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EQUIVALENCE OF SPECIAL MODELS IN ENERGY-DEPENDENT NEUTRON TRANSPORT AND NON-GREY RADIATIVE TRANSFER

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Equivalence of Special Models in Energy-Dependent Neutron Transport and Non-Grey Radiative Transfer

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ABSTRACT

Mathematical equivalence is demonstrated between problems for a separable scattering kernel in energy-dependent neutron transport and for the uniform line-blanketing model of non-grey radiative transfer. As an example, the solution of the Lilne problem is rederived by the method of singular eigenfunctions, with the application of orthogonality relations.
I. INTRODUCTION

As is well known, the theories of radiative transfer (1) and neutron transport (2) bear a close relationship to each other. In particular, there is an obvious equivalence between one-speed neutron transport and monochromatic radiative transfer. This equivalence also extends, as far as time-independent problems are concerned, to the constant cross section approximation in neutron transport and to the grey case in radiative transfer.

Energy-dependent neutron transport and non-grey radiative transfer are in general no longer equivalent, with the possible exception of special models. It is the purpose of this paper to discuss such an exception where a mathematical correspondence can be established in spite of the physical dissimilarity. We specifically refer to the separable scattering kernel in neutron transport (3) and to the uniform line-blanketing (ULB) model in radiative transfer (4).

In the neutron case we shall assume a uniform, nonabsorbing, sourceless medium and isotropic scattering. The assumption of a separable scattering kernel means that the energy distribution of neutrons after scattering is supposed to be independent of the energy before scattering.

The ULB model is a generalization of Chandrasekhar's picket-fence model (5) and is based upon the following three assumptions: a) Local thermodynamic
equilibrium with steady-state radiative equilibrium.
b) Milne-Eddington assumption (b): the spectral shape of
the absorption coefficient is independent of depth (i.e.,
independent of temperature and density). c) Spectral
variations of the absorption coefficient are repeated
within relatively small frequency intervals over the
whole spectrum either periodically or in a random fashion.
We may imagine a spectrum densely populated by spectral
lines, with the lines of each type (height, width, shape)
evenly or randomly distributed over the whole spectrum.

The precise meaning of the last assumption will
be explained in the next section.
II. SEPARATION OF VARIABLES

Under the above assumptions, and for plane geometry with azimuthal symmetry and no time dependence, the neutron problem is described by the following form of the transport equation

\[
\left[ \mu \frac{\partial}{\partial x} + l^{-1}(E) \right] \Phi(x, E, \mu) = \frac{1}{2} \int_0^\infty dE' \int_{-1}^1 d\mu' \sum(E' \rightarrow E) \Phi(x, E', \mu').
\]  

(1)

Here \( \Phi \) is the neutron flux per unit \( dE \) and per unit solid angle, \( \sum(E' \rightarrow E) \) the macroscopic differential scattering cross section per unit \( dE \), and \( l(E) \) the scattering mean free path for neutrons of energy \( E \),

\[
l^{-1}(E) = \int_0^\infty \sum(E \rightarrow E') dE'.
\]

In accord with what is known about neutron scattering, we shall assume that \( l(E) \) vanishes only at \( E \rightarrow 0 \) (like \( E^{1/2} \)) and that it has an upper bound. We take this bound as the unit of length,

\[
\max \{ l(E) \} = 1.
\]

With detailed balance taken into account, a separable \( \sum(E' \rightarrow E) \) can be written as

\[
\sum(E' \rightarrow E) = l_M(E) l^{-1}(E) l^{-1}(E'),
\]

(2)
where

$$M(E) = \left[ \frac{E}{(kT)^2} \right] \exp \left( -\frac{E}{kT} \right),$$

while \( \ell_M \) is defined by

$$\ell_M^{-1} = \int_0^\infty \ell^{-1}(E) \tilde{w}(E) \, dE. \quad (3)$$

In the terminology of the kinetic theory of gases (1), \( \ell_M \) is Maxwell's mean free path for the neutrons.

