JILA REPORT #78

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University of Colorado
Boulder, Colorado
June 30, 1966
This research was supported by the Advanced Research Projects Agency (Project DEFENDER), monitored by the U. S. Army Research Office-Durham, under Contract DA-31-124-ARO-D-139.

This paper is to appear in the Journal of Quantitative Spectroscopy and Radiative Transfer.
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ABSTRACT

Spatial variations of the Doppler width resulting from temperature gradients are shown to play an important role in determining the sensitivity of the radiation field in strong, non-LTE lines to the local kinetic temperature. The consequences of this effect are discussed in connection with: 1) the temperature structure of an atmosphere in radiative equilibrium; and 2) the problem of inferring from line observations the temperature distribution in the surface layers of a star. Large increases in temperature near the surface are shown to lead to substantial non-local effects which depend critically on variations in Doppler width.

* Presented at the Colloquium on Line Blanketing Effects, organized by Commission 36 of the I.A.U., on March 17-19, 1966, at Heidelberg.

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INTRODUCTION

Evidence has accumulated that non-LTE line blanketing does not cause significant changes in the ratio of surface temperature to effective temperature from the value for an unblanketed stellar atmosphere. Chandrasekhar (1935), in his fundamental paper on the Picket-Fence Model, showed that if the lines were formed by pure coherent scattering, the ratio of surface to effective temperature had the same value as for the gray atmosphere. Thomas (1965) showed that the equation of radiative equilibrium would be only slightly influenced by non-LTE lines. Very recently Hummer and Stewart (1966), using the numerical solutions of Avrett and Hummer (1965), found that the source function for a strong, non-LTE line in the surface layers of a semi-infinite atmosphere was controlled primarily by radiation flowing up from great depths in the line wings and was only very weakly dependent on the physical state of the gas near the surface. If the radiation field in a line is insensitive to the local value of the kinetic temperature, it is unlikely to play an important role in establishing the temperature through the condition of radiative equilibrium. In the region where the source function is controlled by radiation from below, the flux in the line is essentially constant.

In the discussion of Hummer and Stewart, it was assumed that the Doppler width and hence the frequency dependence of the line-scattering coefficient does not change with depth. On the other hand, because the values of the line-scattering coefficient for Doppler broadening are quite sensitive to the Doppler width in the region of the line wings, it is reasonable to ask whether variations in the Doppler width will lead to a significant increase
in the sensitivity of the radiation field to the local kinetic temperature, especially in the surface layers of a stellar atmosphere. In addition to its potential importance in controlling the temperature distribution, such an increase in sensitivity would be extremely valuable in attempts to infer the temperature distribution near the surface of the atmosphere from the central portions of strong lines. For this reason we investigate the role played by spatial variations of the Doppler width in the formation of strong lines under conditions in which very large departures from LTE can be expected. A secondary object will be to assess the error made in assuming a constant Doppler width in computing line-source functions and intensities, since this assumption allows substantial simplification in such computations.

We shall first consider an isothermal, isobaric atmosphere and then make slight variations in the temperature throughout the first few optical depths (in the line). A direct comparison of the results obtained using the correct variation of the Doppler width with those for constant Doppler width allows us to isolate the consequences of such variations from the effects of changes in the Planck function. We shall also examine the non-local effects of placing a relatively hot layer near the surface of an atmosphere.

THE LINE TRANSFER PROBLEM

We shall use the standard non-LTE theory of line formation for a two-level atom as outlined, for example, by Avrett and Hummer (1965), who also present extensive numerical results for isothermal atmospheres. It is convenient to use the Doppler width $\Delta_o$ at some fixed temperature as
the unit of frequency. The reduced frequency variable is taken to be

\[ x = \frac{\nu - \nu_0}{\Delta_0} \quad , \quad (1) \]

and the reduced Doppler width at optical depth \( \tau \) is

\[ \delta(\tau) = \frac{\Delta(T_k(\tau))}{\Delta_0} \quad , \quad (2) \]

where \( \nu_0 \) is the line-center frequency and \( T_k(\tau) \) is the kinetic temperature at depth \( \tau \). The normalized line-scattering coefficient is

\[ \phi(x, \delta) = \frac{1}{\sqrt{\pi \delta}} \exp(-x^2/\delta^2) \quad ; \quad (3) \]

an informative graph of this function for several values of \( \delta \) is given by Hummer and Rybicki (1966-a). By making the usual assumption that the emission coefficient has the same frequency dependence as the scattering coefficient, we obtain a source function \( S(\tau) \) which is independent of frequency and which is related to the radiation field by

\[ S(\tau) = (1 - \epsilon) \int_{-\infty}^{\infty} \phi(x, \delta) J_x(\tau) \, dx + \epsilon B \quad . \quad (4) \]

