JOINT INSTITUTE FOR LABORATORY ASTROPHYSICS

PROCEEDINGS

of the

WORKSHOP ON THE INTERDISCIPLINARY ASPECTS OF RADIATION TRANSFER

edited by

R. Goulard

held at the

University of Colorado
on 11 and 12 February 1965

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This Workshop was supported by the office of Naval Research under Grant Nonr (G) 00041-65
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INTRODUCTION

A number of traditionally distinct research fields (astrophysics, meteorology, etc.) share a number of common radiation transfer techniques in both theory and experiment. Hence the often heard statement that "all applications of radiation transfer are actually the same."

The purpose of this Workshop was to probe into this somewhat sentimental comment and to see whether or not the experience accumulated in one field was adaptable to another. State of the art presentations were given on ten distinct specialized areas of application, and a discussion followed each presentation. The table of contents reflects the chronological order of the meeting.

Since spontaneous communication was an essential feature of the Workshop, this record of the presentations and especially of the discussions may not have the polished appearance that an author would wish to produce in a printed publication. For this reason we have arranged for a limited distribution only, any published reference to statements made by a participant should only be made after consultation with him.

The taxing jobs of recording the Workshop sessions and transcribing the recordings were efficiently handled by Mr. Stuart Jordan and Mrs. Janine Enke respectively. The assistance of the Bureau of Continuation Education of the University of Colorado in attending to the needs of the Workshop and of its fifty participants is gratefully acknowledged.

We are also grateful for their ready support to the Office of Naval Research. Not only did it provide the necessary funds for the Workshop, but the very concept and realization of this meeting is due in great part to the enthusiasm of Mr. Morton Cooper of the ONR Fluids Dynamics Branch.
Finally, the constant attention of Mrs. R. J. Low to the organizational aspects of the Workshop is to be credited for the efficient coordination of the efforts of all.

June 28, 1965

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SOME ASPECTS OF RADIATIVE TRANSFER DURING
HYPER VELOCITY REENTRY

by

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The relative importance of radiative energy transfer from the shock heated
gas with respect to convective heating is a strong function of the flight
velocity. It is also a function of the flight altitude. The flight velo-
city and altitudes at which radiative energy transfer becomes important are
shown in Fig. 1. It is seen that radiative energy transfer is not important
for reentry from satellite orbits. It becomes important only at the higher
entry velocities such as those associated with a lunar return and Mars return
entry trajectory. The initial entry velocity for a lunar return trajectory
is approximately 37,000 feet per second. At this entry velocity the shock
layer gas is slightly ionized and radiative heating becomes significant rela-
tive to the convective heating when the reduction of convective heating by
mass injection is considered. At the higher entry velocities required for a
manned Mars mission, the radiative heating can become predominant. It can
be orders of magnitude larger than the convective heating, depending on the
vehicle's shape, flight velocity, and altitude. In this flight regime the
shape of the body will be primarily dictated by the requirement that the
radiative heating be maintained at a tolerable level. This is the flight
regime where radiative heating is important and is the subject of the present
paper.

The basic equations which determine the flow around the reentry vehicle are
the usual conservation equations which we can write in an approximate form
as follows:
\( x \)- mom.

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \rho \left( \mu \frac{\partial u}{\partial y} \right) + O\left( \frac{1}{\sqrt{Re}} \right)
\]

\( y \)- mom.

\[
\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + O\left( \frac{1}{\sqrt{Re}} \right)
\]

energy

\[
\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \rho c_s \left( \frac{\partial P}{\partial x} + \frac{\partial u}{\partial y} \right) + \rho \left( \frac{\partial u}{\partial x} \right)^2 - \frac{\partial P}{\partial y} + O\left( \frac{1}{\sqrt{Re}} \right)
\]

\[
\frac{\partial Q_k}{\partial y} = \int_0^\infty \left\{ \int_0^\infty \mathcal{E}_0 \mathcal{E}_1 (1 - \zeta) d\zeta - 4 \mathcal{E}_0 (\zeta) \right\} dy
\]

cont.

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0
\]
The coordinate system used is shown in the accompanying sketch. The radiation gradient term which appears in the energy equation can be treated in various ways depending on the assumptions made about the optical depth of the shock layer gas.

The classical approach for determining the radiative and convective heating to the surface is to first separate the shock layer flow into an inviscid and a viscous region. The viscous region is the boundary layer which is confined to a region very close to the body. The inviscid equations are then solved to obtain the flow field in the entire shock layer assuming the boundary layer to be vanishingly thin. This inviscid solution is then evaluated at the surface which provides conditions for the boundary-layer solution. This classical approach breaks down if there is transfer of energy by radiation and if it is also assumed that the gas is optically thin. This point can be illustrated by examining simple solutions to the inviscid equations such as that obtained by T. Y. Li (Ref. 1) for the stagnation point flow of a sphere.

According to Li, the velocity component normal to the body surface for \( \bar{\rho} \ll 1 \) is

\[
\frac{\bar{V}}{\bar{c}_e} = -\bar{\rho} \left( \frac{\bar{y}}{\bar{c}_e} \right)^2
\]

The time for a fluid particle to travel from the shock wave along the stagnation streamline to a point \( y \) is

\[
\tau = \frac{\bar{z}}{\bar{c}_e} \left( \bar{y} - \bar{y}_0 \right)
\]

from which it is seen that

\[
\tau \rightarrow \infty \quad \text{as} \quad y \rightarrow \infty
\]

If it is now assumed that the fluid particle radiates as long as it has a finite temperature, the fluid particle will lose all of its energy by radiation as it moves from the shock wave to the body surface. Since the stagnation streamline is also the body streamline, the enthalpy at the wall will also be zero. This means that the enthalpy driving potential for convective heating is zero. We thus come to the conclusion from classical boundary layer theory that the convective heating is zero which is, of course, incorrect.
We can illustrate this point in another way by considering a simple solution to the energy equation for the one dimensional flow of a radiating gas near the stagnation point. The energy equation for an emitting-absorbing gas is

$$\rho v c_p \frac{dT}{dy} = \left[ -4\pi T^4 + \int_0^\infty 2\pi \sigma T^4 \varepsilon(R T - \varepsilon) \, dR \right]$$

which for a non-absorbing gas reduces to

$$\rho v c_p \frac{dT}{dy} = -4\pi T^4$$

Now it is assumed that

$$\rho v c_p = ay^2$$

for which there is some justification from Li's results (Ref. 1).

Using the above relation we can obtain the following expression for the temperature profile in the shock layer

$$T = \left[ \frac{12\pi}{a} \left( \frac{1}{y} - \frac{1}{y'} \right) + \frac{y'}{y^3} \right]^{-1/3}$$

This result shows that

$$T \to -\infty \quad \text{as} \quad y \to 0$$

This anomalous behavior of the temperature profile near the wall is due to the optically thin assumption as was shown by Thomas and Chisnell. Both Thomas and Chisnell have obtained solutions to the energy equation with the self-absorption term included. Typical results from Ref. 3 for the enthalpy profiles are presented in Figs. 2 and 3. Fig. 2 is for a nearly optically thin gas ($\zeta_i \approx 0$) while Fig. 3 is for a slightly absorbing gas ($\zeta_i < 1$). Note that the results for a nearly optically thin gas, Fig. 2, show a non-zero wall enthalpy and the existence of large gradients near the wall. The parameter $\Gamma$ is a measure of the radiation cooling effect and is defined as follows:

$$\Gamma = \frac{\text{energy loss in time } \delta/v_j}{\text{total energy}}$$
The existence of the large enthalpy gradients near the wall for the conditions shown in Fig. 2 imply a coupling between the inviscid flow field and the boundary layer since gradients in the inviscid flow are comparable to those in the boundary layer. For larger optical depths (Fig. 3) the wall enthalpy is well defined and the gradients at the wall are much smaller than in the previous case. There still may be a coupling between the inviscid flow and the boundary layer due to the enthalpy gradients but this is probably a second-order effect. This coupling between the inviscid flow and the boundary layer is a function of the Reynolds number and the optical depth of the shock-layer gas. The Reynolds number is a measure of the inviscid enthalpy gradient at the wall. When the Reynolds number is small, the viscous region can extend sufficiently far out into the shock layer to completely swallow up that part of the inviscid flow where the enthalpy gradients are large. The large enthalpy gradients in the inviscid flow are due to radiation cooling and this inviscid flow region where the gradients are large is referred to as the radiation cooled layer.

In addition to the coupling between the inviscid flow and the boundary layer created by the radiation cooled layer, we can identify two additional coupling effects.

1) The coupling between the loss of energy by radiation and the velocity, temperature, and density profiles in the inviscid region of the shock layer.

2) The inherent coupling between the inviscid and viscous region due to energy loss along those streamlines in the inviscid flow which subsequently enter the viscous flow region.

The first coupling effect is created by the decrease in the total enthalpy of a fluid particle by radiation loss as it travels around the body. The cooling of the flow by radiation-loss increases the density of the shock-layer gas causing the bow shock wave to move in closer to the body. This change in the shock wave position is illustrated schematically in Fig. 4. Although the shock detachment distance can be altered significantly, the shock shape (angle) is much less sensitive to radiation cooling. Hence, the pressure field will be
little affected while the density and velocity will be increased and the
temperature decreased by radiation cooling. Although this coupling is most
prominent in the inviscid-flow region it is still present in the viscous
region. The viscous region, however, is dominated by molecular transport
processes.

The second coupling effect arises because a fluid particle can lose energy by
radiation as it traverses the inviscid region of the shock layer prior to
entering the viscous region. Since the radiation loss along streamlines de-
creases away from the stagnation point (Fig. 4), the total enthalpy of a
fluid particle entering the viscous region will vary along the body. The
total enthalpy at the edge of the viscous region will be a minimum at the
stagnation point. This variation of the edge total enthalpy requires that
the analysis of the viscous region be of a nonsimilar nature. The fluid par-
ticles entering the viscous region have a "past history" and in this sense
the inviscid and viscous flow fields are coupled.

All of the above coupling effects can be accounted for by assuming the entire
shock layer to be a viscous, radiating medium. This is the approach adopted
by Hoshizaki and Wilson (Ref. 4) in their investigation of the convective and
radiative heating to superorbital entry vehicles.

The problem considered in Ref. 4 was to determine, for a given body shape, the
structure of the shock-layer flow field when the shock layer is both viscous
and radiating.

In their analysis an integral method was employed. The governing equations
were first simplified by assuming the shock layer to be thin, i.e., \( \frac{C}{E} \ll 1 \).
In addition, the viscous region is assumed to extend out to the shock wave
which defines the lower limit in Reynolds number. The shock wave itself is
assumed to be a discontinuity. The simplified equations are then integrated
across the shock layer to obtain integro-differential equations in one inde-
pendent variable. The integrals are evaluated by assuming that the velocity
and enthalpy profiles in the shock layer can be represented by suitable poly-
nomials.
The convective and radiative heat transfer distributions on a hemisphere obtained by this method are presented in Fig. 5. Both the convective and radiative heat transfer are reduced by the radiation cooling effect. This reduction in heat transfer is seen to persist around the hemisphere for both convection and radiation. Additional details and numerical results are presented in Ref. 4.

The discussion up to this point has been limited to transparent or grey gases. Self absorption within the shock layer will significantly reduce the loss of energy by radiation as well as the radiative flux to the body. In order to obtain some quantitative results, a few simple numerical calculations were performed. The radiative flux through the shock wave and to the wall were determined for specified shock-layer enthalpy profiles. The shock detachment distance and the pressure and temperature behind the shock were also specified. The radiative fluxes were calculated under the following three conditions:

1. optically thin shock layer
2. grey gas shock layer
3. emitting and absorbing shock layer

The absorption coefficients used in these calculations were obtained by empirically modifying the simple results of Penner and Thomas (Ref. 5). The empirical modifications are justified on the basis that the resulting equations are in better agreement with the more exact calculations of Armstrong (Ref. 6). A sample comparison of the absorption coefficients for nitrogen as obtained from the modified Penner-Thomas results and the exact calculations of Armstrong is presented in Fig. 6.

The enthalpy profiles used in the calculation are presented in Fig. 7. The parameter $c^*$ is merely a form parameter and has no other particular significance. Small values of $c^*$ result in static enthalpy profiles which are characteristic of stagnation point profiles while large values of $c^*$ give profiles similar to those found on the afterbody.
The radiative flux was calculated for the following cases:

Shock detachment distance, \( s = 1, 10 \text{ cm} \)

\[
\begin{align*}
\text{P} &= 1 \text{ atm} \\
T_s &= 11,400 \degree \text{K}, \\
\text{P} &= 0.1 \text{ atm} \\
T_s &= 10,000 \degree \text{K}
\end{align*}
\]

Ratios of the energy flux with self absorption to the energy flux without self absorption are shown in Fig. 8 for \( s = 10 \text{ cm} \). The curve labeled gray gas was obtained by making a gray gas approximation and by using the Planck mean absorption coefficient. The curve labeled spectral was obtained by a spectral integration of the absorption coefficient.

For these conditions, it is seen that the gray-gas approximation results in practically an optically thin gas in that there is little reduction in the radiative flux to the surface by self absorption. On the other hand, a spectral integration over frequency results in a substantial reduction in the radiative flux by self absorption. It is also interesting to note that for small values of the form parameter, \( \Theta^* \), there is little effect of pressure (or density) since most of radiation comes from the upper portions of the shock layer and the uv is absorbed for both pressure levels. For large values of the form parameter, the majority of the radiation comes from the region near the wall. At a pressure level of 1 atm, all of the uv is still absorbed, although the optical path length from the maximum temperature point in the shock layer to the body is considerably reduced from the cases where \( \Theta^* \) is small. At a pressure level of 0.1 atm, the uv radiation is beginning to be transmitted to the wall and the radiative flux is increased over the \( P = 1 \text{ atm} \) value.

Similar results for \( s = 1 \text{ cm} \) are presented in Fig. 9. In Figs. 8 and 9, the radiative flux is that to the wall. Fig. 10 is a comparison of the radiative flux to the wall and through the shock wave. For small values of the form parameter, there is a greater reduction by self absorption of the wall flux.
than the shock flux since the cool layer near the wall does most of the absorbing. For large values of the form parameter, the opposite is true. There is more absorption between the maximum enthalpy point and the shock wave than by the cool layer near the wall. In Fig. 11, the monochromatic intensity as a function of frequency is presented. The monochromatic intensity show clearly the effects discussed above.

**NOTATION**

- $C_p$: complete specific heat of mixture
- $D_i$: diffusion coefficient
- $E_{0i}$: blackbody emissive power
- $e^r$: exponential function
- $h$: complete static enthalpy of mixture, or Planck's constant
- $I_\nu$: specific intensity
- $k$: coefficient of thermal conductivity
- $\rho$: pressure
- $Pr$: Prandtl number
- $q_c$: convective heat flux
- $q_r$: radiant heat flux
- $r$: distance from axis of revolution to surface of body
- $Re$: Reynolds number
- $t$: dummy variable of integration for $\tau_\nu$ and $\tau$
- $T$: temperature
- $T_0$: surface temperature
- $u$: $x$-component of velocity
- $v$: $y$-component of velocity
- $z$: distance along surface
NOTATIONS (cont'd)

\( g \) distance normal to surface
\( \hat{f} \) shock detachment distance
\( \mu \) viscosity, also absorption coefficient
\( \omega \) frequency
\( \rho \) density
\( \bar{\rho} \) density ratio across shock wave
\( \kappa \) Boltzmann constant
\( \tau \) monochromatic optical thickness

REFERENCES:


6. Armstrong, B., private communication
Fig 4. Flow field schematic.
Figure 5: Convective and Radiative Heat Transfer Distribution on a Sphere, $u_\infty = 50,000$ fps, $h = 170,000$ ft, $k = 5$ ft.
Fig. 6 Continuum Absorption Coefficients of Nitrogen
\[
\frac{h}{h_0} = \eta^{1/\theta^*} + \frac{\theta^* - 1}{10} \sin (a\eta^{1/\theta^*} + b\eta^{1/2\theta^*})
\]

\[
b = \pi \frac{\left(1 - \frac{1}{2} \eta^{-1/\theta^*}\right)}{1 - \eta^{-1/2\theta^*}}, \quad a = \pi - b
\]

\[
\tilde{\eta} = \frac{1}{2.9 + 0.1\theta^*}
\]

Fig. 7 ENTHALPY PROFILES
FIG B  RATIO OF RADIATIVE FLUX WITH AND WITHOUT SELF-ABSORPTION
FOR A SLAB OF GAS 10 cm THICK

GRAY GAS

\( \frac{\phi}{\phi_0} = 1.1 \times 10^{-2} \)

\( \rho = 0.2 \text{ atm}; \ T_0 = 14400 \text{ K} \)

\( \frac{\phi}{\phi_0} = 1.2 \times 10^{-3} \)

\( \rho = 0.1 \text{ atm}; \ T_0 = 15200 \text{ K} \)

\( \phi_0 = 12,500 \text{ BTU/ft}^2 \text{ hr} \)

SPECTRAL

STATIC ENTHALPY FORM PARAMETER \( \alpha^* \)
Fig. 9. Ratio of radiative flux with and without self-absorption for a slab of gas 1 cm thick.

Gray Gas

\[ P = 1 \text{ atm, } T_s = 11,400^\circ K \]

\[ P = 2 \text{ atm, } T_s = 10,000^\circ K \]

\[ h = 12,500 \text{ Btu/lb} \]

Spectral
FIG 10 RATIO OF RADIATIVE FLUX WITH AND WITHOUT SELF-Absorption FOR A SLAB OF GAS 1 CM THICK $P = 1$ ATM
TEMPERATURE BEHIND SHOCK = 11,400 °K, $l_1/\sigma = 1.1 \times 10^{-2}$

GREY GAS

SHOCK

WALL

SPECTRAL

$S = 10$ CM (WALL)

STATIC ENTHALPY FORM PARAMETER, $\theta^*$
MONOCHROMATIC INTENSITY AS A FUNCTION OF FREQUENCY, $h\nu$

- $\theta^* = 1.2$
- $\theta^* = 2.0$

$\frac{S'}{S} = 1.1 \times 10^{-2}$
$T_e = 11,400 \, ^{\circ}\text{K}$
$S = 1\, \text{cm}$
$P = 1\, \text{atm}$

WALL
SHOCK

MONOCHROMATIC INTENSITY, Watts/cm$^2$-ev-s

FREQUENCY, $h\nu$ (ev)

$A^* \times 10^{-3}$
DISCUSSION

Chairman: Have these theoretical results been checked experimentally?

H. H. Hoshizaki: We are running an experimental program using a shock tube technique. We measure the intensities from a uniform slab of gas, in a range of densities and temperatures representative of the Fire project. As you know, project Fire is a test vehicle for the Appolo program.

We are looking at the gas in the reflected shock region and we vary the beam path by using baffles. We have looked at the gas with and without a window. The quartz window we use cuts off at about 2000 Å; if you refer to Fig. 6, you see that all the vacuum UV is cut off by this window. In both cases, we measure the emerging intensity by means of a thin film gauge coated with carbon.

We recently obtained two sets of results at 15,000° K, and 1.3 x 10^-2 normal density. They show a considerable increase in intensity when the window is removed, as expected. However, the intensity measured without window is not nearly as large as the one calculated on the basis of an optically thin gas.

J. C. Steward: What data did you use to calculate your optically thin intensity?

H. H. Hoshizaki: We used Armstrong's recent data. If we now multiply this optically thin predicted intensity by the proper calculated self absorption ratio (of the type illustrated in Figs. 8 and 9), we find a reasonable agreement between theory and experiment.
The important point we have thus experimentally verified is that a good deal of self absorption exists for air in the range of properties considered, even though the path lengths are small (\( \sim 1 \) cm.) and the grey gas calculations (Planck mean) predict thin gas.

**Chairman**: Is this true for other flight conditions as well?

**H. H. Hoshizaki**: Yes. We ran another test at Mach 3.1 (\( \sim 17,000^\circ \) K) and higher pressure (200 \( \mu \)Hg). In that case we saw very little difference with and without a window, which indicates that self absorption is cutting out all the vacuum UV. There is almost an order of magnitude reduction from the optically thin case in this case, which shows that the self absorption is extremely important.

**S. C. Traugott**: Do you have thermodynamic equilibrium in all cases?

**H. H. Hoshizaki**: Yes.

**E. M. Sparrow**: What would happen in the case of an actual detached shock layer, where you have shown earlier that the density varies between the shock and the wall?

