplitude mode preference of the star. Linearized pulsation calculations show that many models are simultaneously unstable in two (or even more) modes. Only non-linear calculations hold any promise for permitting a decision to be made as to which mode (if any!) the star will choose to pulsate at large amplitudes. Even such calculations need to be performed using the Baker-von Sengbusch or Stellingwerf techniques. Observationally, it is not possible, as far as this reviewer is aware, to determine the pulsation mode of a Cepheid (the situation with respect to RR Lyrae variables will be discussed below).

C. RR Lyrae Variables

The RR Lyrae variables have received perhaps more theoretical attention in the past decade than even the classical cepheids. An excellent review particularly of the more recent aspects of this work, up to about mid-1971, may be found in Iben [114].

We shall begin our review with the linear, non-adiabatic calculations of Baker [115], the first of their kind as applied to this type of star. These calculations gave strong support to an earlier suggestion by Schwarzschild [116] that the Bailey types a and b were fundamental pulsators, whereas the type c were first overtone pulsators. Baker also found that both modes were simultaneously excited under certain conditions.

The by now classical study by Christy [117] of over 100 RR Lyrae models by use of the nonlinear, nonadiabatic techniques described in § II A (5) marked a large step forward in our understanding of these objects. This study strongly suggested that the mass of a typical RR Lyrae variable was closer to 0.5 M⊙ than to 1 M⊙, as had been supposed heretofore, and also that a value of helium mass fraction Y closer to 0.3-0.4 than to zero would be appropriate for Population II systems.
Many people have been intrigued by Christy's discovery of a new type of theoretical period-luminosity ($\Pi - L$) relation for the RR Lyrae variables. According to this relation, for each luminosity (lower than some limiting value), a definite fundamental period, say $\Pi_{tr}$, is defined, whose value is essentially independent of mass and composition and is given by the approximate formula

$$\Pi_{tr} \approx 0.057L^{0.6} \text{ days},$$

(16)

where $L$ is the equilibrium luminosity in solar units, and $\Pi_{tr}$ is in days. This period $\Pi_{tr}$ has the significance that it is the shortest-period fundamental in which a star of the given luminosity can pulsate for an extended time, either at large or small amplitudes. That is, stars of given luminosity whose fundamental periods exceed the period given by equation (16) would pulsate only in the fundamental mode; whereas, stars of that luminosity whose fundamental periods were smaller than the above could only pulsate in the first harmonic mode.

The relation between $\Pi_{tr}$ and $L$ expressed in equation (16) has subsequently been referred to frequently as the «Christy transition line». The reason is that, for a group of stars all having the same composition and mass, say, equation (16) represents a line on the H-R diagram, sloping down and to the right. This line cleanly separates the corresponding instability strip into two regions: a lower region, containing only first harmonic pulsators; and an upper region, containing only fundamental pulsators. Christy admitted that this line was not infinitely narrow, but that it actually was a narrow band. Within this band, stars should exhibit an «either-or» kind of behavior at limiting amplitude, in the sense that the mode in which the star will pulsate at limiting amplitude depends on the past history of the star, i.e., its direction of evolution; or else a «mixed-mode» kind of behavior, characterized by a more-or-less permanent mixture of the two modes, fundamental and first harmonic. However, his paper gave the impression that this band was so narrow that its thickness could be neglected. We shall return to this point later, in the light of more recent work.

An obvious application of equation (16) comes to mind: If $\Pi_{tr}$ can be determined observationally, then the intrinsic luminosity of the variables in question is given immediately by this equation. This period presumably corresponds to the period separation between the types $a$, $b$ and the type $c$ pulsators in globular clusters such as M3 and $\omega$ Cen. This kind of analysis has been carried out a number of times since Christy's paper appeared (e.g., Sandage [118]; Dickens and Flinn [119]).