Here

\[ \epsilon = \frac{C_{21}}{C_{21} + A_{21}[1 - \exp(h\nu_0/kT_e)]^{-1}} \quad . \quad (5) \]
is the probability per scattering that a photon is lost from the line by collisional de-excitation of the upper level and

\[ B = \frac{2h\nu^3}{c} \left[ \exp\left(\frac{h\nu}{kT_e}\right) - 1 \right]^{-1} \] (6)

is the Planck function at the line-center frequency. \( C_{21} \) and \( A_{21} \) are the collisional and radiative de-excitation rate constants, respectively, and \( T_e \) is the electron temperature. The parameters \( \varepsilon \) and \( B \) will in general depend on depth.

The transfer equation is conveniently written as

\[ \mu \frac{d}{d\tau} I_{x\mu} = \phi(x) \left[ I_{x\mu} - S \right] \] (7)

where \( \mu \) is the cosine of the angle between the direction of propagation and the outward normal and where \( \tau \) is the mean optical depth in the line, related to the geometrical depth \( z \) by

\[ d\tau = \frac{k(z)}{\Delta_o} \, dz \] (8)

Here

\[ k(z) = \frac{h\nu}{4\pi} \left[ N_1(z)B_{12} - N_2(z)B_{21} \right] \] (9)

where \( N_1 \) and \( N_2 \) are the number densities of atoms in the lower and upper levels, respectively. The line-center optical depth is
\[ \tau_{LC} = \frac{1}{\sqrt{\pi}} \int_0^\tau \delta^{-1}(\tau) \, d\tau \]  

(10)

Eliminating \( S(\tau) \) between (4) and (7), we obtain an integro-differential equation in \( I_{xu} \), which we may reduce to a set of coupled first-order differential equations by replacing the integrals over frequency and direction by finite sums. The choice of quadrature formulae and the method of integrating the resulting set of coupled differential equations with two-point boundary conditions are discussed in detail by Hummer and Rybicki (1966-a,b). Briefly, the linear equations are transformed by the method of Rybicki and Usher (1966) (c.f. Rybicki 1965) into non-linear equations with one-point boundary conditions which are stable for numerical integration. In this method \( \varepsilon, B, \) and \( \delta \) may be arbitrary functions of depth.

LOCAL CONSEQUENCES OF SMALL TEMPERATURE VARIATIONS

To make the discussion more concrete we shall consider the following two cases: 1) a line with the atomic parameters and excitation potential of hydrogen Lyman-\( \alpha \); and 2) a line with the atomic parameters of Lyman-\( \lambda \), but having an excitation potential of 1 eV. The collisional de-excitation rate constant used is that given by Hummer (1963) for the 1s \( \leftrightarrow \) 2p transition, which depends on temperature as the 0.2 power. In each case we take \( \Delta_0 \) to be the Doppler width for the line at a temperature of 5000°K, and measure all intensities in units of the Planck function for the line-center frequency at 5000°K. Henceforth we assume that kinetic and electron temperatures are equal and write \( T_e \) as the temperature variable.
We consider a semi-infinite atmosphere at $5000^\circ K$ throughout and then vary the temperature in the first few optical depths, so that the surface temperatures have the values $4500^\circ K$ to $5300^\circ K$, in steps of $100^\circ K$. The temperatures are represented by a function of the form

$$T_e(\tau) = 5000 + b \left[ \exp[q(\tau - \tau_o)] + 1 \right]^{-1}, \quad (11)$$

where $b$ is a parameter chosen to give the desired value of $T_e(o)$. The parameter $\tau_o$ is roughly the thickness of the region in which the temperature differs from $5000^\circ K$. In these examples $\tau_o = 3.0$ and $q = 2.0$. The temperature distributions are plotted for the extreme cases, $T_e(o) = 4500^\circ K$ and $T_e(o) = 5300^\circ K$, in the inset of Figure 1; and the surface values of the Planck function for the two lines are given in columns 2 and 6 of Table I. A very low value of electron density is assumed: $N_e = 10^{10}$, for which $\varepsilon$ is of order $10^{-5}$.