**H. H. Hoshizaki**: In the case I have shown, the density immediately behind the shock was about \( 10^{-3} \) atm. and it was about \( 10^{-2} \) atm. at the wall; it naturally increases in the viscous region. Hence, the self absorption effect will appear first near the wall.

**S. C. Traugott**: How would ablation from the wall change these results?
H. H. Hoshizaki: The existence of a highly radiation absorbing boundary layer would create two opposite effects. On the one hand, some of the radiation emitted by the inviscid shock layer would be absorbed there and thus be prevented from reaching the wall; on the other hand, the energy stored in the boundary layer would increase and a stronger enthalpy gradient would be created at the wall with a correspondingly higher conduction.

We have therefore to expect a trade off between radiative and convective fluxes. The actual computation of these effects is a heavily coupled mathematical problem.

S. C. Traugott: There has been a report by Howe* that shows that you might conversely increase the radiation transfer and decrease the convective transfer at the wall by injecting a cool, highly absorbing gas into the boundary layer, with a small overall reduction of heat transfer.

H. H. Hoshizaki: That is the general conclusion I have also come to; you cannot reduce heat transfer significantly without mass injection.

D. H. Sampson: You seem to have used molecular band radiation in your calculations (Fig. 5) while your shock tube experiments seem to indicate you operate in a region where line emission is predominant.

H. H. Hoshizaki: I would expect that at high temperatures and pressures, line radiation is going to self absorb and won't get out. To really take it in account in the other cases would be very difficult, each line would have to be treated separately. Biberman has made a calculation showing that the results might differ by a factor of two.

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D. H. Sampson: The range of your experiments (around 15 000° K and
density 10^{-3} atm.) is quite interesting because this is the temperature
range where the various theoretical models show the largest discrepancy:
1.1/2 to 2 ev.

Chairman: Is there any other work to be reported on the aerospace applications?

S. Scala: I would like to report two current efforts carried out at the
Space Sciences Laboratory of General Electric.

One is the work of Dr. Warren and his group on experimental
determination of the high temperature properties of air. We use, as do
the Lockheed Laboratories (Dr. Hoshizaki's earlier comments) a shock tube
device, but we look at the incident shock by means of a "shock holder"
rather than at the reflected shock. Also, instead of a quartz window, we
have a "gas" window, which is an interface with the outside maintained at
the same pressure as the inside. Also we use a cavity gage for the intensity
measurement.

On the theoretical side, we have made studies of reentry flow
fields over a wide range of Reynolds numbers. If you are in the high
Reynolds number regime, the heat transfer rate is moderate and the pressure
gradient normal to the surface is still relatively small; none of the
contaminates will get out into the higher temperature region because the
boundary layer is cooling the gas. But as one goes to lower Reynolds numbers,
then the entire region between the surface and the shock wave is viscous.
So even though the ablation products remain within this viscous layer, this
means that they fill the whole space between the shock and the surface.
Similarly, this extremely high mass transfer rate, even in the high Reynolds
number regime, will produce a large pressure gradient normal to the surface. The boundary layer concept breaks down again, and you have to use a low number Reynolds number equation and possibly the complete Navier Stokes equation.

Recently we have been successful in getting a solution to time-dependent Navier Stokes equations where we include compressibility, viscosity, and thermal conductivity. We are in the process now of coupling into this the radiative transfer equations. These solutions are done on a computer. But we do not have anything to report about this at this point.

H. H. Hoshizaki: Is that just for the blunt body?

S. Scala: It is a one-dimensional problem so it could be in spherical coordinates, cylindrical coordinates, or it could be planar coordinates. The equations are essentially the same as long as we have time determinates and one independent spatial variable. It turns out that it is quite a tricky thing to do even though it is a numerical computation.

S. C. Traugott: In what situation is the unsteady approach important?

S. Scala: For many classified and unclassified situations. In stellar dynamics, time dependence would be quite important. Similarly, if an external radiation field is suddenly applied to a reentry vehicle, time dependence becomes important.
SOME PROBLEMS OF RADIATION TRANSFER IN QUIESCENT STELLAR ATMOSPHERES

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I. Introduction

In one sense, this is a strictly classical review paper -- it is concerned wholly with problems of radiation transfer in a quiescent gaseous medium. By contrast, in a symposium such as this which brings together, among others, aerodynamicists as well as astrophysicists, one naturally asks the inter-relation between velocity and radiation fields, particularly from the standpoint of spectroscopic diagnostics. But that aspect, we defer until tomorrow, in Whitney's presentation. Here, I would only outline for you the groundwork of some new developments -- or rather, of a new outlook -- over the strictly classical theory of radiation transfer in quiescent media, which must be considered first, before proceeding to the differentially-moving media, in order to orient your physical ideas. In this sense, the paper is not at all classical. We discard the usual ideas of emission and absorption of radiation following the behavior in a thermodynamic enclosure, where the radiation field represents only a negligible perturbation on the state of the matter in the enclosure. This is only a short summary, because a rather full account will be published by the Colorado University Press in August of this year, under the title: "Some Aspects of Non-Equilibrium Thermodynamics in the Presence of a Radiation Field," written by me. An earlier, less-complete account of the methodology, but applied to the stellar atmosphere, can be found in the monograph: "Physics of the Solar Chromosphere" by R. G. Athay and myself.
Mainly, the developments that I summarize today were motivated by their application to stellar atmospheres. Now I recognize that to many of you, not only are these problems of radiation transfer in stellar atmospheres not completely familiar, but also stellar atmospheres are somewhat strange, relative to your laboratory experience. So let me emphasize to you that the chief problem astronomers have in describing a stellar atmosphere lies in the fact that we have only that information on them which is provided by spectroscopic observations. A priori, we know neither their composition nor their physical state: the analysis must determine each. Also, we are not always familiar with the way to describe the state. For example, where there are velocity fields in stellar atmospheres, we generally have conditions quite unlike anything studied in the laboratory; and we try to learn about the physics of such motions as well as their effect on the state of the atmosphere. Consequently, it is usually the case that astronomers study stellar atmospheres by building a model, based on a certain set of assumptions, which are initially chosen as the simplest that the astronomer thinks he can get away with. Then he modifies them, increasing their complexity, very grudgingly, and only when inconsistency of several observations as interpreted on the basis of this model forces him to do so.

Consequently, there are two points which I must make clear to you. The first point concerns the kind of models that astronomers make to represent an assumed quiescent stellar atmosphere. We ask how physically-consistent are the assumptions defining such a model, quite apart from whether there is any object in the universe actually mimicking such a model. Actually, the whole aspect of radiation transfer which I wish to summarize for you today hangs on this point of physical consistency. The second point is the question whether there actually are any stars for which these models are a good
representation. This is mainly the subject which Whitney will discuss
tomorrow. Early in stellar astrophysics, it was most convenient -- because
of our low level of knowledge of almost everything --- to assume that there
were actually stars having no velocity fields, so that quiescent models
could be introduced. Later it turned out that there are stars for which
this is a good approximation; it also turned out that this is not at all
the case with other kinds of stars. Whitney will consider these points
tomorrow. Today, we concentrate on quiescent models of stellar atmospheres,
and ask about their physical consistency, especially with regard to the
problems of radiation transfer.

Classically, the quiescent models of stellar atmospheres are defined
by three assumptions:

1. Hydrostatic equilibrium

2. Radiative equilibrium, RE; viz, no energy transfer other
   than by radiation

3. Local thermodynamic equilibrium, LTE; viz, all distribution
   functions for microscopic parameters, except that for photons,
   are given by their thermodynamic equilibrium forms, fixed by
   the local kinetic temperature of the electrons. The photon
   distribution function is fixed by the assumption:

\[ \text{emissivity/absorptivity} = B_\nu(T_e) \]  \hspace{1cm} (1)

This together with a solution of the equation of radiative
transfer over the atmosphere.

We can write the equation of radiative transfer in a plane-parallel atmosphere:

\[ dI_\nu = (- I_\nu K_\nu + e_\nu)ds \]  \hspace{1cm} (2)
where \( K_\nu \) is the volume absorptivity; \( e_\nu \), the volume emissivity; and \( I_\nu \), the specific intensity of radiation (ergs cm\(^{-2}\) sec\(^{-1}\) frequency\(^{-1}\) crossing unit area normal to \( I_\nu \)). \( \mathbf{s} \) is the arbitrary direction studied. Sometimes it is convenient to define some standard reference direction, which we usually take as the normal to surface, in which case we write

\[
ds = dr / \cos \theta = \mu^{-1} \, dr \quad (3)\]

where \( \theta \) is the angle of the considered direction with the normal to the surface, \( r \), and \( \cos \theta = \mu \). Then if we define

- **optical depth:**
  \[
d\tau_\nu = -K_\nu \, dr \quad (4)\]

- **source-function:**
  \[
  S_\nu = e_\nu / K_\nu \quad (5)\]

we can write the equation of radiative transfer in the form usually used by astronomers:

\[
\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \quad . \quad (6)
\]

Then we see that the assumptions (2)-(3) become:

- **RE:**
  \[
  \frac{d}{ds} \int F_\nu \, d\nu = 0 = \iint K_\nu I_\nu \, dr \, d\mu - \iint K_\nu S_\nu \, d\nu \, d\mu \quad , \quad (7)
  \]

- **LTE:**
  \[
  S_\nu = B_\nu(T_e) \quad , \quad (8)
  \]

where \( F_\nu \) is the local flux

\[
F_\nu = \int_\mu I_\nu \, d\mu \quad . \quad (9)
\]

We see that the LTE assumption removes the necessity to consider any microscopic details of processes happening in the atmosphere. Inserting Eq. (8) into Eq. (7), we see that given the \( \nu \)-dependence of \( K_\nu \), we have one equation that gives \( T_e \) in terms of \( I_\nu \) at each point in the ensemble.
And we see that we have the formal integral of Eq. (6), viz

\[ I_\nu(t_\nu, \theta) = \int_{t_\nu}^{\tau_\nu_{\text{max}, \theta}} B_\nu(T_e) e^{-|\tau_\nu - t_\nu|/\mu} \, d\tau_\nu/\mu, \quad (10) \]

giving one more relation between \( I_\nu \) and the distribution of \( T_e \) over the atmosphere. Thus, quite apart from any microscopic considerations of anything but the \( \nu \)-dependence of \( K_\nu \), we obtain what astronomers call the model of the atmosphere, \( T_e(\tau_\nu) \). We can convert this distribution of \( T_e \) in terms of optical depth, into a distribution of \( T_e \) in terms of geometrical depth, by using the relation (4). The use of this relation demands a knowledge of \( K_\nu \), which is a microscopic parameter. But again, the LTE assumption permits us to compute the concentrations of all microscopic atomic species at a point, in terms of the macroscopic density and value of \( T_e \) at a point, without asking any details of microscopic interactions. Thus we can say that the assumptions of RE and LTE permit us to reduce the problem of the model of a stellar atmosphere to a macroscopic, rather than microscopic, problem. The primary question we pose, is the validity of such a reduction. So in a phrase, you can summarize the guiding motive of the questioning of the logical consistency of the classical models of stellar atmospheres, which has been going on this past decade and a half, as a questioning of the validity of the dropping of a microscopic approach to stellar astrophysics in the stellar atmosphere. I outline here for you the development of this questioning thus far.
II. A Microscopic Approach to the Question of the Validity of LTE

For the moment, we put aside the question of the validity of the assumption of radiative equilibrium. Suffice it to say, that all the phenomena of the solar chromosphere-corona represent departures from this condition of RE, coming from the presence of aerodynamic dissipation mechanisms. Such things properly belong in the domain of Whitney's talk tomorrow. But the language we require to discuss such problems, must be developed from an inquiry into the validity of the LTE assumption. So today, we concentrate on the validity of LTE.

If we are to ask the validity of the assumption (8), which removes the microscopic approach, the most direct way to proceed is to formulate everything in microscopic terms, and then ask what (8) implies. So first of all, we obtain a microscopic expression for \( S \). Today, we concentrate on the situation in the line-spectrum, because it is more simple to describe, and because it is easily generalized to include the continuum. Then we can write microscopic expression for the emission and absorption of radiation, which give \( e \) and \( K \), as follows, using the Milne-Einstein coefficients to describe the processes:

\[
K = - n_L \phi \ L_U + n_U \psi \ U_L \tag{11}
\]

where the \( n \) are occupation numbers of the two energy levels involved in producing the spectral line, which we call lower, L, and upper, U. The \( B \) are the Milne-Einstein coefficients. The first term represents absorption and the second term represents stimulated emission, in Eq. (11). The coefficients \( \phi \) and \( \psi \) are what we call profile-coefficients, such that, for example, \( n_L \phi \) represents the number of atoms per cm\(^3\) capable of absorbing
at frequency \( \nu \). Likewise we have:

\[
e_{\nu} = n_U j_{\nu} A_{UL} \quad . \tag{12}
\]

Then we see that we can write

\[
S_{\nu} = \frac{2h\nu^3}{c^2} \left\{ \frac{n_L g_U}{n_U g_L} - \frac{\nu}{\phi_{\nu}} \right\}^{-1} j_{\nu} / \phi_{\nu} \quad . \tag{13}
\]

We have used the familiar relations

\[
\frac{A_{UL}}{B_{UL}} = 2h\nu^3 / c^2
\]

and

\[
g_L B_{LU} = g_U B_{UL} \quad .
\]

If we had \( j_{\nu} = \phi_{\nu} = \psi_{\nu} \), we would be able to write:

\[
S_{\nu} = B_{\nu} (T_{ex}) \quad , \tag{14}
\]

where \( T_{ex} \) would be defined by requiring the two energy levels to have a Boltzmann distribution at the temperature-parameter \( T_{ex} \). If all energy levels had the same \( T_{ex} \), we would of course have LTE.

Now in the summary today, I put aside the question of the values of the profile coefficients, and set them equal to unity. It turns out this is a good approximation, that can be justified by microscopic inquiry into what fixes their value (it turns out to be the thermal velocity fields, in the regions of the line-center). But today, in the short time available, I want to concentrate on the essential question of what fixes the value of \( T_{ex} \). So we see that our basic problem is simply to compute the ratios of the occupation numbers, in the actual atmosphere.

Now the one thing we know about stellar spectra, is that their time-scale of variation is very, very large. The spectra remain the same
for years, except in such cases as novae, and even there the time-scale is long compared with that for atomic interaction processes (relaxation and other processes) to occur. So we introduce as the condition to fix the occupation numbers, the condition of a time-steady state for the spectrum produced by the atmosphere, which means time-steady values for the occupation numbers. So our condition is

$$\frac{d n_k}{dt} = 0, \text{ for all } k$$  \hspace{1cm} (15)$$

Eqs. (15) are simply a set of algebraic equations, involving the microscopic interaction rates of passing from one energy level to another. There are of course just two kinds of rate-processes: radiative processes, and inelastic collisional processes. The parameter describing the first of these is the radiation field. The parameter describing the second, is the velocity field. In the simple case studied here, we assume the velocity field to be described by one kinetic temperature, the same for all species present, thus $T_e$. We do not ask what fixes the value of $T_e$ at each point, this question involves the RE, or its alternative, assumption. So we simply assume that we are given $T_e$ and $n_e$ at each point in the atmosphere.

When we come to the radiation field, we have a different kind of problem. If we were to assume that we were given $I_\nu$ at each point in the atmosphere, we would have the problem of insuring consistency between the values of $B_\nu(T_{ex})$ computed from the Eqs. (15), and the value of $I_\nu$ computed from the generalized version of Eq. (10), in which we use $B_\nu(T_{ex})$ instead of $B_\nu(T_e)$ following assumption (8), which we wish to test. Thus, we must at present keep $I_\nu$ as a parameter whose value is to be determined. So we wish to solve Eqs. (15) to determine the value of each of the $n_k$, thus the value of $B_\nu(T_{ex})$, as a function of the various parameters involved.
Now generally, Eqs. (15) are an infinite set of equations, since an arbitrary ion has an infinite set of energy levels. If we take into account the fact that those energy levels near the continuum are effectively populated as though they were in equilibrium with the continuum, since an electron has difficulty knowing if it is free or bound for small binding energy, we can in all practical cases reduce this to a finite set of equations. But even so, the solution must generally be done numerically, because of the large number of parameters. Considerable insight has been gained by considering simplified models of atoms, in which only a few energy levels are considered. So today, let me illustrate the physical points by restricting attention to a two-level atom; viz, an atom having two discrete energy levels and a continuum.

Writing down explicitly the two Eqs. (15), in this special case, we have

**level L:**
\[ n_L(C_{LU} + C_{LC} + B_{LU} \int \bar{I}_\nu \phi_\nu \, d_\nu + R_{LC}) - n_U(C_{UL} + A_{UL} + B_{UL} \int \bar{I}_\nu \phi_\nu \, d_\nu) = F_{CL} \]  
 **(16)**

**level U:**
\[ - n_L(C_{LU} + B_{LU} \int \bar{I}_\nu \phi_\nu \, d_\nu) + n_U(C_{UL} + A_{UL} + C_{UC} + B_{UL} \int \bar{I}_\nu \phi_\nu \, d_\nu + R_{UC}) = F_{CU} \]  
 **(17)**

The \( C_{jk} \) represent inelastic collisional terms; the \( R_{jc} \) represent radiative ionizations from level \( j \); the terms \( F_{ck} \) represent recombination processes to level \( k \) from the continuum. Subscript \( C \) always denotes the continuum. Because the ensemble has a much lower opacity in the continuum than in the line, we assume for this discussion that the continuum radiation field is given, a priori.

Then solving Eqs. (16) and (17) to obtain \( B_\nu(T_{ex}) \) from Eqs. (13) and (14), we can write the resulting expression as:
\[
B_v(T_{\text{ex}}) = \frac{\overline{I}_\nu \phi_\nu d_\nu + \varepsilon B_v(T_e) + \eta B^*}{1 + \varepsilon + \eta}
\]

where
\[
\varepsilon = (1 - e^{-\frac{-h\nu/kT}{e}}) \frac{C_{UL}}{A_{UL}}
\]

and \( \overline{I}_\nu \) is the average value of \( I_\nu \) over solid angle.

If we assume that we can represent the radiation field in the continuum by a Planck-function, with temperature-parameter \( T_r \), we can write:
\[
B^* = B_v(T_r)
\]
\[
\eta = \frac{R_{\text{UC}}}{A_{UL}} \frac{F_{\text{CL}}}{F_{\text{CL}} + F_{\text{CU}}}
\]

For the moment, we neglect collisional ionization relative to radiative in discussing \( B_v(T_{\text{ex}}) \) for the two discrete levels. Now recognizing that we have normalized all the terms in Eq. (18) on the right hand side with respect to \( A_{UL} \), we can interpret each term physically.

The numerator represents the ways of exciting the upper level from the lower. The first term represents radiative excitations in the line; the second term, collisional excitation directly from the ground to the excited level; the third term, a process of excitation from the ground level to the excited level via the continuum. The denominator represents ways of getting from the excited level to the ground level. The term unity represents spontaneous radiative transitions; the second term, radiationless collisional de-excitation; the third term, de-excitation to the ground level via the continuum.

So if we want to know what fixes the value of the source-function, and whether the assumption of LTE is justified, it would appear that all we
need ask is, which are the major terms in the expression (18), and to what
does the expression lead if we include only these major terms in computing
the value of $B_v(T_{ex})$? This was the procedure followed in astronomy for
years. Let us ask what it leads to, and if it makes sense.

First, we note that on this basis, LTE would require the terms
$\epsilon B_v(T_e)$ in the numerator, and $\epsilon$ in the denominator to dominate. Then the
ratio would indeed give LTE. But this is of course the condition that
collisional effects dominate -- the usual result quoted in the literature
for LTE. Unfortunately, it was recognized by Pannekoek as long ago as 1930,
that the low densities in stellar atmospheres insure that the second term
in the denominator and numerator are each only some one per cent of the
corresponding first term, in each case. One can also show the same to hold
for the third term. But if we then neglect all save the first terms in
numerator and denominator we obtain a situation where radiation is only
scattered, never produced in the atmosphere itself. We simply have the
radiation incident from the sub-atmospheric regions scattered in the
atmosphere as it tries to diffuse outward. This leads to a situation where
we obtain a completely black spectral line, for an atmosphere which has
large opacity in the line. But such lines are not observed -- there is
always some residual intensity in the observed lines; and some strong lines
have quite a significant value for this residual intensity. Astronomers
played ostrich for years, relative to this dilemma. They sometimes used
the alternative of adopting a source-function given by the first term alone,
in numerator and denominator, for the resonance lines of atoms. Usually,
for all other lines, they adopted the LTE approximation. The justifications
variously offered were that other processes must exist to drive the results
to the LTE assumption; or that actually the collisional cross-sections were much larger than our best observations and calculations showed.