The next important theoretical development, in this reviewer's opinion, was the discovery by Iben and Huchra [120] of another theoretical period-luminosity ($\Pi-L$) relation, based entirely on linear, non-adiabatic theory, for the RR Lyrae variables. This $\Pi-L$ relation was defined by the intersection of the fundamental and first harmonic blue edges on the H-R diagram, and bore a striking resemblance to the Christy transition line of equation (16). For fractional hydrogen ($X$) and heavy element ($Z$) mass fractions $X = 0.7$, $Z = 0.001$, the Iben-Huchra $\Pi-L$ relation is approximately

$$\Pi_{tr} \approx 0.048L^{0.588}$$

(17)

($\Pi_{tr}$ in days, $L$ in solar units), which is to be compared to the Christy relation, equation (16). The relation exemplified by equation (17) we shall refer to as the «Iben-Huchra transition line», in analogy to the Christy relation.

That the intersection of the fundamental and first harmonic blue edges on an H-R diagram does indeed define a $\Pi-L$ relation (for fixed composition) may be seen
from the following considerations. For a fixed composition, the two blue instability edges are described schematically by functional relations of the form

\[ f_1 (L, M, R) = 0, \]  
\[ f_2 (L, M, R) = 0, \]  
\[ f_3 (L, M) = 0. \]  

where \( L \) and \( R \) denote, respectively, equilibrium luminosity and radius, and \( M \) denotes mass. At the intersection point equations (18) and (19) must both be simultaneously satisfied, so we have, for example, at the intersection point,

\[ f_3 (L, M) = 0. \]  

Thus, the luminosity, radius, effective temperature, fundamental pulsation period, etc., at the intersection point are uniquely determined, given only the mass (and composition, of course). More generally, the relations among the stellar parameters form a one-parameter set along the Iben-Huchra transition line; and the Iben-Huchra \( \Pi-L \) relation, equation (17), is one such relation.

That the Christy and Iben-Huchra transition lines cannot be identical in principle, despite the superficial similarity of the corresponding \( \Pi-L \) relations, can be seen from the following considerations. First, the Christy relation is obtained entirely from large amplitude, nonlinear calculations, without particular regard to blue instability edges. On the other hand, the Iben-Huchra relation is based solely on small amplitude (linear) calculations and blue edges. Second and finally, a more fundamental difference is that, for given composition and mass, the Iben-Huchra transition line becomes a point on the H-R diagram, while the Christy transition line becomes a line (see above). This last statement can be understood, alternatively, by noting that, in general, period \( \Pi \) is determined primarily by mass \( M \) and radius \( R \). According to Christy, his \( \Pi-L \) relation (eq. 16) does not depend on \( M \) or composition. Consequently, given \( M \) and composition, this relation becomes a relation between \( L \) and \( R \) or \( T_e \), i.e., a line on the H-R diagram.

It was conjectured by Iben \cite{Iben1967} that the Christy transition line on an H-R diagram (for, say, a given mass and composition) would pass through the intersection point between the two linear blue edges (on which the Iben-Huchra transition line is based), if both sets of calculations were to be carried to arbitrarily high accuracy and if identical input physics were used in both. Iben's conjecture was discussed further by Cox, Castor, and King \cite{Cox1972}, who concluded that Iben's conjecture was equivalent to the assertion that the Iben-Huchra transition line should actually be a subset of the Christy transition line, and also that the composition dependences reported for these two transition lines were not consistent with this conjecture.

Cox, Castor, and King \cite{Cox1972} also suggested a physical mechanism to account for the Iben-Huchra transition line, based on « nodal quenching » of the first harmonic instability, resulting from the spatial proximity of the node and the \( \text{He}^+ \) driving region. They also stated that some of the same physics might also apply to the Christy transition line. In a recent paper Tuggle and Iben \cite{Tuggle1987} have examined this proposed mechanism in some detail on the basis of linearized calculations. They conclude that this mechanism probably accounts for the relative stabilization of the first harmonic mode at high luminosities, but not at low effective temperatures.

This last difficulty has apparently been cleared up by Stellinwerf \cite{Stellinwerf1988}, who finds that the relative stabilization of the first harmonic mode at lower effective temperatures is caused not so much by reduced driving due to a diminished separation between the \( \text{He}^+ \) driving region and the node (the above nodal quenching), but

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rather by increased damping in the first harmonic mode at lower effective temperatures. This increased damping results from a larger pulsation energy in this mode, for given surface amplitude normalization. The larger pulsation energy and increased damping are both caused by an enhancement of the pulsation amplitude in the deeper, damping regions (again, for given surface amplitude normalization); these larger pulsation amplitudes also push the node in radius amplitude to greater depths, thus cancelling the effects of the above nodal quenching. This increased interior amplitude is caused, in turn, by the discontinuity in the radius amplitude at the H ionization zone. This discontinuity is always present at the H ionization zone, is always in such a direction as to increase the interior amplitude, and becomes larger in magnitude with decreasing effective temperature.