With the kernel (2), equation (1) reads as

$$\left[ \mu \ell(E) \frac{2}{\partial x} + 1 \right] \Phi(x, E, \mu) = \frac{1}{\ell_M} \tilde{w}(E) \int_0^\infty \int_{\mu'}^{\mu} \ell^{-1}(E') \Phi(x, E', \mu') \, d\mu' \, dE' \quad (4)$$

To this equation we are going to apply a modified version of a procedure of Bednarz and Mika (8). The right-hand side suggests that solutions exist of the form

$$\Phi(x, E, \mu) = M(E) \, G(x, \xi), \quad (5)$$

where

$$\xi = \mu \ell(E). \quad (6)$$

After the order of integration is inverted, an integrodifferential equation for \( G(x, \xi) \) follows from Eq. (4),

$$\left( \xi \frac{2}{\partial x} + 1 \right) G(x, \xi) = \frac{1}{2} \int_{\xi}^{1} c(\xi') \, G(x, \xi) \, d\xi'. \quad (7)$$
We have introduced the function

\[ c(\xi) = \ell \int_{\mathcal{M}_E(\xi)} \ell^{-2}(E) M(E) \, dE, \tag{8} \]

where the integration is carried out over the set of intervals \( \mathcal{M}_E(\xi) \) defined by

\[ E \in \mathcal{M}_E(\xi) \quad \text{if} \quad \ell(E) \geq |\xi|. \]

For \( \xi = 0 \) this set covers the full interval \((0, \infty)\), whereas for \( \xi = 1 \) it is normally empty. If \( \ell(E) \) is a monotonically increasing function, \( \mathcal{M}_E(\xi) \) consists of the interval \( E_\xi \leq E < \infty \), defined by \( \ell(E_\xi) = \xi \).

Evidently the function \( c(\xi) \) is even, non-increasing with increasing \( |\xi| \), and positive-valued, with the exception that normally \( c(1) = 0 \). Our assumption of a non-absorbing medium is reflected by the relation

\[ \frac{1}{2} \int_{-1}^{1} c(\xi) \, d\xi = 1, \tag{9} \]

as is easily shown.

Equation (7), which will be our main concern, has the same appearance as the one-speed transport equation, except that there \( c \) is a constant. It is interesting to note that equations of the form (7) appear also in Chandrasekhar's "pseudoproblems", which are associated with problems for monochromatic radiative transfer involving anisotropic scattering (1).

We wish to solve Eq. (4) for slab geometry or for a semi-infinite medium, and for given boundary conditions which describe the distribution of neutrons incident from outside or coming from infinity. Evidently a solution of the form (5) can fit the boundary conditions only if the incident
neutron distribution is already described by a product of this form. Otherwise we need the more general ansatz,

$$
\bar{\phi}(x,E,\mu) = \mathbb{H}(E) \ G(x,\xi) + \bar{\phi}'(x,E,\xi)
$$

(10)

where each term has to satisfy Eq. (4).

The separation (10) is made unique, as we shall see, by requiring that

$$
\int_0^1 \rho^{-2}(E) \bar{\phi}'(x,E,\xi) \ dE = 0
$$

(11)

for all x and \( \xi \). This integral appears on the r.h.s. of Eq. (4), after substitution (6) is made, and after the integration order is inverted. In view of condition (11), that term in Eq. (4) vanishes, so that \( \phi' \) obeys the differential equation

$$
(\xi \frac{\partial}{\partial x} + 1) \bar{\phi}'(x,E,\xi) = 0.
$$

(12)

Another consequence is that the collision density is expressed simply by the r.h.s. of Eq. (7),

$$
2\pi \int_0^\infty \int_0^1 \rho^{-1}(E) \bar{\phi}(x,E,\mu) = 2\pi \rho_x^{-1} \int_{-1}^1 c(\xi) \ G(x,\xi) \ d\xi.
$$

(13)

The solution of Eq. (12) is found in a trivial way, once the boundary conditions for \( \phi' \) are known. For instance, in the case of a slab \((0 \leq x \leq a)\) we get

$$
\bar{\phi}'(x,E,\xi) = \bar{\phi}'(0,E,\xi)e^{-x/\xi}, \quad 0 < \xi \leq \mathbb{L}(E),
$$

$$
= \bar{\phi}'(a,E,\xi)e^{(a-x)/\xi}, \quad -\mathbb{L}(E) \leq \xi < 0.
$$

(14)
These expressions resemble those for virgin flux. Actually, we may imagine a decomposition of the virgin flux into two parts (negative values admitted): one in accord with the ansatz (5), while the other, $\phi'_{x,E}(x,0)$, is such that the corresponding contribution to the scattered flux vanishes.