Our developments of the last 15 years are based on the very simple recognition that all this difficulty was based on a conceptual error. It actually does not make sense to treat all terms in the same way, in the expression for $B_{\nu}(T_{\text{ex}})$, asking which is the dominate term in each numerator and denominator. Such would make sense only if the value of $I_{\nu}$ were known independently of the value of $B_{\nu}(T_{\text{ex}})$, and independently of the values of the other quantities in the expression. In the very earliest models of stellar atmospheres, where the radiation field was supposed to originate in a hot photosphere, and simply be selectively absorbed by a thin, overlying, cool gaseous layer called the "reversing layer," such a model made sense. Over the years, however, astronomers came to recognize that the radiation field must be considered to originate throughout the atmosphere; different spectral regions having different opacities, produced radiation fields arising at different effective depths. Because of a frozen thought process, however, this same recognition was not extended to the $I_{\nu}$ entering the source-function. The non-LTE developments introduced this recognition. The showed that the radiation field is actually determined by the other terms in the expression for $B_{\nu}(T_{\text{ex}})$; $I_{\nu}$ is an implicit function of the remaining terms. Once we recognized this, the procedure was straightforward. A choice of the dominant terms in numerator and denominator rests upon a choice of domination among all terms other than the first. We call all these terms in the numerator source-terms, because they produce photons in the line. All save the first in the denominator we call sink terms, because they annihilate photons in the line.
Thus, consider the physical picture of the detailed microscopic processes that determine the value of the source-function, as we now see them. At a point in the atmosphere, the value of $S_\nu$ is numerically equal to the term $\int I_\nu \phi \, d\nu$ to a high order of accuracy, to some few per cent. So the source-function, hence occupation numbers for the microscopic energy levels, have their values fixed by the radiation field in the spectral line considered. But then the important question is, what determines the value of the radiation field. And here, we see it is the small terms remaining in the expression for $S_\nu$ -- the source and sink terms. But we also recognize that it is not the values of these source-sink terms at a point which determine the value of $S_\nu$ at that point. Rather, it is their distribution over the atmosphere, that builds up the value of the radiation field at every point, thus fixes $S_\nu$ at every point. We see this more explicitly, if we consider explicitly how we actually determine $I_\nu$.

We see that we must use the algebraic expression for $S_\nu = B_\nu (T_{ex})$, given by Eq. (20), in an integration of Eq. (6) over the atmosphere. It is more convenient to eliminate the angular dependence of $I_\nu$ in Eq. (6) because we see that it is only the angular average of $I_\nu$ that determines $S_\nu$. So we replace Eq. (6) by the so-called Eddington-approximation to the transfer equation; viz,

$$\frac{1}{3} \frac{d^2 I_\nu}{d\tau^2_\nu} = I_\nu - S_\nu \quad (22)$$

Then inserting Eq. (18) for $S_\nu$, we see that to solve the resulting equation requires two things:

(1) a distribution over the atmosphere of the source-sink terms, given a priori.
(2) two boundary conditions.

Consider the problem of the boundary-conditions first. We see that one of these is simply the condition of no external radiation field incident on the outside of the atmosphere. For the second, consider what happens at large depths in the atmosphere. Conditions become more uniform at large distances from the boundary, and all gradients of the various quantities become very small. We see that Eq. (10) -- having made the replacement of $B_\nu(T_e)$ by $S_\nu$ -- that if $S_\nu$ is very slowly varying, we have $S_\nu \approx \bar{I}_\nu$. Using the relation in Eq. (18), and noting that the profile coefficients are normalized to unity so that $\int \phi d\nu = 1$, we obtain

$$S_\nu = \frac{\epsilon B_{\nu}(T_e) + \eta B^*}{\epsilon + \eta}.$$  \hspace{1cm} (23)

Now deep enough in the atmosphere, $B^*$ becomes $B_{\nu}(T_e)$, so we have $S_\nu$ approaching $B_{\nu}(T_e)$ as $\tau_\nu$ becomes very large. Thus we obtain as our second boundary condition, $S_\nu$ must approach $B_{\nu}(T_e)$ for large $\tau_\nu$.

Consider the point (1) above. We see that the character of the radiation field must be determined by the character of the distribution of the source-sink terms. Thus we have introduced the possibility of several types of spectral lines, depending upon several types of source-functions. Suppose, for example, that $\epsilon B_{\nu} \gg \eta B^*$, and $\epsilon \gg \eta$. Then we see from Eqs. (18) and (19) that $\epsilon$ and $\epsilon B_{\nu}$ depend only upon the distribution of $T_e$ and $\epsilon$ over the atmosphere and actually upon the effect of collisions. Thus the source-function will depend sensitively upon the distribution of kinetic energy of the electrons over the atmospheric regions where the line is formed. But it will depend upon the whole atmosphere, not just a local region. If, for example, a shock wave occurs
in only a limited region of the atmosphere, the source-function in that region will not be fixed just by the presence of the shock, but there will be an "inertial" effect coming from the influence of the other regions of the atmosphere, where the shock does not exist. Thus we recognize that the source-function is no longer locally controlled; its value at a point depends upon physical conditions throughout a large volume surrounding the point.

Consider a second alternative, that \( \varepsilon B_\nu << \eta B^* \) and \( \varepsilon << \eta \). Then because in the outer atmospheric regions where the spectral line is formed, \( B^* \) and \( \eta \) are constant, depending only upon the radiation field in the photoionization continuum which originates much deeper in the atmosphere, the source-function has little to do with the local properties of the atmosphere. Variations in \( T_e \) and \( n_e \) do not affect its value. We obtain a completely different behavior of a spectral line than in the proceeding case. If we try to infer value of \( T_e \) from this second kind of spectral line, we will be led completely astray.

Thus we see that we can introduce a classification of spectral lines depending upon (i) what kind of terms dominate in the source-sink terms, and (ii) what their distribution through the atmosphere is. We have started to carry out such a classification scheme, which we call the "New Spectroscopy." It is discussed in detail in the monograph mentioned in the introduction, "Non-Equilibrium Thermodynamics in the Presence of a Radiation Field." For example, we call the first type of line discussed above "collision-dominated," the second type, we call "photoionization-dominated." Other categories arise, both within this two-level atom simplified model, and in the cases of more realistic atomic models.
Finally, I would only emphasize a point referred to briefly above, which is probably the most interesting aspect of all this work from a strictly physical point of view. In this non-local control of the source-function, we recognize the presence of one variety of a many-body effect. In contrast, to the usual implication in physics, where the many-body effect refers to the perturbing effect of neighboring ions and atoms, here we see it arises from the presence of photons. We see that the radiation field at a point fixes the value of \( S_\nu \) at a point, for these diffuse, optically-extended stellar atmospheres -- by contrast to the situation in dense atmospheres where the value of \( S_\nu \) at a point is fixed wholly by collisions at that point. But the photons arise at large distances from the point considered. So the situation is as though we had a long-range force-field, reflecting the properties of material far removed from the point where we compute the value of the distribution function, which is the source-function. Again, the point is discussed in detail in the reference monograph cited.

Thus, I trust you see how important are the considerations summarized here, on the non-LTE effects on source-functions and so on distribution functions, both for quiescent atmospheres and for atmospheres containing differential motions. The outline is short, and skips details; however, I hope it gives some sort of overall picture of the physical situation.
DISCUSSION

J. H. Clarke: What is the temperature range which is relevant to stellar problems?

R. N. Thomas: The boundary temperature of the coolest stars lies around 3000° on the basis of the classical model which gives a monotonic decrease from center to the outside. In the hottest normal stars the boundary temperature is some 25 000°. For very hot stars the classical models are inapplicable and temperatures up to some million degrees are found. In the sun the classical temperature is around 4500°. However, there is actually an outward rise, after this minimum, reaching some $2 \times 10^6$ in the corona. This outward rise corresponds to aerodynamic heating of the outer atmosphere as contrasted with the classical model in which such effects were ignored. One of the currently important problems in stellar atmospheres is that of the extent to which a chromosphere-corona occur generally and what are their properties.

J. H. Clarke: What about the densities?

R. N. Thomas: The density where you see the sun at 6000° is about $10^{17}$ cm$^{-3}$ and it goes down to about $10^8$ cm$^{-3}$ in the corona. This range covers the transition from collision domination to non-collision domination. These are the regions of formation of the line-spectrum we are talking about here. Clearly the density drops further as we go further out on the atmosphere. The continuum for stars whose boundary temperature is less than about 10 000° is generally produced by the H$^-$ ion; and the reaction which determines the abundance of the H$^-$ ion seems to be the dissociated attachment of hydrogen with $\text{H}^- : \text{H}_2 + e \rightarrow \text{H}^- + \text{H}$. 
In this case, the cross sections are apparently high enough so that collisions do predominate.

S. Scala: What are the characteristic lengths which are significant in defining regions of a stellar atmosphere?

R. N. Thomas: Suppose I have a uniform atmosphere, for illustration; then consider the following schematic diagram:

\[
\begin{array}{cccccc}
I \\
\text{Vacuum} \\
0 & 1 & \frac{1}{\sqrt{\epsilon}} & \frac{1}{\epsilon} & \tau_0
\end{array}
\]

We consider a spectral line whose center lies at \( \nu_0 \), and the optical depth at the line-center is \( \tau_0 \). Then if we consider a photon produced by collisional excitation from the lower to the upper level of the line, at depth \( \tau_0 \), we can distinguish four regions of the atmosphere:

\( \tau_0 \leq 1 \): In this region of the atmosphere, every photon produced and moving in an outward direction escapes sans absorption. So no radiation field builds up. If we had only an atmosphere of this thickness, the value of the radiation field would be just the flux of the emitted photons. The source-function would just have the value \( \epsilon B_\nu [T_e] \).

\( \tau_0^{1/2} \leq 1 \): Photons produced in this region will generally be absorbed and re-emitted many times before escaping from the atmosphere; but a photon will not be lost, once produced, by collisional de-excitation of the excited atom. Thus the rate of loss of photons from both sides of
gaseous shell satisfying this inequality would be just the rate at which they are produced within the shell. But, the radiation field within the shell would build up because of the repeated absorption and re-emission so that the source-function would indeed depend upon the \( \int \mathcal{I}_v v^2 d\nu \) term and exceed considerably the \( \varepsilon B_v \left( T_e \right) \) value. The above condition comes because \( \varepsilon \) measures the rate of collisional de-excitation per absorption, and generally the photon suffers \( \tau_o^2 \) absorptions and re-emissions before escaping from an optical depth \( \tau_o \).

\[ \tau_o \varepsilon \gg 1: \] Here, the photon produced at \( \tau_o \) is both absorbed and re-emitted, and has a chance of collisional destruction, before it escapes. Thus not all photons produced escape; so the radiation field saturates. The mechanism which would produce the Planck function under LTE now acts to produce the non-LTE radiation field. In this uniform atmosphere, the line saturates at a value which is roughly \( \varepsilon^{1/2} B_v \left( T_e \right) \), instead of \( B_v \left( T_e \right) \) as in the LTE case.

So you see there are several kinds of atmospheric thicknesses or regions—not just the classical turn, according to whether \( \tau_o > 1 \) or not.

R. W. Hillendahl: I think that all your remarks are very pertinent if you are trying to do something where the physics you are interested in are determined by bound-bound transitions. I don't think that this is the case in aerophysics. My experience has been that when you do have a continuum present, the temperature distribution and the flux are determined by the continuum and you can very nearly ignore the lines, not completely, but almost entirely. One gets better results using straight continuous absorption coefficients, for example, rather than putting the
lines in. In the case where you have no continuum, the radiation levels themselves are so small that you don't care. This situation exists over a wide gamut of problems.

R. N. Thomas: You are generalizing too quickly from limited experience. Let us distinguish several kinds of regions.

Consider an ensemble which is optically very thick. Then the continuum will arise from much greater depths than will the lines. In the deep regions, you will have essentially detailed balance in the lines, and all that counts is the continuum in considering the energy transfer. It is this situation to which your remarks really apply. Now near the top of this region, we begin to find departures from Boltzmann distributions in the energy levels, and so even here one does not get a Planck function for the radiation field, even in an isothermal atmosphere. In a rough sort of way, if you are in a spectral region where the continuum opacity arises from bound-free transitions from some bound level \( k \), the intensity of radiation is \( b_k^{-1} B [T_e] \) instead of \( B [T_e] \), where \( b_k \) is the ratio of the actual population of the energy level \( k \) to the population it would have in LTE. The effect is not trivial. In the region of the solar Lyman continuum, e.g., it makes the radiation temperatures some 7000\(^\circ\) instead of 12,000\(^\circ\), which is several orders of magnitude in the intensity. So in such a situation, when you say the temperature distribution and energy flux are determined by the continuum, you are of course correct in your statement if you define what you mean by 'determined'; and of course what you mean by 'temperature'.
Now when you go further toward the outside of the ensemble in this considered optically-thick ensemble, then we come to a region where the continuous opacity is very small, and the lines are produced. The energy density of the continuous radiation field in the continuum doesn't change very much; but the energy balance becomes a strong balance of the energy flux carried in the lines because of this very behavior of the continuum. If, e.g., one has an aerodynamical dissipation of energy in these regions of the ensemble, then the energy balance and the local value of the kinetic temperature depend critically on what happens in the lines. In this region, your comment is simply wrong, provided of course the densities are small enough that the radiation field predominates over collisions—the situation discussed here.

Finally, if you consider the situation of, e.g., the 'fusion-pots' in the laboratory, or the gaseous nebulae or solar corona in astronomy, we have a further situation where you would be in error to follow your reasoning. Here, the collisional excitation of the metallic impurities [and of the oxygen-carbon-nitrogen impurities], followed by their radiation, provides a very strong energy leak which holds down the kinetic temperature of the electrons. Trying to compute temperatures without considering this line-radiation leads to enormous errors, as was found some 30 years ago in astronomy and re-discovered some 10 years ago in the fusion-pots. So, I think you should be careful as to what regions of aerophysics you discuss, when advancing the conclusions you mention.

J.I.F. King: Is there anyplace left in the stellar atmosphere where radiative equilibrium does prevail?
R. N. Thomas: Yes, in the region where the visual continuum is formed, radiative equilibrium is still a very good approximation.

S. C. Traugott: Can you expand around the LTE situation, to use a small-perturbation approach in computing these distribution functions. We find in fluid dynamics, that even in shock waves where you don't expect small-perturbation results to hold, the perturbation-derived Navier-Stokes equations give good results.

R. N. Thomas: There has been a series of papers, by Henyey and Grasberger [Ap. J. 103, 332, 1946 et seq] which explored this possibility. However, one can show that even in principle this approach cannot work [cf the monograph on Non-equilibrium Thermodynamics in the Presence of a Radiation Field cited earlier, Lecture IV]. Basically, this follows because the small-perturbation approach is an expansion about a TE situation for the fluid dynamics situation, but about a perturbation on the TE situation for the low-density, high-$T_e$, large-optical-depth case discussed here and typical of stellar atmospheres. The problem is, that if you start a small-perturbation approach, and iterate, you find a correction each time that is small, but always monatomic until you have iterated so often that you are a very long distance from the starting point.

Chairman: Has any aspect of the non-LTE problem been solved, in a practical way?

R. N. Thomas: Yes, a number of solutions to special cases and categories of cases exist. Cf the references cited.
Chairman: Can you match the solar observations, for example?

R. N. Thomas: Yes, cf the extensive discussion and bibliography in the Thomas-Athay monograph cited.
Radiative Shock Wave Structure

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The previous discussion has shown at least two situations simultaneously involving radiation and shock waves: very high speed re-entry flows and propagation of disturbances in a stellar atmosphere. In a situation involving shocks, one is tempted to think of the classical picture of a shock wave, namely a thin (of the order of collision mean free paths) transition region between an upstream, undisturbed state and a downstream, shock-heated state.

There are, of course, many kinds of situations where this simple picture is know to be inadequate – hydromagnetic shocks of various kinds, shocks in a gas with relaxing internal degrees of freedom – all involving a very much more complicated internal structure. This discussion, and the following one by Professor Clarke, will concentrate on the structure of a plane shock in a radiating gas.

The division of labor is such that I will talk about what one gets with the simplest, but possibly not very real, modification to a classical shock due to radiation: that in a grey gas. A grey gas has the same photon mean free path for all wavelengths. Professor Clarke will describe what happens when this restriction is lifted. Therefore, for the time being, we shall look at a grey gas. Consider only the case where local thermodynamic equilibrium holds throughout the gas. This assumption allows us to relate the radiation intensity to the temperature by using the Planck function as the emission term in the radiative transfer equation. This restriction is one of sufficiently frequent gas particle collisions; in fact, the gas is
taken to be a continuum whose only departures from equilibrium are the small departures which lead to viscosity and heat conduction. Photons may be as much out of equilibrium with the gas as they like. One of the key assumptions not to make about a radiating shock is a priori restrictions on optical thickness. This turns out to vary widely, both with shock strength as well as position in the shock.

There are then two a priori characteristic lengths in the problem: gas particle collision mean free path and a single photon mean free path, independent of photon energy. What is the consequence of several characteristic lengths? To make the idea clearer, consider for a minute a non-radiating gas with some internal degree of freedom, say vibration or dissociation. A typical thing that happens is that as the gas flows through the shock it is first heated in the ordinary way, which just means it takes a few collision lengths to excite the translational degrees of freedom. Thereafter a further change takes place as the internal degree of freedom relaxes toward an asymptotic downstream state so that the shock is broadened. If the distance required to fully excite the internal mode is much larger than a collision length, the processes separate and the picture becomes a conventional shock jump determined by viscosity and heat conduction; this is followed by an inviscid, non-diffusive, and much longer downstream tail. This kind of shock has been called "partly-dispersed" by Lighthill.

With radiation one may expect the same sort of thing. There are two important points. First, since there is a radiant heat flux upstream which is absorbed in the cold gas, thereby heating it, in addition to translational energy loss downstream, there will be flow modifications both upstream as well as downstream. Second, the radiation relaxation distance
which determines the extent of these regions is not known without solving
the problem. It is not the photon mean free path except in special circum-
stances. Nevertheless, it appears sensible to assume that whatever these
upstream and downstream lengths are, they are much larger than a collision
mean free path.

In effect, there is then only one a priori characteristic length.
With this assumption one has an inner, viscous shock and outer, inviscid,
radiation-determined precursor and tail. This is the radiation-dispersed
shock. It is partly-dispersed because of the existence of the inner
shock. The inner shock being very thin, one agrees that the heat flux
across it is continuous, thus the jump conditions across this inner shock
involve the same flux on either side which cancel; hence, the inner jump
is given by the conventional Rankine-Hugonot relations. With this model,
how does one compute the outer profiles?

This problem has been discussed by Prokof'ev, Zel'dovich, and Clarke
(not Dr. Joseph Clarke of Brown University, but another Dr. Clarke - which
is a good thing because the other Clarke's analysis was wrong), and Heaslet
and Baldwin. There is an interesting parallel. Prokof'ev came first, and
he overlooked the existence of an inner shock. Thus all of Prokof'ev's
solutions are what is called "fully-dispersed," Next came Zel'dovich, and
he discussed the necessity of jumps. Then came Clarke, and he independently
left out jumps. Then Heaslet and Baldwin put the jumps back in. One way
of treating the problem is shown in Fig. 1.