The surface values of the source functions, with $\delta = 1$ and with $\delta$ varying consistently with $T_e$, appear in columns 3 and 4 of Table I, respectively, for Lyman-$\alpha$, and in columns 7 and 8 for the 1-eV line. As expected, the values of $S(o)$ for $\delta = 1$ are nearly independent of $T_e(o)$, while the values for the correct $\delta$ show a substantial dependence on $T_e(o)$. It is noteworthy that the rate at which $S(o)$ increases with $T_e(o)$ is nearly the same for both lines, although the respective Planck functions increase at very different rates. This observation confirms the idea that the temperature sensitivity of the line radiation field is due entirely to the Doppler width and not to the changes in the Planck function.
In Figure 1, the Lyman-α source functions near the surface for the extreme cases, \( T_e(0) = 4500^\circ\text{K} \) and \( 5300^\circ\text{K} \), are compared to the source function for the isothermal case. The source functions with \( \delta = 1 \) throughout, which are represented by broken lines, differ only very slightly from the isothermal source function. When the correct dependence of \( \delta \) on depth is included, one obtains the results represented by the solid lines. Although the response of the source function to these temperature changes is still quite small on an absolute basis, it is many times greater than when \( \delta \) is held constant. On the other hand, at \( \tau = 10 \) the source functions lie within 2% of the source function for the isothermal atmosphere.

It is interesting that spatial variations in the Doppler width cause the source function to lie above the \( \delta = 1 \) result in one case and below the \( \delta = 1 \) result in the other. This behavior is a consequence of the fact mentioned above, that the source function near the surface depends primarily on radiation from below. When the temperature near the surface drops, the scattering coefficient in the line wings and hence the amount of radiation intercepted by the surface layers, decreases. When the surface temperature increases, the surface layers see more of the upcoming radiation and the degree of excitation increases.

The line-center values of the normally emergent intensities, computed with the correct variation of the Doppler width, appear in columns 5 and 9 of Table I. For the extreme and isothermal cases, the Lyman-α line profiles are shown in Figure 2. It should be noticed that the emergent intensities near the line center differ somewhat more from the isothermal cases than do the respective source functions. This example illustrates what appears to be
a general rule -- that the radiation field is more sensitive to the local
temperature than is the source function, which may be explained by the
appearance of $\phi(x,\delta)$ in the functional giving $I_{\nu x}$ in terms of $S(\tau)$.

NON-LOCAL CONSEQUENCES OF TEMPERATURE VARIATIONS

In the foregoing examples we have investigated the sensitivity of the
radiation field in the vicinity of the surface to the local temperature.
We now turn to some examples in which quite large temperature variations
are imposed in a region of moderate thickness in order to see the effects
on the radiation field in the rest of the atmosphere. We shall consider a
mock chromospheric situation consisting of a semi-infinite atmosphere at
5000K with surface layers of varying thickness at 10000K. The temperature
distribution is again represented by Eq. (11), with $q = 1.0$. The parameter
$\tau_0$ is used to identify the various models. The Lyman-$\alpha$ Planck functions
for the cases we consider are shown in Figure 3. Calculations were made
for Lyman-$\alpha$ with $N_e = 10^{10}$ and $10^{11}$ and for the 1 eV line with
$N_e = 10^{10}$.

In Figure 4 we have plotted $\log S(\tau)$ against $\tau$ for $\tau \leq 90$, again
using a broken line for the results obtained with $\delta = 1$. As the thickness
$\tau_0$ of the hot region is increased, very large changes in the source function,
extending far beyond $\tau = \tau_0$, are observed. For the cases $\tau_0 = 0.5, 1.0,$
and 2.0 the source functions are increasing right up to the surface,
instead of falling away as in the remaining cases. This behavior results
because the source function is dominated by the photon-creation term $\varepsilon B$
when the hot region is too thin for the scattering term \((1 - \varepsilon) \int_{-\infty}^{\infty} \phi(x) J_\infty dx\)

to build up appreciably.