In view of Eq. (14) it is sufficient to require that condition (11) holds for $x = 0$, $0 < \xi < 1$, and for $x = a$, $-1 < \xi < 0$. After expressing $\phi'(0,E,\xi)$ from Eq. (10), we have

$$
\int_{\mathcal{M}_{\xi}} \mathcal{L}^{-2}(E) \phi(0,E,\xi) \, dE = \mathcal{L}^{-1}_{M} c(\xi) G(0,\xi),
$$

(15)

and a similar equation is obtained for $G(a,\xi)$. In this way, both the supplementary function $\phi'(x,E,\xi)$ and the boundary conditions for $G(x,\xi)$ are uniquely determined from the boundary conditions for $\phi(x,E,\xi)$.

We have tacitly assumed that the integral in (15) converges. This means, for instance, that incident neutron distributions of the slowing-down type ($\phi \propto 1/E$ for $E \to \infty$ if $\ell(E) \to const. \neq 0$) cannot be allowed with the present formalism. Similarly, we must prohibit incident distributions which would make the integral in (15) blow up at $E \to 0$.

Let us now turn to the radiative transfer problem and show that it can also be reduced to Eq. (7) (9,10). For sake of simplicity we again assume that the mean free path is bounded and vanishes at most for isolated values of $\gamma$. 
With \( z \) being the geometrical depth, we define an optical depth \( x \) by aid of the maximum of \( \ell_\nu \) at each depth,

\[
dx = dz / \max \{ \ell_\nu \}.
\]

Under the assumptions (a) and (b) mentioned in the Introduction, and writing \( \ell_\nu \) for the ratio \( \ell_\nu / \max \{ \ell_\nu \} \), we then have the equation (11)

\[
\left( \mu \frac{\partial}{\partial x} + 1 \right) I_\nu(x, \mu) = B_\nu(T(x)),
\]

and the condition for steady state,

\[
\int_0^\infty \ell_\nu \left[ \frac{1}{2} \int_{-1}^1 I_\nu(x, \mu) d\mu - B_\nu(T(x)) \right] d\nu = 0,
\]

where

\[
B_\nu(T) = 2 c^{-2} h^3 \nu^3 (e^{h\nu/kT} - 1)^{-1}.
\]

A reduction of Eq. (16) to a form like (7) is not possible until a certain integration over frequencies is carried out. However, only in the grey case is this an integration over the whole spectrum. In the present case we carry out the integration only over a set of intervals \( M_{dl}\{\nu\} \) defined by

\[
\nu \in M_{dl}\{\nu\} \quad \text{if} \quad l \leq \nu < l + dl.
\]

This defines a new distribution function,
\[ I(x, \ell, \mu) d\ell = \int_{\mathcal{M}_{\mu}^{\{\nu\}}} I_{\nu}(x, \mu) d\nu. \]  

(18)

It's meaning is explained if we imagine the photons grouped according to their mean free paths instead of their frequencies. \( I(x, \ell, \mu) d\ell \) is the part of the total intensity belonging to photons with mean free paths in the interval \((\ell, \ell + d\ell)\). Evidently

\[ \int_0^1 I(x, \ell, \mu) d\ell = \int_0^\infty I_{\nu}(x, \mu) d\nu. \]

Assumption (c) enables us to carry out the integration also on the right-hand side of (16). We think of a subdivision of the whole spectral range of interest into intervals \( \Delta\nu \) small enough so that variations of \( B_{\nu}(T) \) within each \( \Delta\nu \) are negligible (for all \( T \) in the range of interest). Then let \( P(\ell) d\ell \) denote the fraction of an interval \( \Delta\nu \) covered by the set \( \mathcal{M}_{\nu}^{\{\nu\}} \). We may also say that \( P(\ell) d\ell \) is the probability for finding a mean free path \( \ell_{\nu} \) within the interval \((\ell, \ell + d\ell)\) if the frequency is picked at random from \( \Delta\nu \). The precise meaning of assumption (c) is that a subdivision of the mentioned kind exists such that \( P(\ell) \) is the same for all \( \Delta\nu \)-intervals.