On top are the once-integrated, one-dimensional momentum and continuity
equations. w is a dimensionless velocity, t is a dimensionless temperature.
\( \mu \) is viscosity, K is the grey absorption coefficient. \( \epsilon \) is a shock
strength parameter. $\xi$ is a dimensionless radiant heat flux. Without radiation, these two equations determine the structure of a viscous shock. With radiation, in the radiation-dispersed, non-viscous parts of the profile, $\frac{dw}{dx}$ is given by numerator/zero, so that the numerator must vanish. To obtain solutions for the limit of vanishing viscosity, consider the form given below, which is the temperature and heat flux for vanishing right-hand sides of the equations above plus a correction. If and when there is a jump $\delta$ will have to be very large to give vanishing denominator but non-vanishing numerator. Rewriting the top equation to introduce an optical thickness $\rho Kdx$ shows that the quantity $\delta(w)$ is essentially the slope of the velocity profile in $\tau$ space. ($\frac{\mu''K}{u_{-\infty}}$ is the ratio of collision to photon mean free paths.) So once we have $\delta$ the problem is solved by one further integration. To get $\delta$, we must introduce the radiative transfer equation. The differential equation for $\delta$ shown is the result of a differential approximation to the radiative flux which can be briefly described as follows:

It is essentially the moment method of Krook, except it is not restricted to radiative equilibrium. One constructs from the radiative transfer equation a hierarchy of equations for various moments. This is an infinite, indeterminate set. By expanding the intensity in spherical harmonics, and terminating the expansion after a certain number of terms, one finds that the termination gives relations between various moments, and the problem becomes determinate. The first approximation is to keep only two terms. In a one-dimensional situation one can then construct a second order ordinary differential for the first moment, which is the radiative heat flux. It is that equation, which for the special form
considered above for $\xi$, gives the equation for $\delta$ which is shown. Please do not become sidetracked by the use of the differential approximation. By integrating the differential heat flux equation one can show that it is equivalent to approximating an integro-exponential function by a single exponential. (The next approximation, a fourth order equation, is equivalent to approximating it with two exponentials.) The single exponential approximation has been checked numerically by comparing shock profiles obtained with the correct integral expressions for $q_r$, agreement is obtained to within a few per cent.

Now return to the shock problem. In the differential equation the second term is the absorption term, the last term is the emission term, and the first is their difference. Note here that neither viscosity coefficient nor absorption coefficient appear. Solutions in $\tau$ can be found without knowledge of the transport properties. If portions of the shock are transparent one estimates, neglecting absorption, $\delta$ as shown. If there are opaque regions, the last terms balance and $\delta$ is estimated as shown. Without emission, the middle estimate. From the equation on the top right let us estimate a radiation relaxation length $\ell$ by $\rho \ell = \delta^{-1}$. $\ell$ is supposed to be a measure of the extent of the radiation-dispersed tails. Thus, the last terms on the right of the expressions for $\tau$ and $\xi$, which measure the influence of viscosity in the outer portions of the shock are just the ratio of collision to radiative relaxation lengths, a form which one ought to suspect from the very beginning. The radiative relaxation lengths are different depending on the optical properties of the portion of the shock considered, as shown along the bottom. Thus for a portion of the shock which is transparent $\varepsilon \frac{\mu''K}{u_{-\infty}} \frac{16\sigma T^4}{\rho_{-\infty} u_{-\infty}^3}$ being small
signifies the validity of the dispersed model. If opaque, the ratio rather than the product of these quantities enters. A final point: There is a radiation relaxation distance such that $\rho K g T^4 \sim \rho_\infty u_\infty^3$. If the shock has such an emissivity that it radiates according to the kinetic energy flux available to it, that emissivity corresponds to a transparent relaxation distance.

Now let's look at the results that are obtained with this sort of model. Fig. 2 shows some results that have been given by Heaslet and Baldwin. Strong shock means just that; weak and strong radiation refer basically to the temperature upstream. In all these plots, the abscissa is optical thickness. Consider the upper left. Here is a strong shock with weak radiation; it is the case most nearly resembling a non-radiating shock which would have uniform temperature and velocity on either side of the jump, and no radiant heat flux.

Note next the upper-right strong shock with strong radiation. The front tail is nearly 20 photon mean free paths thick, and it is, in fact, mostly opaque. The rear tail is shorter than a mean free path; it is transparent. Note the temperature ahead of the jump — it is identical to the asymptotic downstream temperature. This is an extremely large upstream heating effect. This kind of shock hardly resembles in any way the classical picture of a shock wave. Note that there are solutions without jumps. Thus, both fully-dispersed as well as partly-dispersed shocks can occur. In the lower right is a weak shock moving into a hot gas; it is continuous, viscosity never matters, and it is altogether about 60 photon mean free paths thick. Drastic changes occur in this profile as the shock becomes stronger.
Fig. 3 shows the boundary between partly-dispersed and fully-dispersed shocks. Shocks without jumps are in the lower right-hand region. Plotted up is shock strength $\varepsilon$ defined above, the abscissa depends only on upstream temperature. $\varepsilon = \frac{\gamma - 1}{2\gamma}$ defines the boundary between profiles with and without temperature overshoots. The shock strengths along the dashed curve are quite weak; they correspond to Mach numbers well below two. An interesting observation has been made by Moore. He notes that on the far right the asymptote of the boundary corresponds to a Mach number of $\frac{1}{\gamma}$ behind the shock. In terms of the isothermal speed of sound, this is a Mach number of unity. Any shocks stronger than this become isothermally subsonic downstream; hence, one can loosely argue the deceleration must be discontinuous. Why is there an isothermal speed of sound? As just seen, such a weak shock moving into a hot gas is opaque everywhere; the disturbance is so that in terms of collision lengths the temperature is constant. It is certainly far from involving isentropic changes.

Fig. 4 shows the maximum radiant heat flux in the shock. If there is a jump, this occurs at the jump. Upstream conditions here are cold, corresponding to a small value of the abscissa of the previous figure, and a shock of any strength will have an inner jump. Consider first the black-body curve. This is gotten by simply taking the gas behind the shock to be a semi-infinite, constant temperature slab at the asymptotic downstream temperature. Since the radiation from such a slab goes as $T^4$ which is $M^8$, the flux divided by $\rho_{-\infty} u_{-\infty}^3$, which is what we are plotting here, increases as $M^5$. The flux from the calculated profiles, on the other hand, quickly reaches an asymptote with increasing shock strength. Thus the shock radiates away a fixed fraction, about 1/3, of the available kinetic energy flux far upstream, no matter how strong.
Recall the earlier argument involving a radiant heat flux proportional to $pu^3$, it involves a transparent rear tail. The shock makes itself transparent in back of the jump to prevent radiating away more energy than is fed to it. Note, incidentally, how quickly the asymptote is reached. A Mach number of ten is hardly very strong in the conventional sense, yet a radiating shock of this strength already has the odd structure shown earlier: opaque in front, transparent behind, a temperature overshoot within the inner shock.

There are two things to keep in mind at this point about the shock. First, it is an isolated shock in an infinite medium, far away from any walls. The presence of a solid boundary behind the inner shock such as would be encountered in the radiating flow field of a re-entry body, induces a downstream radiant heat flux which reduces the upstream flux. Since it is the magnitude of the upstream flux which has changed the shock structure, we expect a less drastic modification of the classical shock profile. Second, even in an astrophysical situation where this factor does not arise, one must expect that departures from a one-dimensional configuration must cause a reduction in upstream effects. For the plane shock, all energy radiated upstream is absorbed and converted back. For a more general geometry in which radiant flux and flow direction are not aligned, much of the energy radiated upstream will be lost to the shock.

Now let us look at the situation where the ratio of collision length to radiative relaxation length is no longer so small that inviscid outer profiles are connected by a jump. In this situation, one obtains a picture of what can happen when there are truly two characteristic lengths which may both be important in certain parts of the shock profile. This happens first in that part of the shock with the shortest radiative tail, which
is the transparent one in back. In the case just considered, it was possible to describe the shock in terms of optical thickness. The magnitude of the photon mean free path itself only came into the picture when translating these profiles into physical space. The grey absorption coefficient only mattered in a coordinate transformation. If viscosity and heat conduction are explicitly taken into account, this is no longer so. The grey absorption coefficient must now be specified, in particular its variation with temperature. The sort of function that is encountered is shown in Fig. 5.

Fig. 5 is a plot of various people's calculation for the Planck mean for air, with the exception of the dots at high temperature which are Rossland means calculated by a group at Los Alamos. Rossland and Planck means don't appear to differ from each other by more than one order of magnitude; the variation of either one with temperature is so large that I will simply ignore this and pretend there is one mean. I will further fit it with the solid lines by a convenient empirical equation.

At one particular density, one of these solid lines is reproduced in Fig. 6. It is the curve marked $\rho K_{III}$. There are two others which are purely arbitrary. Their purpose is to construct alternate functions just to see how sensitive the viscous solutions will be to changes in absorption coefficient. $\rho K_I$ is the low temperature variation continued on indefinitely. $\rho K_{II}$ is a happy medium between these two. With these absorption coefficients, I shall now show some results for a viscous, radiating shock in a perfect gas. Numerical computations have been made at several shock strengths, selected in such a way that at the downstream end of the shock the ratio of collision to relaxation length is small but not microscopic, or of order
unity depending on which curve is used. Recall in the following that $\rho K_I$ is the largest and least realistic, while $\rho K_{III}$ is the smallest and most realistic.

Some results are shown in Fig. 7, a phase plane representation which some of you may be familiar with from the structure of a non-radiating shock. Temperature $t$ is plotted against velocity $w$. The asymptotic state in back of the shock is hot and slow; it is, therefore, in the upper left corner and the front is in the lower right. This is for a strong shock moving into a cold gas; the previous dispersed solution is indicated by the dashed curve interrupted by a jump. There is a large temperature overshoot in this profile. Viscous solutions for the various conditions are shown. If the relaxation length is not very much larger than the collision length in back, the temperature overshoot tends to disappear.

The lowest curve is for the largest absorption coefficient $\rho K_I$. In front, for all cases, it is impossible to have viscosity make any difference. The lowest curve is of special interest: It is almost exactly what one gets for a non-radiating gas of vanishing Prandtl number. Such a gas is obtained in the opaque limit, where radiation becomes diffusive, and one may argue that there is then a much enhanced thermal conductivity without corresponding increase in viscosity. This argument has on occasion been used in discussions of radiative shock structure. We see here that it is viscosity, coupled with an unrealistically large absorption coefficient, which can give such a profile. Curiously, examination of the solution reveals that except in the front tail the shock is transparent. Thus the results based on a Rossland diffusion model are reproduced in a transparent portion of the shock if the collision mean free path is large enough.
These same results in physical space are shown in Fig. 8. In all cases the upper curve is radiant heat flux, the middle curve is temperature, and the lower is velocity, all plotted as functions of physical distance through the shock. The lower right gives non-radiating profiles. What one sees here is essentially the inside of the region lumped into a jump in the dispersed model. The upstream tail is so long that one cannot see it on this scale. The profile for $\rho K_{III}$, the most realistic absorption coefficient, is on the upper right; viscosity got important enough to reduce the temperature overshoot somewhat.

These are a few isolated calculations, and now I shall try to place these solutions in perspective. With $\rho K = \rho K_{III}$, how important can viscous effects be? It happens that the example in the upper right is about as viscous as one can make a shock. This is because of the maximum of the function $\rho K$, which means a maximum for the ratio of collision to photon mean free paths. Stronger shocks would be less viscous, as would weaker. These calculations were all made for a perfect gas. In real life, dissociation (if we are concerned with air or some diatomic gas) changes drastically the temperatures behind the shock as well as viscosity. For this reason, a calculation of the ratio of collision to relaxation length has been made for real air as shown in Fig. 9. This figure shows that it will be next to impossible to make a radiating, normal shock in air depart significantly from the dispersed model.

What one does learn, however, is the complicated way in which things can change when multiple characteristic lengths overlap. In this connection, it may be of interest to mention some recent work of Rhyming and Emanuel, in which an inviscid radiating shock is considered with
chemical or vibrational relaxation. Results somewhat similar to those obtained here are found, in that the temperature overshoot can be made to disappear if the ratio of radiation to chemical relaxation lengths is not large. One may speculate with fright on the complications which would arise upon simultaneous consideration of viscosity, chemical relaxation, and radiation, or even what would happen in a real radiating gas, meaning a non-grey gas with not one photon mean free path but a number of them at different wavelengths.
\[\frac{\gamma}{\mu} \frac{d}{dx} \frac{d\mu}{dt} = \frac{(r-1)(1-w^2) - \frac{3}{2} \frac{\epsilon}{1+e} (1-w^2)}{1+e} \frac{1}{1+e} \frac{3}{2} \frac{\epsilon}{1+e} (1-w^2) \]

\[\frac{\gamma}{\mu} \frac{d}{dx} \frac{d\mu}{dt} = \frac{(r-1)(1-w^2) - \frac{3}{2} \frac{\epsilon}{1+e} (1-w^2)}{1+e} \frac{1}{1+e} \frac{3}{2} \frac{\epsilon}{1+e} (1-w^2) \]

\[\delta = \frac{9 \gamma}{\rho_\infty U_\infty^3} \]

\[E = -w + \frac{3}{2} \frac{\epsilon}{1+e} (1-w^2) + \frac{\mu K}{U_\infty} \delta(w) \]

\[\delta = \frac{1}{2} \frac{\epsilon}{1+e} \frac{3}{2} \frac{\epsilon}{1+e} (1-w^2) + \frac{\epsilon^2 \mu K}{U_\infty} \lambda(w) \]

\[\frac{d}{dw} \left( \frac{w^3}{8} \right) + \frac{3}{2} \frac{(r-1)(1+w^2)}{(r+1)e} (1-w^2) - \frac{165 T^4}{\rho_\infty U_\infty^3} \left[ \frac{1}{(r-1)(1+w^2)} \left( \frac{1+e^2}{1+e} \right) \frac{dt}{dw} + \frac{d}{dw} \left( \frac{1+e^2}{1+e} \right) \frac{dt}{dw} \right] = 0 \]

Transverse
\[\delta \sim \frac{165 T^4}{\rho_\infty U_\infty^3} \]

\[\rho_{KL} \sim \left( \frac{165 T^4}{\rho_\infty U_\infty^3} \right)^{-1} \]

Opaque
\[\delta \sim \left( \frac{165 T^4}{\rho_\infty U_\infty^3} \right)^{-1} \]

\[\rho_{KL} \sim \frac{165 T^4}{\rho_\infty U_\infty^3} \]

Figure 1
FULLY DISPERSED: 
PARTLY DISPERSED: 

\[ \varepsilon = \frac{M_\infty^2 - 1}{1 + \gamma M_\infty^2} \]

\[ \varepsilon = \frac{1}{\gamma} \quad \gamma = 1.4 \]

\[ \varepsilon = \frac{\gamma - 1}{\gamma} \]

\[ \varepsilon = \frac{\gamma - 1}{2 \gamma} \]

\[ \frac{16 \sigma T_{\infty}^4}{\rho_{\infty} a_{\infty}^3} \]

Figure 3
Figure 4
Figure 6
Figure 8
Figure 9
Q: In this solution, do you know the upstream and downstream conditions? Are these assumed known and you get the details between known end conditions?
S. Traugott: You know them because they are the usual jump conditions. Far upstream and downstream, there isn't any heat flux. If you apply jump conditions, they are the Rankine-Hugoniot conditions. They are also the same across the inner discontinuity. But that is all that you know as far as jump conditions are concerned.

J.H. Clarke: Ira Cohen and I did some work on this problem using asymptotic expansions in the ratio particle mean free path to photon mean free path, proceeding much as one would in a boundary layer theory. In that framework, you can make inner and outer expansions. We find that for conditions precluding continuous solutions the ordinary viscous heat conducting solution applies in the inner shock to first order. If minus infinity and plus infinity conditions are re-interpreted to mean just ahead of the discontinuity, then to second order you get that the radiative flux must be accounted for in this viscous heat conducting region with a rate of change of radiative heat flux given by the emission minus the absorption from the outer region, that is, the invicid region. So we made a hierarchy where the discontinuity is like a skinny boundary layer and the outer part is like a thick boundary layer. You can match these together for all orders. The only point is that if the smallness of these two ratios is being suspected, the procedure is very simple.

S. Traugott: I agree. This is true if you are going to argue that the characteristic collisions length is pretty small. What would on the other
hand if it turned out that somewhere there was a gas which had a very high absorption coefficient at high temperatures. You would get a very much different profile.

J.H. Clarke: We looked into all of the data that we could find and particle mean free path divided by the photon mean free path always seemed to be 10 to some enormous negative power.

S. Traugott: I can make this number pretty big in one situation which I didn't mention. If you look at properties behind reflected shocks the temperature is high and the shock velocity is relatively low. At realistic but very strong incident shock parameters, you get ratios like one or so. You would have to do the viscous problem.

A.G. Hearn: First, I should really confess that I didn't follow as much of your talk as I should have but there are some things that did disturb me. One is the Planck mean being equal to the Rosseland mean. I have never seen this to be true. Also, I wonder if it would not have been better to take as your absorption coefficient one that is constant in the low temperature region which becomes transparent in the high temperature region. We found this to be crucial in some of the structure of the shock developing in air as a result of atomic explosions, for example. Absorption is high for the low temperatures.

S. Traugott: You may be perfectly right. I have never seen it like that.

A.G. Hearn: You get an opacity process which is your process of scattering at high temperature. It holds down radiation but it does not cause emission. You have to differentiate or you do not get the right answer. It depends
on whether there is emission or absorption.

**S. Traugott:** What I need is a coefficient in the radiative transfer equation. Having made the equation grey, I am trying to approximate this coefficient. Now I know that there are two means, the Rosseland mean and the Planck mean, and they are appropriate when things are thick or thin. I hope that I didn't say that I thought they were equal. I didn't mean it. What I meant was that the difference between them was very, very much less than the variation of either one.

**A. Skumanich:** It can become very serious. I think that one weights the transparent portion of the absorption coefficient and the other weights the opaque portion. This can be orders of magnitude different. Certainly in my experience in aerodynamics it has been. For the astronomers here, one is called the Chandrasekhar mean and the other is called the Rosseland mean. The Planck mean is like the Chandrasekhar mean weighted by the Planckian function. The Chandrasekhar mean is a mean weighed by the specific intensity.

**Comment:** The Chandrasekhar mean is a flux weighed mean, not an intensity weighed mean.

**A. Skumanich:** One more part to my question. Could you display explicitly the expansion used and where exactly you truncated the expansion? You terminate after $P_1$. The reason that I ask this is that if you terminate after $P_1$, you get the radiation diffusion equations.

**S. Traugott:** That's not true. I cut off after two terms when the equation looks like $\frac{d^2q}{dt^2} = 3q - 4\pi \frac{dB}{dt}$. That is the sort of equation that you get.
Q₁: Did you expand the specific intensity in the moments of the spherical harmonics? And there did you truncate after $P_1$?

S. Traugott: Yes. Maybe you are thinking of radiative equilibrium.

Q₁: No. At one point you said that your approximations are equivalent to the approximations of the kernel of the integral equation by a single exponential. Now this is only true if you are using what is called a half-moment. I presume that you were using a half-moment.

S. Traugott: No. I wasn't using a half-moment. All of the moments are from $-1$ to $1$.

Q₂: So you are essentially just using a Rosseland type.

S. Traugott: No.

Q₃: If he is truncating after $P_1$, and he is using the full moment method, it is the Rosseland approximation.

S. Traugott: Excuse me. This is the Rosseland approximation. $q = \frac{4}{3} \pi \frac{dB}{d\tau}$.

Is it?

Q₃: No. You are using the equation of transfer and you are expanding $I$ in spherical harmonics and you are using that to truncate and, so to speak, to terminate your moments. That is tantamount to saying that you are using the Rosseland approximation, that you are using the diffusion approximation.

S. Traugott: I claim that it isn't. But we had probably better argue this out privately, that the equation that you get from the procedure is not the Rosseland approximation.
R.N. Thomas: I think that I could amplify Andy Skumanich's remark. I don't think the Planck to Rosseland ratio is small even on the scale that you showed. I have found ratios of $10^4$ and $10^5$ and this is because of the lines.

R.W. Hillendahl: I am convinced that people who argue this point can resolve every detail by just looking and defining things carefully, whether it is with lines, without lines, and so forth. It is true that the Rosseland and Planck means, say from 5 volts to maybe up to 50 volts temperature, are practically identical for air no matter how you do it. In any other region, at different temperature regions, one can get factors of $10^4$, but none of these are very usable anyway. All I would like to say is that I am very much impressed by what you had to say. We have been doing some radiation hydrodynamics calculations using the transport type equation, the non-grey gas, the real equation of state and lots of things like this, and we got temperature overshoots. We didn't like them, we thought that it was an instability, but we got it anyway. We compared it with the experiment: these are there! Of the nine illustrations that you had for Heaslet and Baldwin, in our problems we have encountered at least eight.