Very close to the surface, the source functions lie below the results for \(\delta = 1\) because the broader emission profile allows more radiation to escape. In the cooler region the correct source function is larger because the radiation flowing out of these regions is partially obstructed by the wider scattering coefficient in the hot region. The extra opacity in the line wings throughout the hot region acts as a partial reflector for the radiation from the lower layers, which drives them toward LTE. We shall refer to this as the reflector effect. From Figure 4 it is also clear that variations in the Doppler width tend to smooth out the source function and hence the radiation field. Comparison of the approximate and exact source functions indicate that in the case at hand neglect of variations in the Doppler width causes errors as large as sixty per cent in the surface layers and larger errors in the deeper layers.

Increasing the electron density and hence \(\varepsilon(\tau)\) by a factor of ten gives the results shown in Figure 5. Comparing Figures 4 and 5, we see that the source functions for the cases with \(\tau_o > 2\), scale almost exactly as \(\varepsilon\), that is, the hot layers behave essentially as independent atmospheres the thicknesses of which are finite and less than a thermalization length \((\sim 1/\varepsilon\) for Doppler broadening). On the other hand the isothermal source function has increased in proportion to \(\sqrt{\varepsilon}\) in the surface layers. The cases with \(0 < \tau_o < 2\) are controlled by both hot and cold parts of the atmosphere and therefore do not scale in a simple fashion, although in the first few optical depths, they do scale essentially as \(\varepsilon\).
The normally emergent intensities for $N_e = 10^{10}$ and $10^{11}$ appear in Figures 6 and 7, respectively. The relative intensities within one Doppler width of the line center are seen to be nearly the same for cases with $\tau_o > 0$ at both densities, since this part of the line is formed in the hot layers. The emission lines at the lower density are wider at a given fraction of peak intensity than are those at the higher density because the thermalization length is larger at the low density, thus leading to the persistence of gross effects from the hot layer extending to larger depths.

Let us turn now to the 1 eV line, the source functions for which are shown in the lower panel of Figure 8. As one would expect, the overall variation of $S(\tau)$ with $\tau_o$ is very much less than for the Lyman-$\alpha$ line. In the top panel of Figure 8 appear the source functions for the isothermal case and the case $\tau_o = 20$, computed with $\delta = 1$. It is now clear that very nearly all of the dependence of the source functions shown in the bottom panel comes in through variations in the Doppler effect discussed earlier. The emergent intensities are shown in Figure 9, where only the extreme cases are drawn in the wings; the remaining cases lie between the two curves. It appears that it would be observationally possible to distinguish between the various cases we have considered.

The extreme cases of Lyman-$\alpha$ and the 1-eV line illustrate the division of strongly non-LTE spectral lines into two classes as regards their sensitivity to the electron temperature in an inhomogeneous atmosphere. In the first class, $\hbar \nu / k T_e > 1$, and most of the response to the temperature structure of the atmosphere arises from the large variations of the Planck function, while variations in the Doppler width make relatively small contributions.
In the second class, $\hbar \nu / kT_e \lesssim 1$, variations of Planck function play a minor role, and the dominant response to temperature variations comes largely through the Doppler width. As lines are formed closer to LTE, that is, as $\varepsilon$ gets larger, the Planck function will dominate more and more and Doppler effects will cease to be important in determining the source function. If collision broadening is appreciable, the changes in the Doppler part of the opacity in the line wings will be masked by the collisional contribution, which will not depend in the same way on temperature. It is possible, however, that in this case the sensitivity to local density variations may be enhanced.

The extent to which the increased sensitivity to the local temperature of the radiation field through Doppler effects will influence the temperature structure of an atmosphere in radiative equilibrium is not yet clear and may well be small. However, from the point of view of structural diagnostics, these effects could be of great importance. It is quite clear that any future attempts to apply the non-LTE theory of line formation to specific models for comparison with observations must include the correct variation of the Doppler width.