An immediate consequence of this assumption is that

\[ \int_{\mathcal{M}_{\nu}^{\{\nu\}}} B_{\nu}(T) d\nu = B(T) P(\ell) d\ell , \]  

(19)
where

\[ B(T) = \int_0^\infty B_x(T) \, d\nu = (\sigma/\pi) T^4. \]

More generally, we have

\[ \int_0^\infty B_x(T) \, f(\nu) \, d\nu = B(T) \int_0^1 f(\ell) \, P(\ell) \, d\ell, \tag{20} \]

for an arbitrary function \( f(\ell) \).

In view of Eq. (19), the \( M_{de} \)-integrated equation of transfer is

\[ (\mu \ell \frac{\partial}{\partial \ell} + 1) I(x, \ell, \mu) = B(T) \, P(\ell). \tag{21} \]

By applying the same reasoning to the condition (17), we obtain

\[ \frac{1}{2} \int_0^1 \ell^{-1} \, d\ell \int_0^1 d\mu \, I(x, \ell, \mu) = \ell_p^{-1} B(T), \tag{22} \]

where \( \ell_p \) is the Planck mean free path,

\[ \ell_p^{-1} = \int_0^1 \ell^{-1} P(\ell) \, d\ell = [B(T)]^{-1} \int_0^\infty \ell_p^{-1} B_x(T) \, d\nu. \tag{23} \]

Equation (22) is used to express \( B(T) \), whereby Eq. (21) becomes

\[ (\mu \ell \frac{\partial}{\partial \ell} + 1) I(x, \ell, \mu) = \frac{1}{2} \ell_p \, P(\ell) \int_0^1 d\ell \int_0^1 d\mu \, \ell_c^{-1} I(x, \ell_c, \mu). \tag{24} \]

In analogy with Eq. (10) we try
\[ I(x, l, \mu) = P(l) G(x, \xi) + I'(x, l, \xi), \]  
with \( \xi = \mu l \) and with \( I' \) obeying the condition

\[ \int_{|\xi|}^1 l^{-2} I'(x, l, \xi) \, dl = 0, \]  

with leads to similar consequences as before [Eqs. (12) - (15)].

As for \( G(x, \xi) \), we find that it obeys Eq. (7) with

\[ c(\xi) = l P \int_{|\xi|}^1 l^{-2} P(l) \, dl. \]  

Again Eq. (9) is valid. In place of Eq. (13) we find

\[ B(T(x)) = \frac{1}{2} \int_{-1}^1 c(\xi) G(x, \xi) \, d\xi. \]  

This completes the demonstration of mathematical equivalence of the two problems. That is to say, in both cases we are left with the task of solving Eq. (7).

Two special cases may be mentioned. By taking

\[ l = \text{const.} = 1, \]

we are led to the constant cross section approximation of Davison (2) and to the grey case, where \( \xi = \mu \) and \( c(\xi) = 1 \). In the latter case, we have \( P(l) = \delta(l - 1) \) and

\[ I(x, l, \mu) = \delta(l - 1) G(x, \mu), \]

where

\[ G(x, \mu) = \int_0^\infty I(\nu, x, \mu) \, d\nu \]

is the total radiation intensity. Similarly, in the neutron problem, \( G(x, \mu) \) turns out to be the energy-integrated angular flux.
The simplest non-trivial case is given by the picket-fence model of Chandrasekhar (5), where

\[ P(l) = w \delta(l - l_1) + (1 - w) \delta(l - 1), \]  

(29)

so that \( c(\xi) \) is a step function. This model has been treated several times by approximate methods (4,11), and recently also by exact analysis. Solutions of the latter kind have been worked out by Stewart (10) as examples of the ULB model. Siewert and Zweifel (12) developed a different approach, based upon Case's method (13) and a two-group scheme. In an analogous way they used a multigroup procedure for the case with more than two terms on the r.h.s. of Eq. (29).
III. EIGENFUNCTIONS

Several analytical procedures for solving Eq. (7) are available: a method given by Chandrasekhar (1), the Wick-Chandrasekhar method, which in the limit leads to the exact result (10), the Wiener-Hopf technique (14, 15), and Case's singular eigenfunction expansions (13), applied to the present problem by Bednarz and Mika (8).

The latter method can still be somewhat simplified by the use of half-range orthogonality relations, which were first derived for one-speed problems (or the grey case) (16). The generalization of these relations to a non-constant c will be explained in the following and then used in the next section to rederive the solution of Milne's problem.