S. Traugott: Then you are really impressed with what they did. Is that right?

R.W. Hillendahl: I am just saying that regardless of the details of how you got there, at least three people have done this problem and all gotten this result. Instead of radiation smoothing a shock front, you get anything but that. I am saying that we have done this from a completely different approach - a non-analytic approach.
S. Traugott: I think that an important point is that you have done something that is non-grey and got something like this.

R.W. Hillendahl: We went at this from what I guess you would call a completely non-analytic point of view, we went at it numerically with a big machine because we wanted a lot of detail. We got an effect and we didn't believe it, but when we compared it with an experiment, it's there. If it isn't there, we differ from the experiment by an order of magnitude. I mean that it has to be there. We have a case which is not particularly classified, I mean that everybody knows that bombs radiate, everybody knows that bombs have shock waves. But if you look at the magnitude of the radiation and you look at the classical shock picture and at a classical temperature profile, you just can't get there. Even though it goes like $T^4$, when you look at the radiation and solve backward you get $T$, if you are off even a little bit in the temperature, you notice it. That is a rather sensitive way to look at this.

S. Traugott: Have you done this sort of a problem spherically?

R.W. Hillendahl: Yes.

S. Traugott: Nevertheless, because of the difference in geometry, I have wondered what would happen to the strong upstream effect if you let it become spherical. I thought that the upstream effect would maybe all go away. I think that you are saying that it doesn't.

R.W. Hillendahl: No, there are many more effects. Everything that you said is certainly there.
S. Traugott: Even in the spherical case?

R.W. Hillendahl: Yes.
The next presentation was made by Dr. J. H. Clarke of Brown University. His review of the available literature on shock wave structure with spectral effects was followed by some new results which he obtained in collaboration with Dr. C. Ferrari of the Politecnico di Torino. Since this presentation followed rather closely a recent report released by Brown University, Dr. Clarke authorized us to reprint this report in the proceedings of the workshop. The discussion which took place after the presentation is reproduced on pp. 140 - 145.

R.G.
On Gas Dynamics with Nonequilibrium Radiative and Collisional Ionization

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The dynamics of a gas undergoing nonequilibrium ionization by both radiative (bound-free) and collisional transitions is formulated. It is shown that collisional ionization rates, if small, must never be completely neglected compared with the radiative rates, so a complementary limit for slow collisional rates does not exist. On the other hand, line radiation may be considered separately from the bound-free radiation. The nonequilibrium emission coefficient obtained bears a significant relation to the equilibrium Planck function. The radiative contribution in the ionization-rate equation can be put in the form of the divergence of a photon particle flux vector, advantageously paralleling the radiative contribution in the energy equation. The ionization rates and emission and absorption coefficients are all evaluated explicitly from statistical results available for the monatomic gas considered. This makes the problem concrete; one also sees that the approach for other cross-sections, transitions, and gases would be analogous in most or many respects. It is shown that spectral complexities can sometimes be removed by expansion in terms of a large thermal parameter, and there emerges rationally a gas which is approximately gray above the edge frequency. The general theory is applied to the strong normal shock wave structured by the nonequilibrium ionization processes. The nature of this problem is examined, the complete structure is computed, the photoionization in
the precursor portion of the shock structure is also determined according to the black-body emitter model for comparison, and the spectral as well as a directional approximation are assessed.

I. INTRODUCTION

The combination of high temperatures, high velocities, and low densities, such as are encountered in hypervelocity flight, has led to the study of such nonequilibrium gas dynamic phenomena as vibrational and dissociative relaxation, and of many other lagging chemical reactions in flows. The approach is from the viewpoint of chemical kinetics. A treatment of this subject together with pertinent references may be found in the recent book by J. F. Clarke⁴. High-temperature phenomena have also stimulated interest in radiative gas dynamics, which deals with the application of the older methodology of radiative transfer to a moving fluid medium⁵,⁶,⁷ (thus inviting comparison with magnetohydrodynamics). Complex enough, current treatments of this subject are commonly restricted to a perfect gas, a gray gas, and especially a gas in local thermodynamic equilibrium, whereby the collisional transitions are considered fast enough that equations for the rate of population and depopulation of states due to radiation and collision need not be written.

However, measurements of nonequilibrium as well as equilibrium emission behind shock fronts in air have been reported⁸,⁹ for a variety of species in connection with the study⁰ of the radiant heating of re-entering bodies by the hot gas cap. (Ionization in the gas cap followed by neutralization in the radiating wake are effects of interest in connection with communication and detection aspects.) The methodology of
radiative gas dynamics has not yet appeared in the literature of this
genus because the gap cap has been considered optically thin and the
radiant energy loss has been considered a small fraction of the convec-
tive energy, the limitations of these two restrictions having been con-
comitantly recognized\textsuperscript{5,6,7}. The problem then becomes one of exper-
imentally identifying the important emitting species and transitions, of
determining the local concentrations, and of inferring the resulting
emission coefficient; it is noteworthy that the source function remains
the Planck function for the transitions considered as long as the emit-
ting species remain in a local thermodynamic equilibrium corresponding
to the current, lagging value of the species concentration.

Nonequilibrium photoionized precursors before strong normal
shock waves in various monatomic gases have been observed in shock tubes
by a number of experimenters. An extensive photoionized region in the
cold gas ahead of a reentering satellite has been noted by S. C. Lin,
and analogous but stronger photoionization has been observed by astronomers
to exist ahead of and far along the sides of entering meteors. The analysis
of an ionizing, radiating one-dimensional flow has been considered by
Ferrari and Clarke\textsuperscript{8,9,10,11} with particular reference to calculating photo-
ionized normal-shock precursors. The distant decay of this precursor has
been analyzed by Wetzel\textsuperscript{12,13}, by beginning with a simple fluid mechanical
model (see references 9 and 13 for the relevant precursor literature).
The precursor research seems to represent the only area where nonequilibrium
radiation has been considered theoretically in a fluid mechanical problem,
although a pertinent radiation model for nonequilibrium molecular gases
has recently been reported by Hansen\textsuperscript{14}. 

On the other hand, astrophysicists have known since 1929 that collisional transitions are too slow to maintain local thermodynamic equilibrium in stellar atmospheres. But they have been primarily concerned with nonequilibrium radiative transfer not involving macroscopic rate processes, viz. a nonequilibrium steady state\textsuperscript{15}. More recently, astrophysical gas dynamic phenomena have received attention. In connection with shock waves in stellar atmospheres, Whitney and Skalafuris\textsuperscript{16} have considered the photoionized precursor preceding a normal shock wave in atomic hydrogen and made rough estimates of the rest of the shock structure. Goldsworthy\textsuperscript{17} and Axford\textsuperscript{18} have considered the same problem, in connection with the interstellar medium, for the more intense case wherein the gas is completely ionized in a "front" preceding the shock wave.

Since the appearance of our earlier precursor work\textsuperscript{8,9,10,11}, we have been actively interested in the broader gas dynamic ionization problem that intrinsically involves both nonequilibrium, nongray radiative transfer and a nonequilibrium reaction. There has naturally been considerable feedback from this experience and from the results obtained, and the first part of this paper, the first of a series, includes an entirely new and more comprehensive formulation of the broader problem that starts from the same statistics for the production and disappearance of ions. The formulation becomes much clearer physically and more manageable mathematically when the most appropriate parameters and directions have been discerned, sometimes from many possibilities. For the convenience of the reader who may be familiar with the earlier work (although this is not necessary), we have organized the material so that the first part of the paper parallels Ref. 9.
The choice of a monatomic gas makes the problem conceptually clearer; the ionization rates and the emission and absorption coefficients can then be determined theoretically from available statistics; the results are then directly relevant to some of the experiments and phenomena noted above. The use of other radiative or collisional cross-sections, transitions, statistical models, radiation models, or gases would not change the present approach in many respects. Even in the case of air for the temperature and density range of interest, the principal emission is due to free-bound transitions and bound-bound transitions in monatomic nitrogen; other writers emphasize the importance of free-bound radiation above 12,000°K in air.

The first part of the paper is concerned with the radiation model (mainly the role of line radiation), the statistical processes and model, and the formulation together with deductions and interpretations. Next a very convenient approximation for the spectral aspects of the problem is proposed. The radiative and photon particle fluxes are then given for one rectilinear dimension. The rest of the paper represents an application of the theory to normal shock wave structure. As in the case of other high-energy nonequilibrium flow problems, this interesting application also illuminates particularly well the effects considered because of the steep transitions appearing therein. Since Sect. XV mainly consists of a physical discussion of the shock structure, the reader may wish to scan it now.
II. THERMODYNAMIC RELATIONS

We consider a partially ionized monatomic gas, assuming that the atoms are singly ionizable at most. The mixture is regarded as one of perfect gases wherein the number densities of atoms, ions, and electrons are respectively denoted by \( n_a \), \( n_i \), and \( n_e \). The mass of an electron is considered negligible with respect to that of an ion, charge separation is neglected (so that \( n_e = n_i \)), and no electric or magnetic fields are present.

The equation of state for the mixture then is

\[
p = \rho RT(1+\alpha), \tag{2.1}
\]

where \( R = k/m_a \) is the gas constant per unit mass of atoms \((k\) being the Boltzmann constant and \( m_a \) the mass of an atom), \( p \) is the pressure, \( \rho \) is the mass density, \( T \) is the translational temperature, and \( \alpha \) is the degree of ionization, here equal to the mass fraction of ions; thus

\[
\alpha = \frac{n_i}{(n_a + n_i)} = \frac{\rho_i}{\rho}. \tag{2.2}
\]

The internal energy per unit mass of mixture \( E \) may be written

\[
E = (3/2)RT(1+\alpha) + E_c + \alpha RT_j, \tag{2.3}
\]

where \( E_c \) is the electronic energy of atoms and ions and \( T_j \) is an ionization temperature defined by the relation \( \epsilon_j = kT_j \), \( \epsilon_j \) being the energy required to remove an outermost electron from an atom in the ground state.

In (2.1) and (2.3) the electrons have been assumed in translational equilibrium with the atoms and ions, whereas \( \alpha \) is considered
generally to lay behind the local thermodynamic state. The electronic energy of the \( n_a \) atoms and the \( n_i \) ions in unit volume is considered in local thermodynamic equilibrium.

III. EQUATIONS OF CONTINUITY, MOMENTUM, AND ENERGY

If the effects of viscosity, thermal conductivity, and diffusion are neglected, the equations of continuity, momentum, and energy for the mixture are, respectively,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 , \tag{3.1}
\]

\[
\frac{\partial}{\partial t} (\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V} \mathbf{V}) = -\nabla \rho , \quad \text{and} \tag{3.2}
\]

\[
\frac{\partial}{\partial t} [ \rho \left( \frac{1}{2} \mathbf{V}^2 + E \right) ] + \nabla \cdot [ \rho \mathbf{V} \left( \frac{1}{2} \mathbf{V}^2 + E + p/\rho \right) ] = -\nabla \cdot \mathbf{F} , \tag{3.3}
\]

where \( \mathbf{V} \) is the velocity and \( \mathbf{F} \) is the radiative energy flux vector, whose divergence gives the energy flux (per unit volume) out of an element. This flux consists of emission minus absorption of incoming radiation, and the latter accounts for the characteristic action at a distance. The radiative mechanisms are implicit in the expression for \( E \). Radiation pressure has been neglected in (3.2) and (3.3); the radiative energy density has been omitted from (3.3).

Concerning the formalism of radiative transfer, consider a reference surface \( \sigma \) with normal \( \mathbf{n} \) and elementary area \( d\sigma \) in the neighborhood of point \( P \) (see Fig. 1). Consider a ray that subtends an elementary solid angle \( d\sigma \) and passes through \( P \) in the direction of the unit vector \( \hat{\xi} \); this is oriented with respect to \( \mathbf{n} \) by the polar
angle $\theta$ and the meridional angle $\varphi$. Then the flux of radiant energy in time $dt$ across $d\sigma$ within the ray and in the frequency interval $dv$ is given by

$$ I_\nu \ dt \ \ell \cdot nd\sigma \ d\omega \ dv, \hspace{1cm} (3.4) $$

where $I_\nu$ is the specific intensity of radiation. The spectral radiative energy flux for all such rays per unit area of $\sigma$ and unit time is

$$ q_\nu = \int_{\lambda l} I_\nu \cos \theta \ d\omega, \hspace{1cm} (3.5) $$

where $d\omega = \sin \theta \ d\theta \ d\varphi$. Summing over $\nu$, the radiative flux is then

$$ q = \int_{0}^{\infty} q_\nu \ dv. \hspace{1cm} (3.6) $$

The associated radiative flux vector is given by the relations

$$ F = \int_{0}^{\infty} F_\nu \ dv, \hspace{1cm} F_\nu = \int_{\lambda l} I_\nu \ \ell \ d\omega \hspace{1cm} (3.7) $$

so that $n \cdot F = q$ on $\sigma$. When the field is in thermodynamic equilibrium $I_\nu = B_\nu$, the Planck function, which is independent of direction: the fluxes $q_\nu$ and $F_\nu$ then vanish.

IV. THE RADIATION MODEL

We limit the theoretical model of the gas to temperatures not in excess of $0.1 T_j$ or $0.2 T_j$, $T_j$ being of order $2 \times 10^5$ $^\circ$K. Indeed we shall make expansions for large $T_j/T$. For low densities the equilibrium value of $a$ can then reach or exceed 0.5, although the population of atoms in the first excited electronic state remains around 0.1 or less of those in the ground state. The ions are less excited. The error due
to taking as constant the ratio of partition functions of the atoms and
and the ions should not be great according to Alpher\textsuperscript{19}. If \( h\nu_j \approx \epsilon_j \), then
the edge frequency \( \nu_j \) is the minimum frequency photons can have if they
are to ionize atoms out of the ground state (the edge frequency is lower
for excited atoms whose low population is partially offset by a higher
ionization probability per atom). Suppose \( \nu_m \) is the value of \( \nu \) for which
\( B_{\nu}(T) \) vs. \( \nu \) is maximum. Since \( \nu_m/\nu_j = 2\hat{\nu}(T/T_j) \), it is apparent that most
of the radiant energy, at least for radiative equilibrium, is distributed
over the part of the spectrum below \( \nu_j \). Therefore a relatively small part
of the total radiant energy is associated with the radiative ionization
process in the temperature range considered.

The spectral absorption coefficient per unit volume \( K_{\nu} \) for the
mixture of atoms, ions, and electrons is shown schematically in Fig. 2
for the densities and highest temperatures of interest; by definition,
\( K_{\nu} \) depends on the nonequilibrium variable \( a \). Free-free continuum radia-
tion is neglected. The bound-free continuum absorption contribution
associated with photoionization of atoms is shown as the broken line and
one sees the edge frequencies mentioned above. In the statistical calcu-
lations discussed in Sect. V, a single, average-level atom (only one
edge frequency) is adopted and the integration required extends only
from \( \nu_j \) to \( \infty \). The broadened absorption lines associated with excita-
tion of atoms are suggested by the solid lines in Fig. 2. The wings
are sketched more accurately but the tall part of the line has been re-
placed for present purposes by a simple bar in a manner inspired by the
old quantum mechanics. The main line is due to resonance radiation of
atoms. The lines for the ions fall mainly to the right of \( \nu_j \) and are
not shown. For the present study of nonequilibrium ionization we shall
now virtually dispose of the line radiation on one of two bases; the fulfillment of either would validate our results for a specific application.

The equation of radiative transfer along the generic ray of Fig. 1 is the energy balance

$$\frac{\partial I_v}{\partial s} = K_v(S_v - I_v),$$  \hspace{1cm} (4.1)$$

where $s$ is the coordinate shown and $S_v$ is the source function; this would be equal to $B_v$ if only line radiation were present since the electronic states are in a local thermodynamic equilibrium corresponding to the current value of $a$. We observe in (4.1) that radiation is transferred independently at each $v$, and in Fig. 2 that the characteristic heights of the bars, wings, and continuum curve commencing at $v_j$ are disparate. Although the absorption lines extend to the right of $v_j$, these two observations permit the radiative flux due to each absorption component to be discussed separately, and the results then matched in an asymptotic sense. Let us then divide the radiative flux vector at the point $P$ according to the relation

$$F = F_\ell + F_{\ell\ell} + F_j,$$  \hspace{1cm} (4.2)$$

where the subscript $\ell$ refers to the tall part of the lines (i.e. the bars), $\ell\ell$ refers to the line wings, and $j$ refers to the continuum. Associated with each contribution is a suitable frequency-averaged absorption coefficient with units of \((\text{length})^{-1}\). Taking the reciprocal we introduce an associated average radiative absorption length (also called average photon mean free path) denoted by $\lambda$. The disparate ordinates in Fig. 2 then lead to
\[ \lambda_{\ell \ell} \gg \lambda_j \gg \lambda_{\ell} \] \hfill (4.3)

Subject to some qualification, the ionization (or deionization) process will occur in a region the extent of which is of order \( \lambda_j \). From (4.3), this region of interest will be optically thin with respect to \( \ell \ell \)-radiation and optically thick with respect to \( \ell \)-radiation. Taking \( \lambda_j \) as the unit of length, the established theory then gives relations over the region of interest of the form\(^{20}\)

\[ F_{\ell \ell} = \text{const} [1 + O(\lambda^{-1}_{\ell \ell})], \] \hfill (4.4)

and

\[ F_{\ell} = -\frac{1}{3} \lambda_{\ell} (\sum_{\nu} \frac{dR_{\nu}}{dT}) \nu T[1 + O(\lambda_{\ell})]. \] \hfill (4.5)

To lowest order \( F_{m \ell} \) is constant\(^{20}\) in the region of interest and \( F_{m \ell} \) is zero, unless a very steep gradient in temperature (as in a shock) occurs.

The \( \ell \)-radiation has thus been put in the genus of the three transport effects, and the ionization process appears embedded in a much larger region of thermal transfer, which, to be sure, determines the value of the constant in (4.4). It is significant that we arrive at three relevant average absorption coefficients. To the next order of approximation, all three would enter the problem and the dependence of \( \lambda_{\ell \ell} \) and \( \lambda_{\ell} \) on \( \alpha \) would require that the problem for the ionization process be solved first.

As an illustration let us consider the strong normal shock wave. For what concerns the cold gas upstream of the discontinuity, the radiative transfer resembles that between a semiinfinite hot emitting gas at rest and a semiinfinite cold absorbing gas at rest when the two are separated by a transparent window. With the relaxation behind the
discontinuity thus overlooked, black-body radiation passes through the window, so \( I_\nu = B_\nu \). The bars in Fig. 2 effects deep notches in the curve of \( B_\nu \) vs. \( \nu \) over a very short distance \( \sim \lambda_{\ell} \). Then photoionization of the cold, unexcited gas occurs over a layer whose thickness is \( \sim \lambda_{\ell} \). But most of the energy travels over a great distance \( \sim \lambda_{\ell} \) before it is finally absorbed. An example of a radiation-structured shock wave in which the inner discontinuity is further structured into an optically thin layer by introducing a much smaller mean free path has been given recently by Cohen and Clarke.

The range of problems embraced by the present formulation is quite broad. Should the argument leading to (4.2)-(4.5) fail to apply to sufficient accuracy in a specific application, account may be taken of the following. Whereas all three components in Fig. 2 enter the fluid energy balance (3.3), only the bound-free component discussed affects the production and disappearance of ions and enters the nonequilibrium ionization-rate equation to be developed subsequently. If, as we often find, the radiative term is small in the former equation but significant in the latter equation, then the line radiation drops out of the problem considered.

V. STATISTICAL FOR PRODUCTION AND DISAPPEARANCE OF IONS

Denote by \( \eta_0 > 0 \) the mass of ions produced per unit time per unit volume. Dewan has summarized (see his sources and Unsold) the statistical mechanical calculations of the particle encounters contributing to \( \eta_0 \) that will be adopted here. Thus, the basic assumptions are that (i) ionization occurs only by atom-electron collision and
absorption of a photon by an atom (photoionization), (ii) recombination is due only to ion-electron-electron collision and spontaneous and induced radiative recombination, (iii) atoms and ions are considered to have only one average electronic energy level (on the basis of Sect. IV, this is not considered greatly different from the ground level here). Pertschek and Byrom have observed that other particle collisions become important for ionization when the initial supply of electrons is very low, but we find that photoionization tends to create the electrons required to trigger the atom-electron collision reaction.