A study of RR Lyrae models has been completed by Stellingwerf [48], [49] on the basis of a new technique (see § II A [5]) for directly obtaining limiting amplitude solutions of the nonlinear pulsation equations. First, Stellingwerf carried out a preliminary survey, using identical model parameters as Christy [117] used, identical zoning, artificial viscosity formula, and the Christy opacity formula. The only significant difference is that Stellingwerf included radiation pressure, whereas Christy did not. The main results of this preliminary survey are shown in Figure 2 (which is Fig. III-6 in Stellingwerf [48]), based on the assumed parameter values \( M = 0.578 M_\odot, Y = 0.3, \) and \( Z = 0.002 \) (the same as assumed by Christy [117]). In this figure (which is a log L-log \( T_\epsilon \) diagram) the two solid lines denote the linear fundamental and first harmonic blue instability edges (FBE and 1HBE, respectively); the dashed line (short dashes) represents an estimated red instability edge, based on an assumed width of the strip of \( \Delta \log T_\epsilon = 0.06; \) and the dash-dot line represents the Christy transition line, equation (16), for this mass. The two lines with long dashes bracket the region of «either-or» behavior (labelled « F or 1H » in the figure), in which both limiting amplitudes — fundamental and first harmonic — are stable limit cycles; this means that the mode that a star will pulsate in at large amplitudes depends on its past history. In the regions above the top dashed line and below the bottom dashed line (labelled « F » and « 1H », respectively, in the figure), only fundamental or first harmonic pulsations, respectively, can persist at large amplitudes.

It is seen that the region of «either-or» behavior is actually much wider than found by Christy [117], although the middle of this region is described roughly by the Christy formula, equation (16). Stellingwerf states that the Christy transition line might pass through the intersection point (in accordance with Iben’s [121] conjecture) if radiation pressure had been included in Christy’s calculations. It may be noted that Stellingwerf’s two dashed lines pass through the intersection point, in full agreement with Iben’s [121] conjecture.

However, such a wide «either-or» region does not agree with observations, according to Stellingwerf. For, suppose a star with a luminosity (say \( L/L_\odot = 1.6 \)) well below that corresponding to the intersection point entered the strip from the left with nearly horizontal evolution. Such a star would begin as a first harmonic pulsator and would remain so until it stabilized at the red edge. On the other hand, if such a star entered from the right, it would also be a first harmonic pulsator because first harmonics generally have larger growth rates than fundamentals at small amplitudes. Such a star would again be a first harmonic pulsator during its entire traversal of the strip. Thus, essentially all stars having luminosities below that corresponding to the intersection point would be first harmonic pulsators, and one should therefore observe no globular clusters containing both fundamental and first harmonic pulsators. This expectation is known to disagree with observations.
Fig. 2.—RR Lyrae instability region on an H-R diagram using Christy's [11] opacity formula and conventional artificial viscosity formula (from Stellingwerf [17], with kind permission of the author). The lines with long dashes delineate the region in which the pulsation mode at large amplitudes depends on the past history of the star. See text for further explanation.
Fig. 3. — Same as Fig. 2, except with King Ia opacities and the new Stellingwerf [44] artificial viscosity formula (from Stellingwerf [44], with kind permission of the author). Lines of constant period are also shown. See text for further explanation.
The Stellingwerf primary survey used the new King Ia opacities \((X = 0.70, Z = 0.001)\), and also a new artificial viscosity formula whose overall effect is to reduce the viscous damping in the interior and not to affect this damping in the exterior where shock waves may develop. The main results of this survey are summarized in Figure 3 (which was constructed from Figs. III-20 and III-23 of Stellingwerf \cite{48}). The notation and symbolism are the same as in Figure 2. It is seen that the main difference is that the "either-or" region is now much narrower (say \(\sim 300^\circ\text{K}\)) in the region where luminosities of RR Lyrae variables are thought to lie (say \(1.7 \leq \log \frac{L}{L_\odot} \leq 1.9\)). This fact will clearly permit a single globular cluster, for example, having RR Lyrae variables all of about the same luminosity, to possess both fundamental and first harmonic pulsators, as observed. Moreover, the kind of situation depicted in Figure 3 can, according to Stellingwerf, nicely account for the main observed properties of the two Oosterhoff types of clusters, if a suggestion made by van Albada and Baker \cite{123,124} is adopted. (Oosterhoff type I clusters are characterized by a mean period of \(\langle \Pi_{ab} \rangle = 0.55\) of the Bailey types \(a\) and \(b\) variables, with about 20 per cent of the variables being type \(c\); whereas Oosterhoff type II clusters have \(\langle \Pi_{ab} \rangle = 0.64\) and about 50 percent of the variables being type \(c\); see van Albada and Baker \cite{123}). The suggestion is that the variables in Oosterhoff type I clusters, being generally redder than type II clusters, are evolving predominantly to the blue. Their shorter fundamental periods and greater numbers of fundamentals are thereby qualitatively (and roughly quantitatively, according to Stellingwerf) accounted for by the situation depicted in Figure 3. The type II cluster variables, on the other hand, are, according to this suggestion, predominantly evolving to the red. Such a direction of evolution would clearly result in a larger fraction of first harmonic pulsators, and also in longer average fundamental periods, than in type I clusters.