The eigenfunctions \( \phi(\eta, \xi) \) are introduced by the ansatz

\[
G(x, \xi) = \phi(\eta, \xi) \ e^{-x/\lambda},
\]

whereby the following equation is obtained from (7):

\[
(\eta - \xi) \phi(\eta, \xi) = \frac{\pi}{2} \int_{-1}^{1} c(\xi') \phi(\eta, \xi') \ d\xi'.
\]  

(30)

It is convenient to impose the normalization condition

\[
\int_{-1}^{1} c(\xi) \phi(\eta, \xi) \ d\xi = 1.
\]

Multiplying both sides of (30) by \( c(\xi) \ d\xi \) and integrating we find, in view of Eq. (9), that

\[
\int_{-1}^{1} c(\xi) \phi(\eta, \xi) \xi \ d\xi = 0.
\]  

(31)
This means that the net current (net flux) corresponding to $\phi(\eta, \xi)$ vanishes.

A continuous set of singular eigenfunctions corresponds to the eigenvalues in the interval $-1 < \eta < 1$,

namely

$$\phi(\eta, \xi) = \frac{\pi}{2} \mathcal{P} \frac{1}{\eta - \xi} + \frac{\Lambda(\eta)}{c(\eta)} \delta(\eta - \xi),$$

where $\mathcal{P}$ indicates that the principal value of any integral over $\xi$ or $\eta$ must be taken, and

$$\lambda(\eta) = 1 - \frac{\pi}{2} \mathcal{P} \int_{-1}^{1} \frac{c(\xi)}{\eta - \xi} \, d\xi.$$ 

Furthermore there is one discrete eigenfunction,

$$\phi(\infty, \xi) = \frac{1}{2},$$

corresponding to the double discrete eigenvalue $\eta = \infty$.

The weight function needed in the orthogonality relations for the set $\phi(\eta, \xi)$, $0 < \eta < 1$ and $\eta = \infty$ will be constructed in terms of Chandrasekhar's $H$-function corresponding to the "characteristic function" $\frac{1}{2} c(\xi)$.

A convenient definition is (17)

$$\frac{1}{H(-z)} = \frac{1}{1 - z} \exp\left\{ \frac{1}{2\pi i} \int_{0}^{1} \ln \frac{\Lambda^+(\eta)}{\Lambda(\eta)} \left[ \frac{1}{\eta - z} - \frac{i}{\eta} \right] \, d\eta \right\}, \quad (32)$$

which is derived from related expressions of Chandrasekhar (1) and Case (13), and where

$$\Lambda^\pm(\eta) = \lambda(\eta) \pm \frac{1}{2} i \pi \eta c(\eta)$$

are the boundary values of the function

$$\Lambda(z) = 1 - \frac{\pi}{2} \int_{-1}^{1} \frac{c(\xi)}{z - \xi} \, d\xi.$$
From Eq. (32) the following identities ensue for the H-function and for its boundary values $H^\pm(-\xi)\,$

$$
\lim_{\xi \to 0} H(-\xi \pm i\varepsilon), \quad 0 \leq \xi \leq 1:
$$

$$
\frac{1}{H(-\xi)} = \Lambda(\xi) H(\xi), \quad \quad (33)
$$

$$
\frac{1}{2\pi i} \left[ \frac{1}{H^+(\xi)} - \frac{1}{H^-(\xi)} \right] = -\frac{1}{2} \xi c(\xi) H(\xi), \quad \quad (34)
$$

$$
\frac{1}{2} \left[ \frac{1}{H^+(\xi)} + \frac{1}{H^-(\xi)} \right] = \lambda(\xi) H(\xi). \quad \quad (35)
$$

It is also useful to know the behavior of $H^1(-z)$

for large $z$ (14),

$$
\frac{1}{H(-z)} = -\frac{\beta_1}{z} - \frac{\beta_2}{z^2} - \cdots \quad |z| > 1, \quad \quad (36)
$$

with

$$
\beta_n = \frac{1}{2} \int_0^1 \xi^n c(\xi) H(\xi) \, d\xi. \quad \quad (37)
$$

In particular (1)

$$
\beta_0 = 1, \quad \beta_1 = \left[ \int_0^1 \xi^2 c(\xi) \, d\xi \right]^{1/2},
$$

and (8)

$$
\frac{\beta_2}{\beta_1} = 1 - \frac{1}{2\pi i} \int_0^1 \ln \frac{\Lambda^+(\eta)}{\Lambda(\eta)} \, d\eta. \quad \quad (38)
$$