Then if \( S^{(2)}_{1}(\epsilon_{i}, \epsilon_{b})d\epsilon_{b} \) is the elementary collisional ionization cross-section, corresponding to an interval \( \epsilon_{b} \) to \( \epsilon_{b} + d\epsilon_{b} \) of the energy of the departing bound electron, and to an energy \( \epsilon_{i} \) of the incident free electron, the number of ions produced per unit time per unit volume by collision is \(^{21}\)

\[
\frac{m_{ic}}{m_{1}} = n_{a}n_{e} \int \int_{\epsilon_{i}}^{\epsilon_{j}} f(\epsilon_{i})(2\epsilon_{i}/m_{e})^{1/2}d\epsilon_{i} \int_{0}^{\epsilon_{i} - \epsilon_{j}} S^{(2)}_{1}(\epsilon_{i}, \epsilon_{b})d\epsilon_{b}, \quad (5.1)
\]

where \( m_{1} \approx m_{a} \) is the mass of an ion, \( m_{e} \) is the mass of an electron, and \( f(\epsilon_{i}) \) is the distribution function for the energy of free electrons.

If \( S^{(1)}_{2}(\epsilon_{1}, \epsilon_{2}) \) is the collisional recombination cross-section, corresponding to the energy of the two colliding free electrons in the range \( \epsilon_{1} \) to \( \epsilon_{1} + d\epsilon_{1} \) and \( \epsilon_{2} \) to \( \epsilon_{2} + d\epsilon_{2} \) respectively, the number of ions disappearing per unit time per unit volume by three-body collision is \(^{21}\)

\[
\frac{m_{nc}}{m_{1}} = -n_{2}^{2}n_{i} \int \int_{0}^{\epsilon_{1}} f(\epsilon_{1})f(\epsilon_{2})(2\epsilon_{1}/m_{e})^{1/2}(2\epsilon_{2}/m_{e})^{1/2}S^{(1)}_{2}(\epsilon_{1}, \epsilon_{2})d\epsilon_{1}d\epsilon_{2}. \quad (5.2)
\]
If $\psi_1^{(2)}(v)$ is the probability per unit time per unit solid angle that an atom absorbs an ionizing photon in a radiation field of unit specific intensity, the number of ions produced per unit time per unit volume by photoionization is 

$$\mathcal{M}_{ir}/m_i = n_a \int_{v}^{\infty} \psi_1^{(2)}(v) dv \int_{\text{lm}} I_v dv.$$  (5.3) 

The excess photon energy is carried off by the departing electron. The associated ionization cross-section is given by $\sigma_{ir} = h\nu \psi_1^{(2)}$. 

Finally, if $a_2^{(1)}(\epsilon)$ is the probability per unit time per incident free electron in the energy range $\epsilon$ to $\epsilon+d\epsilon$ that an electron is captured by an ion and a photon is emitted spontaneously, and if $\beta_2^{(1)}(\epsilon)$ is the corresponding probability for induced radiative capture, we have 

$$\mathcal{M}_{nr}/m_i = -n_e n_i \int_{0}^{\infty} f(\epsilon) a_2^{(1)}(\epsilon) d\epsilon - h^{-1} n_e n_i \int_{0}^{\infty} f(\epsilon) \beta_2^{(1)}(\epsilon) d\epsilon \int_{\text{lm}} I_v dv.$$  (5.4) 

The energy of the photon emitted is related to the energy of the incident electron by $h\nu = \epsilon_j + \epsilon$.

The cross-sections $S_1^{(2)}$ and $S_2^{(1)}$ are related by the formula

$$(h\nu n_e / h^3) (Q_1 / Q_a) \epsilon_j \epsilon_2 \beta_2^{(1)} = \epsilon_2^{(2)} a_1^{(2)}.$$  (5.5) 

where $Q_1$ and $Q_a$ are the electronic partition functions of the ions and atoms, respectively. The following relations, analogous to those between the Einstein coefficients of absorption and emission, also hold:

$$a_2^{(1)} / \beta_2^{(1)} = 2h\nu^2 / c^2; \psi_1^{(2)} / a_2^{(1)} = [(m_e c^2) / (h\nu)^3] (Q_1 / Q_a) (2\epsilon / m_e)^{1/2}.$$  (5.6)
Equations (5.5) and (5.6) are valid in the same sense that (2.3) is valid, and illustrate that the nonequilibrium processes being considered are out of equilibrium in a restricted sense, in terms of statistical mechanics.

VI. EQUATION OF RADIATIVE TRANSFER IN THE CONTINUUM

For each ion produced by photoionization in accord with (5.3), a photon is absorbed; for each ion removed by radiation recombination in accord with (5.4), a photon is emitted. From (5.3), the number of photons absorbed per unit time per unit volume per unit solid angle per unit frequency interval is evidently equal to \( n^a_1 \psi^{(2)}_1 i_\nu \). From (5.4) and the relation \( h \nu = \varepsilon + \epsilon \), the corresponding number of photons emitted is equal to 

\[
\frac{1}{4} \pi^2 n^1_2 \psi^{(1)}_2 h + \frac{1}{4} \pi^2 n^1_1 \psi^{(1)}_1 h.
\]

Multiplying each expression by the energy per photon \( h \nu \), we arrive effectively at the definitions of absorption coefficient (times \( i_\nu \)) and emission coefficient, respectively. Then the equation of radiative transfer in the continuum is

\[
\frac{\partial i_\nu}{\partial s} = \frac{1}{4} \pi^2 n^1_2 \psi^{(2)}_1 h \left[ a^{(1)}_2 (\epsilon) + b^{(1)}_2 (\epsilon) i_\nu \right] - n^a_1 \sigma_{ir}(\nu) i_\nu,
\]  

(6.1)

since \( \sigma_{ir} = h \nu \psi^{(2)}_1 \). The second term in the emission coefficient gives the induced radiative emission.

VII. NONEQUILIBRIUM IONIZATION-RATE EQUATION

The equation of the conservation of ions,

\[
\rho \frac{D \eta}{D t} = \eta_0 = \eta_{ic} + \eta_{nc} + \eta_{ir} + \eta_{nr},
\]  

(7.1)

gives the ionization rate when \( a \) lags behind the local equilibrium value, since the four contributions to \( \eta \) have been given in Sect. V.
We can obtain another form of (7.1) that is significant and generally more useful by essentially using the procedure in Sect. VI backwards. That is, integrate (6.1) over all directions and use the standard manipulation on the left side; the result is

\[ \mathbf{v} \cdot \mathbf{F} = n v n f(\varepsilon)[a_2(\varepsilon) + \ln^{-1} (1) \rho_2(\varepsilon)] I_\nu \int_{\nu} \mathbf{F} d\omega - h\nu n v a_1 \psi(\nu) I_\nu \int_{\nu} \mathbf{F} d\omega, \]

Then divide by the photon energy \( \nu \) and integrate over frequency. By comparison with (5.3) and (5.4), we obtain

\[ m_a^{-1} (\mathbf{m}_\mu + \mathbf{m}_{n\nu}) = - \nu \int_{\nu} d\nu (\nu \nu)^{-1} \mathbf{F}, \]  

(7.2)

wherein the integral on the right may be called the photon particle flux vector. Equation (7.2) is a direct consequence of the first sentence in Sect. VI. Substituting (7.2) in (7.1), we obtain for the rate equation

\[ \frac{\partial}{\partial t} (\rho a) + \mathbf{v} \cdot (\rho \mathbf{v} a) = - m_a \nu \int_{\nu} d\nu (\nu \nu)^{-1} \mathbf{F} + \mathbf{m}_\mu + \mathbf{m}_{n\nu}. \]  

(7.3)

The radiation now enters both (3.3) and (7.3) as the divergence of a flux vector. Moreover, if we were to approximate \( \nu \mathbf{v} \) by \( \nu_j \) in (7.3) and multiply (7.3) by \( \nu_j / m_a = \nu_j R_T \), then both (7.3) and (3.3) would have on the right the radiation term \( \mathbf{v} \cdot \int_{\nu_j} d\nu \mathbf{F} = \mathbf{v} \cdot \mathbf{F} \) and on the left the ionization energy term \( \mathbf{v} \cdot (\mathbf{m}_\mu R_T) \), in virtue of (2.3). These correspondences in (3.3) and (7.3) are important and are studied in Sect. XVII.

If the flow is steady and if it is rarefied enough or cool enough in some region that the collision processes can be neglected there, then (7.3) gives a divergence-free quantity:
\[ v \cdot \left[ \frac{m_a}{m_i} V_i + \int_{V_j}^{\infty} dv(h\nu)^{-1} F_{\nu} \right] = 0. \]  

(7.4)

Since the first term is \( n_i V_i \), (7.4) means that the particle flux of the "mixture" of ions and ionizing photons is conserved. Evidently, the steady collisionless rate equation (7.4) is readily integrated in special cases. For example, in rectilinear one-dimensional flow the flux over a surface perpendicular to \( x \) is

\[ m_a^{-1} p u a + \int_{V_j}^{\infty} dv(h\nu)^{-1} q_\nu = \text{const}, \]

(7.5)

where \( u \) is the component of \( V \) in the x-direction, and, from (3.1), \( p u \) is a constant (say greater than zero). Consider again the strong normal shock wave. Upstream of the discontinuity both collision processes can usually be neglected if \( a \ll 1 \), and (7.5) gives the flux of photoionized atoms due to the negative counterflux of photons that originate from the radiative recombination process occurring downstream of the discontinuity. If we imagine conditions (as in stellar applications) where both collision processes are small downstream of the discontinuity as well and then tentatively neglect them, (7.5) applies in consequence from \(-\infty\) to \( +\infty \) (since \( a \) and \( q_\nu \) are continuous at a discontinuity). At \( +\infty \), the photon particle flux must vanish for equilibrium. Equation (7.5) shows then that steady-state transition from one equilibrium state of ionization to another one by a purely radiative ionization and deionization process is impossible, since \( a(\infty) = a(-\infty) \); the gas may become photoionized by the internal radiative flux, but it would have to revert; even slow collision processes are ultimately crucial. This result is physically quite plausible.
EQUATIONS (7.1) AND (7.3) CAN BE MADE EXPLICIT BY INSERTING
APPROPRIATE CROSS-SECTIONS IN THE EXPRESSIONS FOR \( \eta_{ic} \), \( \eta_{nc} \), \( \eta_{ir} \), AND
\( \eta_{nr} \) IN SECT. V. THOSE INTEGRALS THAT DO NOT CONTAIN \( I_v \) CAN THEN BE
INTEGRATED \(^{21,9}\), BECAUSE THE DISTRIBUTION FUNCTION FOR THE ENERGY OF THE
FREE ELECTRONS IS MAXWELLIAN IN ACCORD WITH SECT. II:

\[
f(\varepsilon) = 2\pi^{-1/2}(\kappa T)^{-3/2}\varepsilon^{1/2}\exp(-\varepsilon/\kappa T).
\]  

\[\text{(8.1)}\]

FOR THE CROSS-SECTION \( S_1^{(2)} \), WE USE THAT OF THOMSON \(^{23,21,9}\):

\[
S_1^{(2)}(\varepsilon_i, \varepsilon_b) = \pi\varepsilon\varepsilon_b[\varepsilon_i^2(\varepsilon_i + \varepsilon_b)^2]^{-1}.
\]  

\[\text{(8.2)}\]

HERE \( e \) IS THE CHARGE OF AN ELECTRON. FROM (5.5) AND (8.2),

\[
S_2^{(1)}(\varepsilon_1, \varepsilon_2) = 2^{-1/2}\varepsilon_1^{3/2}\varepsilon_2^{-1/2}(\alpha/\alpha_1)^{1/2}[\varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2)^2]^{-1}.
\]  

\[\text{(8.3)}\]

INSERTING THE RESULT FOR \( \eta_{ic} + \eta_{nc} \) IN (7.3), WE OBTAIN FOR THE RATE EQUATION

\[
\frac{\partial}{\partial t}(\rho a) + \nabla \cdot (\rho Va) = -m_a\nabla \cdot \int_0^\infty dv (hv)^{-1}F_v + \int_0^\infty \frac{dv}{m_v}v \int_0^\infty dv (hv)^{-1}F_v + \alpha(1-a)\rho c_{ic}^{-1/2}\exp(-T_j/T)[1-2(T/T_j) + \ldots] - \alpha^2\rho c_{nc}^{-1/2}[1-2(T/T_j) + \ldots],
\]  

\[\text{(8.4)}\]

WHERE

\[
\eta_i = \frac{m_a^{-1}\rho a}{m_a^{-1}\rho (1-a)},
\]  

\[\text{(8.5)}\]

\[
c_{ic}^{-1} = 2^{3/2}m_a^{-1/2}e^{1/2}m_a^{-1}e^2, \quad c_{nc}^{-1} = 2^{-1/2}m_a^{-1}e^{-1}m_a^{-1}e^2(\alpha/\alpha_1)e_j^2.
\]  

\[\text{(3.6)}\]

THE EXPONENTIAL INTEGRAL \( E_1(T_j/T) \) OCCURS IN \( \eta_{ic} \) AND \( \eta_{nc} \) AND THIS
HAS BEEN APPROXIMATED \(^{9}\) FOR LARGE \( T_j/T \) BY THE ASYMPTOTIC EXPANSION
\[ E_{(1)}(x) = \int_x^\infty t^{-1} e^{-t} \, dt = x^{-1} e^{-x(1-x^{-1}+2x^{-2}-\ldots)}. \]  

(8.7)

To determine \( a_{2}^{(1)} \), we use the associated cross-section for spontaneous radiative capture of electrons by ions obtained by Stueckelberg and Morse\(^{24,21,9}\) and find

\[ a_{2}^{(1)}(\varepsilon) = \mathcal{A} \varepsilon^{-1/2}, \]

(8.8)

where \( \varepsilon \) is in ergs, and

\[ \mathcal{A} = 7.53 \times 10^{-19} \sum_{r=1}^{\infty} A(r) \text{gm}^{1/2} \text{cm}^{1/2} \text{sec}^{-2}. \]

(8.9)

The \( A(r) \) are constants depending on the total quantum number \( r \) and the summation extends over the electronic energy levels [see item (iii) in Sect. V]. The summation in (8.9) is somewhat uncertain in magnitude but Dewan has proposed the value unity. If this value is not used, recourse may be made to measured cross-sections to determine this constant. From (5.5) and (8.8) there results

\[ \beta_{2}^{(1)} = 2^{-1} \mathcal{A} c^{2} \varepsilon^{-1/2} h^{-1/2} \psi_{1}^{(2)} = 2^{1/2} \mathcal{A} c^{2} m_{e}^{3/2} (Q_{i}/Q_{a})(\hbar \nu)^{-3}. \]

(8.10)

Then from (5.3), (5.4), (8.1), (8.8), and (8.10) we obtain

\[ m_{ir} + m_{nr} = (1-\alpha) \rho c_{ir}^{-1} \int_{j}^{\infty} \rho v^{-3} \int_{h}^{\infty} \rho d\omega I_{\nu} - \alpha^{2} \rho c_{nr}^{-1} \nu^{-1/2} \]

(8.11)

\[ = -m_{a} v \int_{j}^{\infty} \rho (\hbar \nu)^{-1} d\nu, \]

where

\[ c_{ir}^{-1} = 2^{1/2} c h^{-3/2} \mathcal{A}(Q_{i}/Q_{a}), \quad c_{nr}^{-1} = 2n^{-1/2} k^{-1/2} m_{a}^{-1} \mathcal{A}. \]

(8.12)
The induced radiative recombination has been neglected in (8.11) for this reason: if $a$ has the local equilibrium value, one knows the effect is accounted for by the factor connecting the absorption coefficient and the Rosseland reduced absorption coefficient, the factor being $[1 - \exp(-h\nu/kT)]$. Now $h\nu > h\nu_j$ and $1 > \exp(-T_j/T)$ when $(T_j/T) > 1$ according to Sect. IV. The effect is evidently completely negligible, a result not likely to be reversed by nonequilibrium considerations. If the divergence term is eliminated between (8.11) and (8.4), we get an explicit version of the rate equation in the form of (7.1).

If the collision reaction time is much smaller than either the radiative reaction time or the flow transit time in (8.4), then $a$ would not normally lag the local state. Omitting the nondimensionalization step, let $C_{nc} \rightarrow 0$ such that $C_{nc}/C_{ic}$ and the other terms in (8.4) are of order unity. The bracketed series expansions cancel out and we get

$$\frac{a^2}{E} = (C_{nc}/C_{ic})\rho^{-1} \exp(-T_j/T),$$

(8.13)

which is the Saha Equation. More generally, when nonequilibrium ionization is under way, $a_E$ signifies what the equilibrium value of $a$ would be at the local density and temperature.

It is very useful to recast (8.4) in the traditional normalized form of chemical kinetics, using (8.13). Multiplying and dividing the two collision terms by the collisional recombination term, we get

$$\frac{\partial}{\partial t}(\rho a) + \mathbf{v} \cdot (\rho \mathbf{v} a) = -m_a \mathbf{v} \cdot \int_{-\infty}^{\infty} d\nu (\nu) \nu^{-1} F_{\nu} + \rho a \tau_c^{-1} [(1-a) a_E^2 (1-a_E) - a^2],$$

(8.14a)

where

$$\tau_c = \frac{C_{nc} \rho^{-1}}{T (1-2(T/T_j) + \ldots)^{-1}}$$

(8.14b)

is the local characteristic time of the collision ionization. It will give a good numerical indication of the actual time for collisional relaxation if the term in brackets in (8.14a) is not too far from unity. If all radiative effects are neglected, (8.14) together with (2.1), (2.3), and (3.1)-(3.3) then reduce to the usual formulation of a nonequilibrium flow.
IX. EXPLICIT EQUATION OF RADIATIVE TRANSFER IN THE CONTINUUM

In (6.1) we neglect $\beta_2^{(1)} I_\nu$, express $a_2^{(1)}$ in terms of $\sigma_{ir}$ by means of (5.6), insert (8.1) wherein $\varepsilon = \nu_0 - \nu_j$, and take note of (8.13). After employing some algebra and some of the ancilliary relations given, we obtain

$$\frac{\partial I_\nu}{\partial s} = n_a \sigma_{ir} \left[ \frac{\nu}{(1-\nu)} \right] \left[ \frac{(1-a_0)}{a_0^2} \right] B_\nu(T) - I_\nu, \quad (9.1a)$$

wherein the Planck function has been approximated according to

$$B_\nu(T) = \frac{2h\nu^3c^{-2}}{\exp(h\nu/kT) - 1} \approx \frac{2h\nu^3c^{-2}}{\exp(-h\nu/kT)}, \quad (9.2)$$

since $\nu \gtrsim \nu_j$ and $\exp(T_j/T) \gg 1$. We have that $n_a \sigma_{ir} = K_\nu = \lambda_\nu^{-1}$, where $\lambda_\nu$ is the photon mean free path at frequency $\nu$, so (9.1a) is now displayed in the language of (4.1). We see at a glance how the present nonequilibrium source function is related to the equilibrium source function $B_\nu$: if, in (8.14), the collisional reaction time is very small compared with both the radiative reaction time and the flow transit time, then (8.14) gives $a = a_0$, and (9.1a) gives $B_\nu$ for the source function; the problem then reduces to the usual formulation of radiative gas dynamics corresponding to local thermodynamic equilibrium of the gas.

To show that the Planck function need not appear at all in a nonequilibrium source function, one may repeat the derivation of (9.1a) retaining the induced emission and not otherwise approximating for large $(T_j/T)$. The result is shown in the Appendix.