Finally, the dashed line in Figure 3 to the right of which is labelled "aperiodic" marks the blue boundary of full amplitude behavior in which neither fundamental nor first harmonic are stable limit cycles: the star, if initially placed in either pure mode, tries to switch to the other mode. This is the region where the final, "steady state" motion is a more-or-less permanent mixture of the two modes. One such RR Lyrae variable, AC Andromedae, is known which exhibits just this kind of behavior (Fitch \cite{122}). The longer of the two periods observed for this star (0.711; the shorter period is 0.525) suggests that this star probably lies near the red edge of the instability strip, and thus provides some observational support for Stellingwerf's results in this regard. (However, it should be remembered that Stellingwerf's models were purely radiative; the presence of appreciable convection could conceivably modify these results.)

A survey of RR Lyrae variables is also nearing completion by Spangenberg \cite{126}, based on the conventional initial value techniques, but making use of a much improved treatment of radiation flow in the outermost layers. While this work is not yet completed, one can at least say that there are no severe contradictions with Stellingwerf's \cite{48} results.

The main moral to be drawn from these recent results seems to be that the modal situation is considerably more complex than was suggested by Christy's \cite{117} original investigation, and that the Christy II-L relation should be used only with extreme caution, if at all, in inferring luminosities of RR Lyrae variables.
D. W Virginis and RV Tauri Variables

The evolutionary status of the W Virginis variables seems to have been clarified to a large extent by the calculations of Schwarzschild and Harm \[^{127}\], along with the supporting observational evidence by Wallerstein \[^{128}\]. According to Schwarzschild and Harm, these stars are ascending the asymptotic branch in their evolution, and thermal instabilities in the helium burning shell source cause the stars to execute blueward loops into and across the instability region on the H-R diagram. These stars presumably become W Virginis variables during their traversals of this region. More recent calculations by Sweigart \[^{129}\] do not confirm the Schwarzschild-Harm calculations in detail, but it is not yet clear that there is really a fundamental contradiction.

Linearized pulsation calculations carried out by Cox and King \[^{98}\] for these models confirmed the suspected instability referred to above, and yielded fundamental periods (the fundamental was the only mode excited) ranging between some 3^2 and 14^2, depending on which portion (lower or upper) of the loop the star was on. These models had very short e-folding times for amplitude growth (~ 10-20 periods), in general agreement with some earlier calculations by Christy \[^{130}\] on a slightly different model.