ACKNOWLEDGMENTS

The work reported here was carried out at the Joint Institute for Laboratory Astrophysics and was supported by the Advanced Research Projects Agency (Project DEFENDER), monitored by the U. S. Army Research Office-Durham under Contract DA-31-124-ARO-D-139. The authors are grateful to the Fellows of JILA for their hospitality during the summer of 1965 and for computer time made available subsequently.
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TABLE I. Surface values of Planck functions, source functions, and line-centered normally emergent intensities for Lyman-α and for the 1-eV line. B, S, and I are expressed in units of the appropriate Planck function at 5000°K.

<table>
<thead>
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<th>T_e (°K)</th>
<th>B (°K)</th>
<th>10^3 S (°K)</th>
<th>10^3 S (°K)</th>
<th>10^3 I_o (°K)</th>
<th>B (°K)</th>
<th>10^3 S (°K)</th>
<th>10^3 S (°K)</th>
<th>10^3 I_c (°K)</th>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td>h_v = 1.0 eV</td>
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FIGURE CAPTIONS

Figure 1. Source functions for Lyman-α corresponding to temperature distributions in the inset, in units of the Planck function at 5000°K. Broken lines indicate results with $\delta = 1$.

Figure 2. Normally emergent intensities for Lyman-α in units of the Planck function at 5000°K. Frequency is measured from line center in hydrogen Lyman-α Doppler widths at 5000°K.

Figure 3. Planck functions for Lyman-α. In present units $B(\infty) = 1$.

Figure 4. Lyman-α source functions with $N_e = 10^{10}$ corresponding to Planck functions in Figure 3. Broken lines indicate results with $\delta = 1$.

Figure 5. Lyman-α source functions with $N_e = 10^{11}$ corresponding to Planck functions in Figure 3.

Figure 6. Normally emergent intensities for Lyman-α with $N_e = 10^{10}$.

Figure 7. Normally emergent intensities for Lyman-α with $N_e = 10^{11}$.

Figure 8. Source functions for the 1-eV line in units of Planck function at 5000°K. The curves marked $\tau_o = 0.0$ are, of course, identical.

Figure 9. Normally emergent intensities for the 1-eV line. Curves not shown in full lie between completed curves.
$T_e(0) = 5000 \, ^\circ K$

$T_e(0) = 5300 \, ^\circ K$

$T_e(0) = 4500 \, ^\circ K$

$N_e = 10^{10}$

Figure 1
Figure 2
Figure 3
Figure 4
\[ \tau_0 = 20.0 \]
\[ \tau_0 = 10.0 \]
\[ \tau_0 = 5.0 \]
\[ \tau_0 = 2.0 \]
\[ \tau_0 = 1.0 \]
\[ \tau_0 = 0.5 \]
\[ \tau_0 = 0.0 \]

\[ T_e(\infty) = 10^4^\circ K \]
\[ T_e(\infty) = 5 \times 10^3^\circ K \]
\[ N_e = 10^{11} \]

Figure 5
Figure 6

\[ T_e(0) = 10^4 \, ^oK \]
\[ T_e(\infty) = 5 \times 10^3 \, ^oK \]
\[ N_e = 10^{10} \]
Figure 7

\[ \log I_x(0) \]

\( \tau_0 = 20.0 \)
\( \tau_0 = 10.0 \)
\( \tau_0 = 5.0 \)
\( \tau_0 = 2.0 \)
\( \tau_0 = 1.0 \)
\( \tau_0 = 0.5 \)

\( \tau_0 = 0.0 \)

\( T_e(0) = 10^4 \, \text{K} \)
\( T_e(\infty) = 5 \times 10^3 \, \text{K} \)
\( N_e = 10^{11} \)
Figure 8

The graph depicts the behavior of \( \log S(\tau) \) as a function of \( \tau \), with different values of \( \tau_0 \) marked on the curves. The parameters given are:

- \( T_e(\infty) = 10^4 \text{K} \)
- \( T_e(\infty) = 5 \times 10^3 \text{K} \)
- \( N_e = 10^{10} \)
Figure 9

\[ I_x(0) \]

\[ \tau_0 = 20.0 \]
\[ \tau_0 = 10.0 \]
\[ \tau_0 = 5.0 \]
\[ \tau_0 = 2.0 \]
\[ \tau_0 = 1.0 \]
\[ \tau_0 = 0.0 \]

\[ T_e(0) = 10^4 \text{oK} \]
\[ T_e(\infty) = 5 \times 10^3 \text{oK} \]
\[ N_e = 10^{10} \]