To preserve generality, we still have not made use of the Stueckelberg and Morse cross-section. Using (8.5) and (8.10), we now subjoin to (9.1a) the evaluation

$$n_a \sigma_{ir} = \lambda_\nu^{-1} = \rho(1-\nu)x_\nu; \quad x_\nu \equiv \frac{h}{m_a} \frac{1}{c \nu} \frac{1}{\nu^2}, \quad (9.1b)$$

where $x_\nu (= n_a^{-1} \sigma_{ir})$ is the absorption coefficient per unit mass of atoms.
X. INTERPRETATIONS

The problem is now formulated analytically by (8.14) and (9.1) together with the equations of Sects. II and III. But at this point it is well to identify certain parameters on which the problem hinges physically; all of these will arise naturally in the subsequent treatment. Let \( x \) be the value of \( x_v \) for \( v = v_j \); it will soon be shown that the relevant average absorption coefficient may be approximated by \( x \) if the optical path is not large. Correspondingly, let \( \lambda_j^{-1} = \rho(1-\alpha)x \). It is convenient to write

\[
x_v = x(v_j/v)^2.
\]  

(10.1)

The relative importance and speed of the competing collisional and radiative reactions in different regions during the ionization process is evidently of great interest, and it is well to identify explicitly the radiative reaction time already referred to. It is not readily displayed in (8.14) in the divergence of the photon particle flux, but we may fall back on the connection given by (8.11). However, this is equivalent to integrating (9.1) over all directions, multiplying by \((hv)^{-1}\), and integrating over frequency [after having inserted (10.1)]. The result of this calculation is

\[
-m_a v \int_j^\infty dv (hv)^{-1} F_v = (1-\alpha)\rho(R_T j)^{-1} \int_j^\infty dv (v/v_j)^{-3} \int_4^\infty d\omega \int_v^\infty \frac{d\omega d\nu}{\nu}\frac{d\omega d\nu}{\nu} - a^2 \rho \pi (RT_j)^{-1} \frac{d\omega d\omega}{\omega} \alpha_a(1-\alpha_E) B_{j3},
\]  

(10.2)

where

\[
B_{j3} = \int_j^\infty dv (v/v_j)^{-3} = 2c^{-2}k v_j^3 T \exp(-T_j/T).
\]  

(10.3)

We now form \( \tau_r \), the local characteristic time of radiative ionization, in (10.2) in the same way that \( \tau_c \) was formed in (8.14). The result is
\[ -m_{\alpha_{0}} \int_{\nu}^{\infty} \frac{\rho_{\nu} \nu}{\omega_{\nu}} \left( 1 - \alpha_{E} \right) \frac{\alpha_{E}^{2}}{\left( 1 - \alpha_{E} \right) \left[ \frac{1}{2} \int_{0}^{\infty} \frac{\nu}{\omega} \omega d\omega \right]^{3}} \nu \omega d\omega \nu \nu = \rho_{\nu} \int_{\nu}^{\infty} \frac{\alpha_{E}^{2}}{\left( 1 - \alpha_{E} \right) \left[ \frac{1}{2} \int_{0}^{\infty} \frac{\nu}{\omega} \omega d\omega \right]^{3}} \nu \omega d\omega \nu \nu \]

where

\[ \tau_{1}^{-1} = \frac{4 \pi \alpha_{E} B_{3}}{\left( 1 - \alpha_{E} \right) \alpha_{E}^{2}} \]

Equation (10.1a) may be substituted into (8.1a) and the two reactions compared on the same basis. However, the italicized statement at the end of Sect. VII must be borne in mind.

Substituting (8.13) and (10.3) into (10.1b), we find

\[ \tau_{r} = C_{\nu} \frac{\rho_{\nu}^{2}}{\nu r^{2}} \]

In subsequent work, it is \( \left[ \frac{1 - \alpha_{E}}{\alpha_{E}^{2}} \right] B_{3} \) that appears and (10.1b) and (10.5) should be regarded as an interpretation and evaluation of this term. If we neglect the term in brackets in (8.11b) and combine (8.11b) with (10.5), we find the connection

\[ \tau_{c} = C_{\nu} \frac{C_{\nu}^{2}}{\nu r^{2}} \]

XI. GENERAL FORMAL SOLUTION FOR \( I_{\nu} \)

Equation (9.1) can be solved formally for \( I_{\nu} \) in the usual way.

The optical path (roughly distance divided by photon mean free path) is defined by

\[ d\tau_{\nu} = \rho (1 - \alpha) x_{\nu} ds, \quad \tau_{\nu} = \int_{0}^{s} \rho (1 - \alpha) x_{\nu} ds. \]

Here \( s \) and \( \tau_{\nu} \) are true (signed) coordinates with origin at the generic "influenced" point \( P \) in Fig. 1. Then \( I_{\nu} \) at \( P \) is
wherein $\tau_v < 0$ is the coordinate of the generic "influencing" point $P'$ on the incident ray, and the integration sums such influences on the ray considered with direction $l$. If $\tau$ is the value of $\tau_v$ for $v = v_j$, then $\tau_v = \tau(v_j/v)^2$.

From (3.3) and (8.1b), it is evidently necessary to calculate $F_j = \int_{\nu_j}^{\infty} d\nu f_v$ and $\int_{\nu_j}^{\infty} d\nu (h\nu)^{-1} f_v$, from (11.1) to obtain the two radiative contributions to the problem. From (3.7) this leads, in general, to a volume integration of the influences plus two distinct frequency integrations. It is the latter fact that makes the problem particularly difficult; however, this and the other spectral difficulties of the problem can be neatly removed when it is possible to approximate asymptotically for large $(T_j/T)$. Accordingly, we resolve the problem into two cases:

Case 1 - the thickness (or dimensions) of the region is comparable to or less than the photon mean free path $\lambda_j$ in the sense that $|\tau| \sim 1$. The ratio $(T_j/T)$ is sufficiently large that terms of order $(T/T_j)$ are negligible compared with terms of order unity.

Case 2 - the restrictions of Case 1 are not imposed. In the next section we approximate for Case 1.

XII. FREQUENCY INTEGRATION OF $I_v$ FOR CASE 1

From (11.1b), we obtain

$$\int_{\nu_j}^{\infty} d\nu (h\nu)^{-m} I_v(P) = \int_{\nu_j}^{\infty} d\nu (h\nu)^{-m} \frac{\alpha}{(1-\alpha)} \frac{(1-\alpha)_{1/2}}{E^2} e^{\tau_v} ,$$

(12.1)
where we have reverted to the coordinate \( s \) in order to reverse the order of integration. Here \( m = 0 \) and \( m = 1 \) correspond to the 2 frequency integrations of interest (Sect. XI). We now consider the problem of evaluating the integral

\[
\int_{v_j}^{\infty} d\nu (\nu - v_j)^{m} e^{\nu - v_j} = 2c^{-2} \int_{v_j}^{\infty} d\nu (\nu - v_j)^{1-m} e^{-h\nu/kT} e^{\nu_j^2/\nu^2}.
\]  

(12.2)

Since \( \tau \approx 1 \), we expand according to

\[
e^{\nu_j^2/\nu^2} = 1 + \nu_j^2/\nu^2 + \nu_j^4/2\nu^4 + \ldots.
\]  

(12.3)

We see that we are obliged to consider the integral of various frequency moments of \( e^{-h\nu/kT} \) for large lower limit \( \nu_j \approx (T_j/T) \). An indication of the kind of simplification that can be obtained may be seen by examining the asymptotic expansion (6.7) of \( E_1(x) \) for large \( x \approx (T_j/T) \); we have, in fact, already approximated the collision reaction for large \( (T_j/T) \) with (6.7). The exponential \( e^{-t} \) falls so rapidly as \( t \) increases from \( x \) that the variation of \( t^{-1} \) from its initial value \( x^{-1} \) is unimportant. In fact, if we approximate the moment \( t^{-1} \) by \( x^{-1} \) and take it out of the integral, then we obviously obtain correctly the first term in the series.

To obtain an indication of the error of such an approximation, let \( m = 1 \), insert (12.3) into (12.2), and set

\[
t = h\nu/kT.
\]  

(12.4)
Then
\[ f_1 = 2c^{-2\tau} e^{2\tau T/T} \int f_j e^{-T_j/T} \, dt - ^{\frac{1}{2}} \int \frac{h^2 \tau}{1 - \tau^2} \tau^2 \int \frac{e^{\tau}}{1 - \tau^2} \, dt \]
\[ \text{(12.5)} \]

Now integrate each integral by parts to form \( E_{(1)} \) and then insert in the result (8.7), using 5 terms. The result is
\[ f_1 = 2c^{-2\tau} e^{-1/kT \nu_j^2} e^{-T_j/T} \{ 1 + \tau [1-2(T/T_j) + O(T^2/T_j^2)] \}
\[ \text{(12.6)} \]
\[ + \frac{1}{2} \tau^2 [1-4(T/T_j) + O(T^2/T_j^2)] + \ldots \} \]

Comparing to (12.3), we see that the expansion in braces is summable as \( e^\tau \) to the lowest order as \( (T_j/T) \to \infty \), signifying that the variation of \( e^\tau \nu_j \) with \( \nu_j \) in (12.2) and (12.3) is negligible to this order. But the approximation is evidently nonuniform in distance and the error increases with each term in the exponential series; moreover, 4 or 5 terms are needed in the expansion of \( e^\tau \) for \( \tau = -1 \) to get sufficient accuracy. However, (12.6) displays a fortuitous result: the first-order correction terms \(-2\tau(T/T_j) \) and \(-2\tau^2(T/T_j) \) cancel for \( \tau = -1 \) and the first first-order correction then appears in the fourth term of the exponential expansion in braces. The discussion of (12.2) and (12.3) for \( m = 0 \) is similar but the asymptotic result for \( \tau = 0 \) is not exact, as above, but involves the approximation \( 1 \gg (T/T_j) \). Further indication of how this error of the lowest-order approximation increases with distance can be obtained from the precursor calculations in Sect. XVIII.
A little further consideration leads to the central observation that the integral

$$\int_{v_j}^{\infty} dv \, v^{-n} e^{-h v / k T} \sim \int_{(T_j / T)}^{\infty} dt \, t^{-n} e^{-t} \quad (12.7a)$$

behaves like

$$v_j^{-n} \int_{v_j}^{\infty} dv \, e^{-h v / k T} \sim (T_j / T)^{-n} \int_{(T_j / T)}^{\infty} dt \, e^{-t} \quad (12.7b)$$

to the lowest order in \((T_j / T) = (h v_j / k T)\) as \((T_j / T) \rightarrow \infty\), where the values of the integer \(n\) of interest here are \(-1, 1, 2, 3, 4, \ldots\).

The proof for \(n = 2, 4, \ldots\) is, in effect, provided by (12.6). The error clearly increases with \(|n|\). The lowest-order asymptotic evaluation of the integral \(f_m\) reduces to use of three word statements which are corollaries of (12.2), (12.3), and (12.7). Corollary I: the gas is effectively gray above \(v = v_j\), i.e. \(\chi \rightarrow \chi\) so that \(\tau_v \rightarrow \tau\); the result also holds for power laws other than (10.1), particularly for Kramer's absorption coefficient. Corollary II: the two required frequency moments (corresponding to \(m = 0\) and \(m = 1\)) of \(I_v\) in (12.1) or of \(F_m\) in (3.3) and (3.14) are now merely proportional; thus

$$\int_{v_j}^{\infty} dv \, (h v)^{-1} I_v = (h v_j)^{-1} \int_{v_j}^{\infty} dv \, I_v, \quad \int_{v_j}^{\infty} dv \, (h v)^{-1} F_m = (h v_j)^{-1} F_m, \quad \int_{v_j}^{\infty} dv \, (h v)^{-1} M_m = (h v_j)^{-1} M_m \quad (12.8)$$

and we shall have to consider only the moment \(m = 0\) henceforth.

Corollary III: \(B_v\) behaves like \(2 h v_j^2 c^{-2} \exp(-h v / k T)\).

The frequency integral in (12.1) may now be evaluated by inspection and, in fact, it is easier to return to (11.1b) and write, for \(m = 0\),
\[ I_j(P) \equiv \int_0^\infty dv I_v(P) = \int_0^\infty d\tau \frac{a_0^{2}(1-a)}{(1-a)^2}\left(\frac{a_0^2}{a_E^2}\right)B_j\tau, \]  \hfill (12.9a)

where

\[ B_j = \int_0^\infty dv B_v = \int_0^\infty dv v^2 h_v^3 e^{-\frac{v^2 h_v}{kT}} = 2c^2 v^3 kT e^{-\frac{T_j}{T}}. \]  \hfill (12.9b)

The integral \( \int_0^\infty dv (hv)^{-1} I_v \) would be \( (hv_j)^{-1} \) times (12.9a) and this result may also be obtained by substituting the lowest-order version of (12.6) into (12.1). The approximate version of \( B_j \) above is evidently the same as the integral \( B_j^3 \) occurring in Sect. X.

It is clear that the asymptotic expansion can also be carried out to two terms in \( T_j/T \), as has been indicated in the collisional counterpart (8.14); but a second approximation for the spectral aspects [and in (8.14)] is considerably more complex. We find in Sect. XVIII that the first approximation, for \( \tau \leq 3 \) and for \( T_j/T = 10 \) and 12, gives an adequate description in a severe test whereas the second approximation would give very good numerical accuracy: the approximation is \( 1 \gg 2(T/T_j)^2 \) at \( \tau = 0 \). In the case of a slab whose thickness is comparable to \( \lambda_j \) (and \( \tau \gg 1 \)), those glancing rays for which \( \theta = \pi/2 \) will travel many \( \lambda_j \) to traverse the slab and the above restriction \( \tau \leq 1 \) will be nonuniform in \( \theta \). The required modification of the procedure of this section is to integrate (11.1b) over all directions before integrating over frequency and approximating in accord with Case 1. The lowest-order results above are unchanged by this consideration. The directional integration requires treatment of a specific problem, whereas it has been our intention to exhibit the approximation for large \( T_j/T \) in general.
At this point we do specialize the problem to one rectilinear dimension and calculate the fluxes required in Case 1 and Case 2. For a multidimensional problem, one would presumably take care of the complex directional integration by use of the Milne-Eddington directional approximation. We shall, in fact, be able to assess the accuracy of this approximation from some subsequent one-dimensional calculations.

XIII. DETERMINATION OF RADIATIVE FLUX FOR ONE RECTILINEAR DIMENSION: CASE 1

For one rectilinear dimension, the reference surface \( \sigma \) in Fig. 1 is chosen perpendicular to the x-axis such that \( n = i \). The only component of \( \mathbf{E} \) that enters is \( \mathbf{E} \cdot \mathbf{n} = q_v \). For Case 1, we require

\[
q_j = \int_I d\omega q_v = \int_I d\omega \cos \theta \int_I d\omega I_j = \int_I d\omega \cos \theta I_j, \quad (13.1)
\]

where \( I_j \) is given by (12.9a). The manipulation of the coordinate \( \tau, \theta, \) and \( \phi \) and other rearrangements required here are standard, so that only the result need be given. In terms of the rectilinear optical path defined by

\[
d\eta = \rho(1-\alpha)dx, \quad \eta = \int_0^x d\rho(1-\alpha)x, \quad (13.2)
\]

one finds

\[
q_j(\eta) = 2\pi \int_0^\infty d\eta' \{ [a^2/(1-\alpha)][(1-\alpha)k]a^2_j B_j \} \text{sgn}(\eta-\eta') E_2(|\eta-\eta'|), \quad (13.3)
\]

where the source function in braces is a function of \( \eta' \), and where \( E_n(t) \), the nth integro-exponential function, is defined by

\[
E_n(t>0) = \int_0^1 du u^{n-2} \exp(-t/u), \quad n = 1, 2, 3, \ldots \quad (13.4)
\]
Equation (13.3) sums the contributions to the radiative flux through the "influenced" reference plane with coordinate $\eta$ that are made by the "influencing" emission from each parallel slab with coordinate $\eta'$; the kernel $\operatorname{sgn}(\eta-\eta')B_{\nu}(|\eta-\eta'|)$ gives the direction and attenuation. If $q_j$ is replaced by $q$, $B_j$ is replaced by $B = \int_0^\infty dvB_{\nu}$, and $a = a_j$, then (13.3) becomes the standard expression for the one-dimensional radiative flux in a gray gas, a result in accord with Corollary I, Sect.XII.

XIV. DETERMINATION OF FLUXES FOR ONE RECTILINEAR DIMENSION: CASE 2

For Case 2, we require both the radiative flux ($m = 0$) and the photon particle flux ($m = 1$). The frequency integration of (11.1b) is carried out without simplification whereas the directional integration is the same as required for Case 1. One finds

$$\int_j^\infty dv(h\nu)^{-m}q_{\nu}(\eta) = \int_0^\infty d\omega \cos \theta \int_j^\infty dv(h\nu)^{-m}\bar{\nu} =$$

$$\left(\frac{a}{1-a}\right)^2 \int_j^\infty dv(h\nu)^{-m}\bar{\nu}\left(\frac{v_j\nu}{v}\right)^2 E_2\left(|v_j/v|^2 |\eta-\eta'| \right),$$

where $\eta$ is given by (13.2). The first two terms in brackets are functions of $\eta'$. The remainder of the $\eta'$-integrand represents the two very formidable "kernels"; if the variable of integration therein is changed from $v$ to $(v/v_j)$, the integral kernels are functions of $|\eta-\eta'|$ and $[T(\eta')/T_j]$. The factors $(v_j/v)^2$ come from $x_\nu = \kappa(v_j/v)^2$; thus, if Corollaries I and III of Sect.XII are applied to (14.1) for $m = 0$, it simplifies to (13.3).
Concerning the two cases formulated, only (14,1) holds when the contributions of distant influencing slabs are important to the problem (in the sense $|\eta-\eta'| >> 1$). From (10,15b) we see that $T^{-1}$ appears explicitly in (12.9a) and (13,3) [since $B_{j3} = B_j$ therein], but the opportunity for this direct interpretation does not present itself in (14,1). We shall see that (13,3) admits of analytic considerations, whereas (14,1) can be dealt with by means of numerical calculations of the iterative type; shock-wave structures thus obtained are presented in Sect. XIX.

XV. DISCUSSION OF SHOCK WAVE STRUCTURED BY IN EQUILIBRIUM IONIZATION

Consider steady rectilinear flow in the positive x-direction, and especially the strong normal shock wave in a cold gas, a vital problem for the phenomena considered. The governing equations are (2,1), (2,3), the integrated versions of (3,1)-(3,3), (8,14), and either (13,3) or (14,1). With no external radiation source, the pure gas dynamic shock would correspond to a jump from one equilibrium state of the gas and radiation field to a second equilibrium state thereof. The equations are (2,1), (2,3), the jump versions of (3,1)-(3,3), the statement $\alpha = \alpha_0$ replacing (8,14), and the statement $q_v = 0$ replacing (13,3) or (14,1). But when account is taken of the fact that the ionization reaction occurs at a finite rate and that associated internal radiative transfer occurs, then the transition from the said initial state (1, say) to the said final state (4, say) is dispersed or structured. For reasons of exposition, we shall describe first an archetype of the ionizing shock, mentioning later some complications and modifications.

The problem then exhibits a discontinuity and we take the origin of $x$ and $\eta$ here, denoting by 2 conditions just upstream
of the discontinuity and by 3 conditions just downstream of the discontinuity. The portion of the shock structure extending from 1 to 2 we call the precursor, and that from 3 to 4 we call the tail. The ionization reaction in the hot gas downstream of the discontinuity is accompanied by the radiative recombination process and some of these photons reach the cold atoms upstream, in accord with (13.3) or (14.1) with \( m = 1 \), and photoionize the atoms, in accord with (7.5); radiative and collisional recombination can usually be neglected in the precursor, over which collisional ionization is absolutely negligible. The radiative flux falls from zero at state 1 to a minimum downstream of 3 and then returns to zero at 4. The thickness of the main precursor region is of order \( (\lambda_j)_1 \) although the asymptotic decay of a far upstream is of considerable physical interest\(^9,13\). The temperature increases only slightly over the precursor, overshoots the equilibrium value \( T_4 \) at the discontinuity (by a factor of 3 or 4, as we shall see, and perhaps exceeds the temperature restriction of the theoretical model), and then falls gradually to \( T_4 \); this overshoot is characteristic of the pure chemical relaxation problem and also of the radiative transfer problem in a gas in local chemical equilibrium\(^4\). If the collisional reaction is very fast just downstream of the discontinuity, \( \alpha \) will climb steeply toward \( \alpha_0 \) (with \( T \) falling steeply) over an inner region of thickness \( u_4 (\tau_c)_4 \ll (\lambda_j)_4 \); the thickness of the tail will be \( \sim (\lambda_j)_4 \). If the collisional reaction is not so fast, the tail will thicken. Even if the collisional reaction tends to be fast, enough electrons must have been created over the precursor to trigger it. Thus, even a small amount of precursor photoionization can have a large effect on the subsequent shock structure.
Some indication of the relative importance of the radiative and collisional reactions downstream of the discontinuity can be obtained by evaluating (10.5) for \( \tau_r \) and (8.15b) for \( \tau_c \) (neglecting the factor in brackets). To eliminate \( m_a \) we refer the density \( \rho \) to \( \rho_{NTP} \), the density at normal temperature and pressure. The dependence on temperature is not strong so we set \( (T_j/T) = 10 \). In \( \tau_r \) we set \( \sum A(r) = 1 \).