The most recent and most detailed nonlinear pulsation calculations for this kind of star are those of Davis \[^{131}\], \[^{132}\], who treated the flow of radiant energy through the relatively extended atmospheres by a proper radiative transfer calculation. These calculations showed excellent agreement with observed light curves for these stars, even as to the shoulder on the descending portion of the light curve.

The nonlinear model studied by Christy \[^{130}\] revealed some properties (alternating high and low maxima in the light curve) normally observed in RV Tauri variables, so this model (and Christy's calculations) may actually have been more representative of an RV Tauri variable than of a W Virginis variable.

The location and width of the W Virginis instability strip on the H-R diagram has recently been studied observationally by Demers and Harris \[^{133}\]. The main result is that this strip is considerably wider than the Cepheid strip, and also that its central line lies at lower effective temperatures (by several hundred degrees K.) than for the Cepheids.

E. Delta Scuti Variables and Dwarf Cepheids

These very short period variables (periods of a few hours) lie very near (and may even cross) the main sequence, in the middle and late A's. This region is just where the Cepheid instability strip would cross the main sequence if extended down this far. This fact suggests that the basic instability mechanism for these variables is the same as in the Cepheids, RR Lyrae variables, etc., i.e., He^+ ionization in the envelope.

This last conclusion has been confirmed in the calculations of Baglin, Breger, Chevalier, Hauck, le Contel, Sareyan, and Valtier \[^{134}\]. (This paper is an excellent and very thorough review of most of the observational and theoretical material that is presently known about these stars). The main conclusion of these and other calculations (see Cox \[^{1}\] for additional references) is that these stars are probably somewhat more massive than the sun (say ~ [1.5-2.5] M_\odot), and that they are only
slightly evolved and are now evolving through the lower extension of the Cepheid instability strip toward the «red giant» regions of the H-R diagram. However, a small fraction of these stars may also be pre-main sequence stars evolving toward the main sequence (see Breger [136] and Toma [138]). Not all stars in the region of the δ Scuti variables on the H-R diagram are evidently pulsating. Possible reasons have been suggested by Baglin *et al.* [134]; Breger [137]; and Cox, King, and Tabor [138].

The data regarding the dwarf Cepheids are apparently too meagre to permit the conclusion to be drawn that these stars are physically distinct from the δ Scuti variables.

Linearized pulsation calculations by Castor [139], [140], have shown that as many as the first four or five modes of radial oscillation can be simultaneously excited in these stars. Moreover, the growth rates in these higher harmonics are quite large, corresponding to *e*-folding times on the order of a year or so. It is clear that nonlinear calculations, probably using the Baker-von Sengbusch or Stellingwerf techniques, are badly needed for these stars.

### F. Long Period (Mira) Variables

The only calculations performed in recent years bearing directly on these luminous red supergiants, having periods between some 100 days and 700 days, are those of Langer [141] and Keeley [142], [143]. (Recent calculations by Wood [144] and Stry [145], on ejection mechanisms for the formation of planetary nebulae, indirectly confirm some of Langer’s and Keeley’s conclusions.) Langer’s calculations were based on linear, non-adiabatic pulsation theory, whereas Keely resorted to the fully nonlinear, dynamical (initial value type) calculations of the kind referred to in § II A (5).

Despite the rather drastic differences in computational techniques, the above two authors’ results were, overall, rather similar. The fundamental, as well as the first and/or second harmonics, were overstable, with very fast growth rates (*e*-folding times for radius amplitude of only a few periods). The main driving agent was the H ionization region, which operates in a way qualitatively similar to the operation of the He+ ionization region in Cepheids, RR Lyrae variables, etc. The H ionization region is effective and operative in these stars because the convection, which plays a dominant role in determining the structure of practically the entire envelope, becomes inefficient in the outer parts of this ionization region. Hence, the layers above this region, though convectively unstable, are essentially in radiative equilibrium.