We may remark that the expression for $\beta_1$ is simplified by substituting the r.h.s. of Eqs. (8) or (27), and inverting the order of integrations. The result is
\[
\beta_1 = \left( \frac{1}{3} l_T^3 l_R \right)^{1/2} \quad \text{for the neutron case,} \tag{38a}
\]
\[
= \left( \frac{1}{3} l_T^3 l_R \right)^{1/2} \quad \text{for the ULB model.} \tag{38b}
\]

Here the transport mean free path and the Rosseland mean free path \( l_r \) are used, which for the two models under consideration are given by

\[
l_{tr} = \int_0^\infty l(E) N(E) \, dE \tag{39a}
\]

and \( l_R = \int_0^1 l R(l) \, dl \),

respectively. The role of these two parameters in the diffusion approximation is well known: \( D = \frac{1}{3} l_{tr} \nabla \) for neutron transport, and \( D = \frac{1}{3} l_R \) for radiative transfer.

Essentially as in one-speed theory \( (16), (19) \), we verify by a straightforward manipulation based upon the identities \( (34) \) and \( (35) \) that a proper weight function for half-range orthogonality is \( \xi c(\xi) H(\xi) \).

(In the one-speed case with \( c = 1 \) this weight is proportional...
to the function \( \psi(\mu) \) used in some of the neutron literature, (16). The orthogonality relations and a set of normalization integrals can be presented in condensed form if the following step function is introduced:

\[
\Theta(\eta) = \begin{cases} 
0 & \text{for } 0 < \eta < 1, \\
1 & \text{otherwise.}
\end{cases}
\]

For any \( \eta, \eta' \in (-1, 1) \), we have:

\[
\int_0^1 \phi(\eta, \xi) \phi(\eta', \xi) \xi c(\xi) H(\xi) d\xi = \eta c(\eta) \Lambda(\eta) \Lambda(\eta) H(\eta) \delta(\eta - \eta') \left[ 1 - \Theta(\eta) \right] 
- \eta' \phi(\eta, \eta' \eta') \Theta(\eta) - \eta \phi(\eta, \eta) \Theta(\eta'),
\]

(40)

\[
\int_0^1 \phi(\eta, \xi) \xi c(\xi) H(\xi) d\xi = -\frac{\eta}{H(\eta)} \Theta(\eta),
\]

(41)

\[
\int_0^1 \xi \phi(\eta, \xi) \xi c(\xi) H(\xi) d\xi = -\eta \beta_1 - \frac{\eta^2}{H(\eta)} \Theta(\eta).
\]

(42)

The half-range completeness theorem of Bednarz and Mika (3) guarantees that an arbitrary function \( \psi(\xi) \), given in \( 0 < \xi \leq 1 \), can be expanded as

\[
\psi(\xi) = \frac{1}{2} a_1 + \int_0^1 A(\eta) \phi(\eta, \xi) d\eta.
\]

(43)

The orthogonality relations facilitate the determination of the expansion coefficients \( a_1 \) and \( A(\eta) \).
For the special case \( \psi(\xi) = \delta(\xi - \xi_0) \), to which we are led in the half-space albedo problem, the following expansion is obtained:

\[
\frac{\delta(\xi - \xi_0)}{E_0 c(\xi_0) H(\xi_0)} = \frac{1}{2} \int_0^1 \frac{\phi(\eta, \xi_0) \phi(\eta, \xi)}{c(\eta) H(\eta)} \, d\eta. \tag{44}
\]

This is a closure relation and represents a concise statement of the half-range completeness property of the eigenfunctions. Mika's proof of this completeness can be substituted by a direct proof of the closure relation, which is carried out along similar lines as the proof of orthogonality.

If for a given problem we are asking for the intensity emerging from the surface of the medium, we must determine the values of an expansion like (43) for negative \( \xi \). To this end, we multiply both sides of (43) by \( \phi(-\xi', \xi) c(\xi) H(\xi) \, d\xi, \, 0 < \xi \leq 1 \), and integrate over \( (0,1) \).