Fundamental periods were far too long ( > 1000 days) to agree with observed periods. The reason these periods are so long is that in these enormous envelopes the ionization zones of hydrogen and helium are very extended and occupy an appreciable fraction of the total volume of the envelope. Consequently, the average value, say ⟨Γ1⟩, of Γ1 for the whole envelope is only slightly greater than 4/3; and it is known that, for the fundamental mode of a star of given mass and radius, the period varies approximately as ⟨⟨Γ1⟩ − 4/3⟩−1/2. Both Langer and Keely concluded that good agreement with observed periods of these stars could be achieved if it was assumed that they were pulsating in the first or second harmonic mode.
G. Rapid Blue Variables

The rapid blue variables have been studied extensively in recent years (observationally) by Warner and collaborators (specific references can be found in Osaki and Hansen [146] and in Warner and Brickhill [147]). These variables, of which DQ Her is probably the best known example, appear to be for the most part post-nova stars, probably hot white dwarfs. Their periods range from some tens to some hundreds of seconds (71s for DQ Her).

The importance of these stars to pulsation theory is that they are the first known objects whose interpretation points very strongly to nonradial (g+ mode) oscillations, as has been suggested by Warner and Robinson [148], Chanmugam [149], and Osaki and Hansen [146]. One reason for this statement is that periods of radial oscillations of white dwarfs are too short (a few to a few tens of seconds) to account for the observed much longer periods. On the other hand, g− mode periods can be arbitrarily long.

The most thorough theoretical study of this suggestion is that of Osaki and Hansen [146], who computed the quadrupole (l = 2) nonradial oscillation characteristics in adiabatic theory of cooling white dwarf models having luminosities in the approximate range $10^{-4} \text{L}_\odot - 10 \text{L}_\odot$. They also studied the dissipation of pulsation energy by the usual radiative damping mechanisms by means of the quasi-adiabatic approximation (quite accurate for these kinds of stars), and in addition the energy losses through gravitational radiation.

Perhaps the most interesting part of this study is the discovery of a period-luminosity (Π-L) relation for the g+ modes (g+ modes are dynamically stable) in a cooling white dwarf. This relation is such that, below a certain luminosity, the fainter the star, the longer is the period.

This Π-L relation can be understood approximately as follows. It is known that the angular oscillation frequency $\omega_\theta^2$ of g modes, for a star of given mass and radius, is approximately proportional to the quantity A :

$$\omega_\theta^2 \propto - A,$$

where

$$A \equiv \frac{1}{\rho} \frac{d \rho}{dr} - \frac{1}{\Gamma_1} \frac{d P}{dr}$$

for a spherical star, P and ρ denoting, respectively, the total pressure and density. Equation (22) can also be written in the form

$$A = - \frac{\chi_T}{\chi_\rho} (\nabla_{ad} - \nabla)/H_P,$$

where $\chi_T \equiv (\partial \ln P/\partial \ln T)_\rho$, $\chi_\rho \equiv (\partial \ln P/\partial \ln \rho)_T$, $H_P = -(d \ln P/dr)^{-1}$ is the local pressure scale height, and $\nabla = (d \ln T/d \ln P)$ denotes the logarithmic temperature gradient with respect to pressure ($\nabla_{ad} = [\Gamma_2 - 1]/\Gamma_2$). It is clear that A is related to the K. Schwarzschild convective stability criterion, and that A < 0 in convectively stable regions, A > 0 in convectively unstable regions. Now, $\chi_T \to 0$ in highly degenerate white dwarfs, i.e., the stellar material approaches neutrality with respect to stability against convection. This is exactly what happens in a sufficiently cool, cooling white dwarf: as it cools it becomes increasingly degenerate, and so A → 0. From equation (21), then, it is seen that $\omega_\theta^2 \to 0$, i.e., the g-mode periods increase, also as the star cools.
The various provisos «sufficiently faint», «sufficiently cool», etc., have been inserted in the foregoing because the above Π-L relation does not hold during the hottest, most luminous phases of the cooling white dwarf. In these phases the radius R of the star may be significantly larger than the «zero-temperature» radius R₀, toward which R approaches asymptotically as the star cools. Since the periods of nonradial modes also obey the period-mean density relation (only with different Q values than for radial oscillations), this initial (fairly rapid) decrease in R will similarly produce an initial decrease in the g-mode periods, until degeneracy begins to set in and essentially «freezes» R at values only slightly in excess of R₀.

Osaki and Hansen [146] argue that this kind of Π-L relation is just what is needed to account for the observed properties of the rapid blue variables.