By aid of Eqs. (40) and (41) the "law of diffuse reflection" is obtained in the form \( (12) \)

\[
\psi(-\xi) = \xi^{-1} H(\xi) \int_0^1 \psi(\xi') \phi(-\xi, \xi') \xi' c(\xi') H(\xi') \, d\xi'. \tag{45}
\]

(with \( \xi \) and \( \xi' \) interchanged). Specializing again to the albedo problem, we immediately have an expression for the generalization of Chandrasekhar's S-function. Rewritten in terms of this function, equation (45) assumes a familiar form \( (1) \).

In applications of the relation (45) one must bear in mind that it applies only to functions \( \psi(\xi) \) permitting pure half-range expansions of the form (43).

In all cases (such as slab problems, the Milne problem, and problems involving interior sources) where the
description of the radiation field leads to additional terms in the expansions, those terms must be transferred to the left-hand side, and the resulting sum substituted for \( \psi(\xi) \).

IV. SOLUTION OF MILNE'S PROBLEM

In radiative transfer, we particularly want to solve Milne's problem, which is defined by the boundary conditions

\[
I(0, \mu) = 0 \quad \text{for} \quad 0 < \mu \leq 1, \quad (46)
\]

\[
I(x, \mu) \propto x \quad \text{for} \quad x \to \infty. \quad (47)
\]

The first boundary condition leads to the conclusion that \( I' \) in Eq. (25) now vanishes, and that \( G(0, \xi) = 0 \) for \( 0 < \xi \leq 1 \). We therefore have only to solve Eq. (7) for boundary conditions corresponding to (46) and (47).

We try the expansion

\[
G(x, \xi) = C \left[ (x - \xi) + \ell_x \right] + \int_0^1 A(\eta) \phi(\eta, \xi) e^{-x/\eta} \, d\eta. \quad (48)
\]

The term \( C(x - \xi) \) is a particular solution of Eq. (7), chosen to satisfy the boundary condition at infinity.
The arbitrary constant $C$ obviously is proportional to the net flux. Taking into account Eq. (31), we obtain

$$\Pi' = -2\pi \int_0^1 dl \int_1^\infty I(x, l, \mu) \mu \, d\mu = \frac{4}{3} \pi l_R C. \quad (49)$$

The expansion coefficients in (48) are immediately determined by setting $x = 0$ and applying the half-range orthogonality relations (40) - (42), supplemented by Eq. (37) with $n = 1, 2$. In particular, for the extrapolation distance $l_{\text{ex}}$ we find (3, 14)

$$l_{\text{ex}} = \beta_2 / \beta_1. \quad (50)$$

Once the expansion coefficients are known, $B(T(x))$ is obtained from Eq. (28),

$$B(T(x)) = C \left( x + l_{\text{ex}} \right) + \frac{1}{2} \int_0^1 A(\eta) \, e^{-x \eta} \, d\eta. \quad (51)$$

A comparison with Eq. (49) shows that in the asymptotic region indeed $\Pi' \propto + \frac{1}{3} l_R \text{grad}(4\pi B)$, in accord with a remark in Section III (18).

We are then left with the computation of $I_\gamma(x, \mu)$, a task specific to radiative-transfer problems. The "formal solution" of the equation of transfer provides the answer (1, 6):

$$I_\gamma(x, \mu) = \int_{0, \infty}^x B_\gamma(T(x')) \exp \left( -\frac{x - x'}{\mu l_\gamma} \right) \frac{dx'}{\mu l_\gamma}. \quad (52)$$

The lower bound of the integral is $0$ for $\mu > 0$, and $\infty$ for $\mu < 0$. 
If we are interested solely in the surface values of \( I(x, \ell, \mu) \) and of related quantities, we apply the identity (45) to \( \psi(\xi) = G(0, \xi) + C \xi \). In view of Eq. (42) we have the result (1, 3, 14)

\[
G(0, -\xi) = C \beta_1 H(\xi). \tag{53}
\]

Multiplication by \( \frac{1}{2} c(\xi) \, d\xi \) and integration then leads to an expression for the surface temperature

\[
B(T(0)) = C \beta_1. \tag{54}
\]

In view of the Eqs. (38b) and (49), the ratio of the surface temperature and the effective temperature (defined by \( B(T_e) = F \)) is given by (18)

\[
\frac{T(0)}{T_e} = \left( \frac{3}{16} \frac{\ell^4}{\ell_R^2} \right)^{1/8} \leq \left( \frac{3}{16} \right)^{1/8} = 0.8112, \tag{55}
\]

with the latter value applying only to the grey case (1).

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