They also find that all modes are damped, with gravitational radiation being the most effective damping mechanism of all. Presumably, these various modes are excited by a nova outburst, whereupon they gradually die out.

H. Beta Cephei Variables

These stars, the hottest variables known (effective temperatures ~ 20,000-25,000°K, periods ~ 4-6 hours), remain an intriguing enigma to astrophysicists. Not only is the basic destabilizing agent unknown, even the nature of the motion (radial vs. nonradial) and the exact evolutionary stage involved are not known with certainty (see Osaki [156] and Lesh and Aizenman [151]-[153]).

So far, at least four theories have been proposed to account for the instability, the last two of which have not yet been published.

The first theory, proposed by Stothers and Simon [154], was based on the assumption that these stars were members of close binaries, and that their outer layers had been enriched in helium-rich elements by mass transfer from the companion. This enhancement of helium in the outer layers of the β Cephei stars resulted in a diminution in the relative mass concentration of the star. This diminution in turn produced an enhancement of the central relative pulsation amplitude of radial pulsations, and this enhanced central pulsation amplitude permitted the interior thermonuclear reactions to produce enough driving to overcome the radiative damping in the outer stellar layers. In this way an over-stability owing its origin to nuclear driving was produced. However, a considerable body of observational evidence (summarized in Cox [1] and in references therein) seems to weigh heavily against this theory.

The second theory is due to Osaki [52], and involves a model with a rapidly rotating convective core. In such a convective core, if nonrotating, the (small) superadiabatic gradient produces dynamically unstable g modes (g⁻ modes), which are presumably just a manifestation of the convective motions. It is known, however (see § II B [1]), that rotation of such a core will lift the azimuthal degeneracy and result in overstable g modes in the convective core. Osaki finds that, by choosing the rotation speed of the core properly, it is possible to achieve resonance with the f (Kelvin) mode of nonradial oscillations of the radiative envelope. With a rough theory of the coupling of the core to the envelope, he argues that the f mode (which has periods close to those that are observed) can be excited. He cites a number of observations of these stars which this theory will explain.

The third theory, due to Aizenman and Cox [72], involves no special or ad hoc
assumptions about the stars in their nonpulsating states. In this theory it is proposed that thermal imbalance effects on the nonradial oscillations of stars in the overall contraction or hydrogen shell ignition phases, when the star departs considerably from thermal equilibrium, may either cause or contribute to the instability (it was shown by Davey \textsuperscript{[25]} that these effects are too small to excite radial oscillations). However, recently completed calculations (see Aizenman, Cox, and Lesh \textsuperscript{[24]}) on a 10 M\(_{\odot}\) model in the appropriate evolutionary phases show that these thermal imbalance effects, though destabilizing during the overall contraction phases, never exceed some 10\(^{-5}\) of the total effect (due to nuclear driving and radiative damping) for the \(p\) and \(f\) modes, and a few per cent for the \(g^+\) modes. This proposed mechanism must therefore be discarded as inadequate.

The fourth theory is based on a recent finding by Aizenman, Cox, and Lesh \textsuperscript{[24]} that some of the higher \(g^+\) modes (\(g_{3+}, g_{4+}\), and perhaps \(g_{4+}\)) are excited by nuclear driving in the hydrogen shell source during (at least) the early phases of shell hydrogen burning. The growth rates of these excited \(g^+\) modes are fast enough to result in a build-up of the pulsation amplitude in a time (a few thousand years) smaller than the evolutionary times in these phases. While the periods of these \(g^+\) modes are some 1.5-2 times longer than observed periods of \(\beta\) Cephei stars, it is argued by the authors that nonlinear mode coupling to the normally highly dissipative \(f\) and/or \(p\) modes (which have the right periods) might serve as a means of dissipating the pulsation energy and thus resulting in a limiting amplitude. This period discrepancy is somewhat disturbing. Nevertheless, as the authors point out, this mechanism involves no special or \textit{ad hoc} assumptions about the nonpulsating state of the star.

Regrettably, it appears that the basic instability mechanism of these stars must be considered still not fully understood.

IV. SOME UNSOLVED PROBLEMS OF PULSATION THEORY

In this section we shall list and discuss briefly what, in this reviewer's opinion, are some of the outstanding remaining problems in pulsation theory as applied to pulsating stars.

A. Convection

Convection is, in this reviewer's opinion, the biggest of the remaining problems of pulsation theory. What we mean is, of course, \textit{time-dependent} convection, since, as was pointed out in § II B (4), convective turnover times in at least some parts (the outer layers) of pulsating stars are thought to be of the same order of magnitude as the pulsation periods themselves. Convection may, as has been suggested by Baker and Kippenhahn \textsuperscript{[21]}, play a dominant role in determining the red edge of the instability strip. It is also conceivable, as has been suggested by van Albada and Baker \textsuperscript{[123]}, \textsuperscript{[124]}, that full-amplitude pulsation might tend to suppress convection. (After all, pulsation and convection can be considered as two modes, a radial mode and a \(g^-\) mode, and the suggested suppression could be thought of as a nonlinear coupling of such modes.) If this were the case, the red instability edge might be somewhat cooler for stars evolving to the red than for stars evolving to the blue. Such an effect (with stars evolving toward the red) might conceivably account for the existence of mixed mode behavior in certain stars (*) (see § III B and the paper

(*) The author is indebted to R. F. Stellingwerf for this suggestion.

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in this Symposium by King, Hansen, Ross, Stellingwerf, and Cox). This kind of behavior has so far been found theoretically only in stars having radiative envelopes lying near the empirical red edge of the instability strip.

B. Cepheid Masses

The by now classical "Cepheid mass discrepancy" discussed in § III A remains a disturbing problem. In view of all the uncertainties involved, both in theory and observation, it is not entirely clear to this reviewer that a real discrepancy exists. However, as pointed out by King, Hansen, Ross, and Cox [110], it is possible that a problem as severe as the solar neutrino problem might "... be lurking in the wings here".

C. Physics of Nonlinear Modal Behavior

The recent calculations of Baker and von Sengbusch [46], von Sengbusch [47], and Stellingwerf [48], [49] using the new techniques of nonlinear computation referred to in § II A (5) show that the large (limiting) amplitude modal behavior of pulsating stars is complex and in general cannot, except in certain essentially trivial cases (such as when only one mode is excited), be predicted on the basis of linear theory alone. It would be most desirable indeed if this modal behavior, or at least some aspects of it, could be understood in physical terms. Some preliminary thoughts and conjectures along these lines have been offered by Stellingwerf [48], but it must be admitted that we are still far from a physical understanding of these matters.

D. Stable Stars in the Instability Strip

The apparent existence of several nonpulsating stars well within the instability strip (Fernie and Hube [155], Schmidt [107], Schmidt, Rosendhal, and Jewsbury [108]) is another disturbing problem that has not yet been accounted for in a completely satisfactory manner. A helium deficiency in the envelope has been suggested to account for these stable stars by Cox, King, and Tabor [135].

E. Period Changes

The occurrence of small, erratic period changes in pulsating stars is well documented (Fernie [156]). Such period changes cannot result from evolutionary changes, and so must arise from factors not yet taken into account in pulsation theory. It has recently been suggested by Laskarides [157] that some of these period changes in RR Lyrae variables might be due to mass loss.

F. The Instability Mechanism of the Beta Cephei Stars

Despite some of the rather promising theories of the instability mechanism of the β Cephei stars discussed in § III H, this mechanism, whatever its nature, must be considered, in this review's opinion, as not yet resolved satisfactorily.

G. Effects of Rotation and Magnetic Fields

The effects of rotation and magnetic fields (and possibly of other perturbations as well) on stellar pulsations, remain to be worked out in more detail and generality.
In particular, the problem of the pulsational stability of these objects, which involves non-adiabatic effects, has not yet been tackled.

V. SUMMARY AND CONCLUSIONS

We have attempted to review the present status of pulsation theory (§ II), and also the status of its application to at least the more common types of pulsating stars (§ III). While it is our opinion that, overall, the theory and its application are in a fairly satisfactory state, a number of disturbing problems still remain (some of which are summarized in § IV). It is our fond hope that, by the time this Colloquium is over, some of these problems will either have been solved, or else that one or more groups will be actively at work on some of them.

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