The result is

\[
\tau_r = 0.788 \times 10^{-7} (\rho/\rho_{NTP})^{-1} \text{sec. ; } (\Omega_a/\Omega_i)\tau_c = 1.64 \times 10^{-9} (\rho/\rho_{NTP})^{-2} \text{sec.}
\]

(15.1)

These values are indicative for any monatomic gas, although we have inserted for \( T_j \) the average between argon and helium \( (\simeq 1.34 \times 10^5 \text{K}) \), and for \( T_j^2 \) in \( \tau_c \) the average of the squares. If \( (\rho_1/\rho_{NTP}) = 10^{-3} \), then \( (\rho_4/\rho_{NTP}) = 10^{-2} \) and (15.1) gives

\[
\tau_r = 0.788 \times 10^{-5} \text{ sec. ; } (\Omega_a/\Omega_i)\tau_c = 1.64 \times 10^{-5} \text{ sec.,}
\]

(15.2)

suggesting that the two reactions become competitive downstream of the discontinuity at this density.

We should also evaluate the time \( (\lambda_j/u_1) \). We assume \( (T_j/T) = 10 \) as above. For \( a = 0.5 \), the energy equation between 1 and 4 gives for hypersonic flow \( u_4 = 0.305 k^{1/2} w_1^{1/2} T_1^{1/2} \). Then

\[
(\Omega_4/\Omega_a)(\lambda_j/u_4 \tau_r) = 8.78 \times 10^{-2} W_a^{1/2}
\]

(15.3)

for any density, where \( W_a \) is the atomic weight. Equation (15.3) gives 0.176 for helium and 0.556 for argon, suggesting that \( (\lambda_j/u_4 \tau_r) \) is not so different from unity if \( a \) does not approach unity.
The discontinuity would be dispersed a few particle mean free paths by diffusion, thermal conductivity, and viscosity, so it is important to compare the photon mean free path \( \lambda_j = \frac{1}{n_e \sigma_{ir} (\nu = \nu_j)} \) to \( \lambda_{aa} \) and \( \lambda_{ea} \), respectively the atom-atom mean free path and the electron-atom mean free path. Using elementary kinetic theory with viscosity cross-sections, we find for helium and argon respectively

\[
\frac{\lambda_j}{\lambda_{aa}} = \frac{90}{x} ; \quad \frac{\lambda_j}{\lambda_{ea}} = \frac{101}{x},
\]

(15.4)

where \( x \equiv \frac{\Omega_1}{\Omega_2} \sum_{r=1}^{\infty} \lambda (r) \) has been evaluated from measured cross-sections as 0.12 for helium and 1.25 for argon. Equations (15.4) are independent of density and virtually independent of temperature. Because the electrons are more mobile than the atoms, one also has

\[
\frac{\lambda_{ea}}{\lambda_{aa}} = 5.6,
\]

(15.5)

wherein, however, the Ramzaner effect in the noble gases has not been considered. It appears that the particle free paths are too small to disperse sensibly the archetypal radiative shock, although dispersion might modify the thinner layer \( \sim u_T^2 \tau_c \) described above in connection with a fast collisional reaction.

To be specific, the archetypal conditions we have had in mind are near \( \rho_1 = 10^{-3}, T_1 = 300^\circ K, (T_j/T_1) = 10 \); these values are typical of relevant shock tube research and are suggested by (15.2). If \( \rho_1 \) is reduced sufficiently or if \( T_1 \) is increased sufficiently, then the precursor photoionization becomes very great and three-body recombination becomes important there. Calculations suggest that the shock structure discussed can become highly distorted or that the discontinuity
discussed can disappear. That will happen if the collisional reaction
tends to be relatively very slow is unclear (in the light of the italicized
statement at the end of Sect. VII), but perhaps it is stimulated by a very
high temperature spike. The source function in (13.3) can become very
large downstream, but the ratio of \(|q_j|\) to the convective flux in the
energy equation can never grow very large. This means that the integra-
tion in (13.3) must suppress the source function by (i) causing the in-
terval of integration over the tail to be very small, (ii) balancing the
flux from the tail with a positive flux from a thickened and heated pre-
cursor, or (iii) doing both. This suggests the topology of the strong
shock–strong radiation case discussed in Ref. 4. Thus it is the ratio
of the source function to the convective flux that becomes important in
the energy equation.

XVI. EQUATIONS OF VON-NEUMANN SHOCK STRUCTURE: CASE 1

The approximation is \([1 \gg (T/T_j)\) or \(2(T/T_j)\)] but the first
error quoted actually increases with distance and, from Sect. XII, the
restriction on \((T/T_j)\) applies especially to the region of principal
emission; the second error quoted refers to the local temperature in
(8.14b). From (2.3) the enthalpy per unit mass is

\[
H = (5/2)RT + E_c + aT_j\left[1 + (5/2)(T/T_j)\right]
\]  

(16.1)

and this has to be approximated for consistency by replacing the factor
in brackets by unity.

Then the equations of state, continuity, momentum, energy,
ionization rate, and radiative flux respectively are
\[ p = \rho RT (1 + a), \quad (16.2) \]
\[ \rho u = \text{const} = G = \rho卢 + \rho卢^2, \quad (16.3) \]
\[ p + \rho u^2 = \text{const} = GV = \rho Lu + \rho Lu^2 = \rho Lu + \rho Lu^2, \quad (16.4) \]

\[ G[(5/2)RT + E_c + (1/2)u^2] + \{G\alpha RT_j [1 + \tau(T/T_j)] + q_j\} + q_{\ell \ell} \]
\[ = \text{const} = (1/2)GV^2 + q_{\ell \ell} = G[H_L + (1/2)u_L^2] + q_{\ell \ell} = G[H_L + (1/2)u_L^2] + q_{\ell \ell}, \quad (16.5) \]

\[ \frac{d}{d\eta} [G\alpha RT_j + q_j [1 + \tau(T/T_j)]] = GT_j \chi^{-1} \tau_c^{-1} a_\alpha [a_\alpha^2/(1 - a_\alpha)] - [a_\alpha^2/(1 - a_\alpha)], \quad (16.6) \]
\[ q_j(\eta) = 2\pi \int_0^\infty d\eta ' [a_\alpha^2/(1 - a_\alpha)] [(1 - a_\alpha)/(\alpha_\beta^2)] \beta_j \text{sgn}(\eta - \eta') E_2(|\eta - \eta'|). \quad (16.7) \]

In obtaining (16.6) from (8.14a), we have taken \( \eta \) as the independent variable, multiplied by \( RT_j \) and written

\[ h \int_j \int_j^\infty d\nu (h\nu)^{-1} q_j = q_j [1 + \tau(T/T_j)] \quad (16.8) \]

in accordance with Corollary II; the term in braces is then \textbf{identical} with the term in braces in (16.5) to lowest order. From (4.44) \( q_{\ell \ell} \) is a constant and cancels out of (16.5) and hence out of the problem. We regard \([a_\alpha^2/(1 - a_\alpha)]\) as evaluated by (10.4b), \( \tau_c \) as evaluated by (10.5), \( \beta_j \) as given by (12.9b), and \( \tau_c \) as given by (8.14b); then (16.2)-(16.7) are six equations in the six variables \( q_j, a, T, \rho, u, p \). \textbf{Four} equations being algebraic, we may eliminate four variables and formulate the problem such that the nonlinear integral equation must be solved simultaneously with the first-order, nonlinear differential equation. Note, in particular,
that \( q_j \) can be eliminated from (16.6) by solving (16.5) for \( q_j \) instead of inserting \( q_j \) as given by (16.7).

Furthermore, the integral equation can be replaced by a second-order differential equation by making a directional approximation (to accompany the spectral approximation here in use) for the radiative transfer:

\[
\mathcal{E}_2(|\eta-\eta'|) = \int_0^1 d\mu \exp(-|\eta-\eta'|/\mu) = a \exp(-b|\eta-\eta'|),
\]

where \( a \) and \( b \) are positive constants of order unity. For \( a = 1 \) this amounts to assuming that the average direction of origin of the radiation \( \mu = \cos \theta = b^{-1} \) is independent of the slab distance; for \( b = 1 \), the average direction is always parallel to the x-axis. By differentiating (16.7) twice and inserting (16.7) in the result, one obtains

\[
\frac{1}{b^2} \frac{d^2}{d\eta^2} q_j(\eta) = q_j(\eta) + \frac{1}{b} \frac{d}{d\eta} \{\ln(a/b)[a^2/(1-a)][(1-a_E)/a_E^2]B_j\}.
\]

(16.10)

The precursor calculations presented subsequently suggest that this directional approximation is a good one. Equation (16.10) is the one-dimensional version of the equation in \( F_j \), that one obtains from the Milne-Eddington approximation [provided \( a \) and \( b \) in (16.10) are suitably chosen].

XVII. SOLUTION OF NORMAL SHOCK PRECURSOR: CASE 1

Let us assume temporarily that the inner discontinuity or shock within a shock exists. If both collision processes are neglected over the precursor, we have seen that the ionization-rate equation is integrable; (16.6) gives

\[
Gq_{RT_j} + q_j[1+0(T/T_j)] = \rho \xi q_{\text{RT}_j} = 0.
\]

(17.1)
The constant of integration has been taken as zero because of its extremely small magnitude: for the conditions of the archetype, \( a_1 \sim 10^{-12} \) whereas \( a_2 \sim 10^{-2} \) or \( 10^{-1} \). Now \( E_c \) is negligible over the precursor, and \((17.1)\) simply cancels from the energy equation \((15.5)\) the terms in braces, i.e. the effects of ionization and radiation. This cancellation must be examined more carefully, however, since the presence of the \( O(T/T_j) \) terms leads to a further restriction beyond \( l >> O(T/T_j) \).

Solving \((17.1)\) for \( q_j \) and inserting the result in \((16.5)\), we find

\[
\frac{G[(5/2)RT + (1/2)u^2]}{G} = (1/2)GW^2,
\]
\[
\frac{G[(5/2)RT + (1/2)u^2]}{G} = (1/2)GW^2,
\]

and, further, that the \( 0 \)-term can be written \( O(GaRT) \); estimates show that the coefficient represented by the "0" is not far from unity. Since the lead terms, with respect to which the \( 0 \)-terms have been neglected, have disappeared from \((17.2)\), the two \( 0 \)-terms should be restored. Specifically, the expansion in \((T_j/T)\) leading to \((15.3)\), and hence to \((17.1)\), should be carried to second order when \((17.1)\) is inserted in \((16.5)\). This complication can, however, be circumvented by introducing the further restriction

\[
a \ll 1.
\]

This removes the subject \( 0 \)-term from \((17.2)\) but restricts the density as well as the temperature, and hence precludes study of very strong precursors.

In evaluating the argument of the last paragraph, several points must be born in mind. As noted at the beginning of Sect. XVI, \( T \) in \( O(T/T_j) \) in \((17.1)\) refers to the region of principal emission and
is about $10^2$ times the temperature over the precursor [the notation in (17.2) does not reflect this disparity]. Also the hypersonic flow argument that the enthalpy is much less than the kinetic energy can not be used in (17.2). The reason is that (17.1) and hence (17.2) also apply at state 3, from the jump conditions. While the situation discussed persists until state 3, it disappears downstream of 3 because the collision processes then enter and the right member of (17.1) becomes nonzero; the restriction (17.3) is then lifted. Finally, we note that the largest terms in (16.5) over the precursor are $(1/2)\mu_1^2$, $\gamma a T_j$, and $q_j$; the last two are removed by (17.1).

In virtue of (17.3), the effects of ionization and radiation have disappeared from the system (16.2), (16.3), (16.4), and (17.2) so that these uncoupled equations refer to a flow of atoms only and give, over the precursor, the solution $T = T_1$, $\rho = \rho_1$, $u = u_1$, $p = p_1$. This result, in fact, proves the existence of the inner discontinuity, until now assumed, to the order of approximation stated.

When such a discontinuity exists, the general jump conditions, denoted by [ ], connecting states 2 and 3 for Case 2 are seen to be

$$ p = \rho RT(1+\alpha), \quad [\rho u] = 0, \quad [\rho u^2] = 0, \quad \text{(17.4a,b,c)} $$

$$ [(5/2)RT(1+\alpha) + \alpha_c + (1/2)u^2] = 0, \quad [\alpha] = 0, \quad \text{(17.4d,e)} $$

$$ [\int_0^\infty dv (hv)^{-m} q_v] = 0; \quad m = 1, 2. \quad \text{(17.4f)} $$

The condition (17.4a)-(17.4e) are evidently the same as the nonequilibrium problem with collisional processes but no radiation. Therefore nontrivial solutions thereof exist and the associated sound speed is given by

$$ a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{T,\alpha}. $$

Applying these jump conditions to the present Case 1
with \( a \ll 1 \), we note that the conditions (17.4a)-(17.4d) are consistent with the system mentioned in the first sentence of the preceding paragraph [it is consistent with the argument given to restore \( E_c \) in (17.2), bearing in mind that \( E_c \) is negligible over the precursor but perhaps necessary over the tail]. Free of the nonequilibrium problem, the classical conditions (17.4a)-(17.4d) are then associated with a sound speed given by \( a^2 = (\partial p/\partial \rho)_s \), also a consistent result since the first named speed tends to the second named sound speed as \( a \to 0 \). From \( T_1, \rho_1, u_1, \) and \( p_1 \) we can calculate a \textit{a priori} the values of \( T_3, \rho_3, u_3, \) and \( p_3 \). This provides four of the initial values needed to start the solution of the simultaneous equations downstream of the discontinuity.

The precursor problem now reduces to solving for the remaining variables \( \alpha(\eta) \) and \( q_j(\eta) \), and these are governed only by (17.1) and either (16.7) or (16.10). If emission (radiative recombination) is also neglected over the precursor, then (16.7) gives

\[
q_j(\eta \leq 0) = -2\nu_0 \int_0^\infty d\eta' \left[ a^2/(1-\alpha) \right] \left[ (1-a x)/a_x^2 \right] B_j E_2(\eta' - \eta) < 0, \quad (17.5)
\]

where \( q_j = -\alpha \gamma T_j \alpha \), while the relevant solution of (16.10) is simply

\[
q_j(\eta \leq 0) = q_j(0)e^{b\eta}, \quad \text{so} \quad \alpha(\eta \leq 0) = \alpha(0)e^{b\eta}. \quad (17.6)
\]

For negative \( \eta \), we have from the prior results that

\[
\eta = \int_0^x dx \rho_1(1-\alpha) \alpha = \rho_1 x = x/(\lambda_j)_1. \quad (17.7)
\]

Equation (17.5) ties the precursor solution to an integral over the tail solution. On the other hand, the matching of the precursor to the tail by means of the jump conditions is much simpler when the
directional approximations (17.6) and (16.10) are adopted in the respective regions, because the matching is effected by adjusting only a single constant \([\text{either } a(0) \text{ or } q_j(0)]\).

With the neglect of emission over the precursor, (17.5) does not depend explicitly on the values of the other four variables, previously established, over the precursor. Formally then, we can write the analogous solution by Case 2. The variables \(\alpha(\eta)\) and \(\int_0^\infty dv(hv)^{-1} q_v(\eta)\) are governed by (17.1) [with the photon particle flux restored therein by means of (16.8)] and (17.1) for \(m = 1\) (wherein \(\eta'\) ranges from 0 to \(\infty\)). Whereas this solution is formally free of (17.3), the existence of the shock within a shock is simply assumed.

XVIII. PRECURSOR PHOTOIONIZATION: BLACK-BODY EMITTER MODEL

The precursor results obtained in the last two paragraphs can be used to assess the accuracy of the spectral and directional approximations by comparing with the prediction of Case 2, which would be exact within the framework of the model assumed. Since it is clearly necessary to know the structure of the tail, we may, for the purpose of this comparison, make a simple assumption. Therefore, let us assume that all quantities are constant\(^9\) over the tail, corresponding to the equilibrium value at state \(h\). As one would expect and as the calculations verify, this is tantamount to assuming\(^13\) black-body emission. This is a severe test of Case 1: the minimum error at \(\eta = 0\), where the radiative flux is a maximum, will be poorer than the value quoted in Sect. XII (i.e. \(1 \gg T/T_j\)) because (12.1) is, in effect, replaced by

\[
\int_0^\infty dv(hv)^{-m} I_v(P) = \int_0^\infty dv(hv)^{-m} B_v(T_4) e^{v/T_4}, \quad \pi/2 < \theta < \pi.
\]
The comparison of solutions aside, the black-body emitter model is a very crude representation of the nonequilibrium structure. Surprisingly enough, it will give good precursor predictions under certain restrictive circumstances, as shown in the next Section.

From (17.5) we find easily

\[ a(\eta \leq 0)/a(0; \infty) = 2\Xi_{J}(-\eta), \]  

(18.1)

where \( 2\Xi_{J}(0) \equiv 1 \) and \( a(0; \infty) \) denotes the degree of ionization at \( \eta = 0 \) for \( (T_{j}/T) \to \infty \):

\[ a(0; \infty) = 2nc^{-2}a \chi_{J}^{-1}(T_{j}/T_{j})\exp(-T_{j}/T_{j}). \]  

(18.2)

Since \( q_{j}(0) \) has the black-body value \(-\gamma(\lambda_{j})\), (17.6) gives for \( b = 1 \)

\[ a(\eta \leq 0)/a(0; \infty) = e^{-\eta}. \]  

(18.3)

The result, according to Case 2, is

\[ a(\eta \leq 0)/a(0; \infty) = 2(T_{j}/T_{j})\exp(T_{j}/T_{j})\int_{1}^{\infty} d(\nu/\nu_{j})(\nu/\nu_{j})^{2}\exp[-(\nu/\nu_{j})(T_{j}/T_{j})]E_{3}[-(\nu_{j}/\nu)^{2}\eta]. \]  

(18.4)

From this one finds readily that

\[ a(0)/a(0; \infty) = 1 + 2(T_{j}/T_{j}) + 2(T_{j}/T_{j})^{2}, \]  

(18.5)

which shows the error at \( \eta = 0 \) of Case 1. To make the aforesaid comparison, (18.4) has been normalized to \( a(0; \infty) \) instead of the intrinsic quantity \( a(0) \). The integral in (18.4), a function of \( \eta \) and \( T_{j}/T_{j} \), can
be evaluated asymptotically as \( \eta \to -\infty \) (and for large \( T_j/T_h \)) by the method of steepest descents. The result gives

\[
a(\eta \to -\infty)/a(0;\infty) = \frac{4(\pi)^{1/2}(2T_h/T_j)^{7/6}}{\exp(T_j/T_h)(-\eta)^{1/2}\exp[-3(T_j/2T_h)^{2/3}(-\eta)^{1/3}].
\]

(18.6)

Wetzel has studied especially the spectral aspects of the photoionization precursor of a normal shock using a photon-gas kinetic model, and the black-body emitter model ab initio. His result\(^{13}\), for the inverse \( v^2 \) absorption coefficient (10.1), is the same as (18.4) except that \( 2E\sqrt{\eta v^{-2}} \) is replaced by \( \exp(\sqrt{\eta v^{-2}}) \). Wetzel has also proposed\(^{25} \) the following asymptotic approximation, analogous to (18.6), of his aforesaid result:

\[
a(\eta \to -\infty)/a(0;\infty) = \frac{2^{1/3}(\pi/3)^{1/2}(T_h/T_j)^{1/3}}{\exp(T_j/T_h)(-\eta)^{5/6}\exp[-3(T_j/2T_h)^{2/3}(-\eta)^{1/3}].
\]

(18.7)

It is clear that Wetzel's theory embodies a directional approximation akin to (but different from)(16.9) with \( b = 1 \), so that his approximation of (18.4) will overpredict the degree of ionization in the precursor. On the other hand, the spectral approximation of (18.4) represented by (18.1) will underpredict the ionization because \( E \) in (18.4) actually increases over the interval of integration; the larger the value of \( -\eta \) (for fixed \( T_j/T_h \)), the more important is this increase of \( E \) to the value of the integral. Finally, (18.3) will overpredict a relative to (18.1) and tend to compensate for the underprediction of (18.1) relative to (18.4).
Equation (18.4) has been evaluated over the interval 
$0 - \eta < 3$ for $T_j/T_4 = 10$ and 12 on the Brown University IBM 7070. The 
results are shown in Fig. 3 and compared with the result (18.1) of 
Case 1, and with the result (18.3) of Case 1 plus the directional approximation. The error of the spectral approximation is evident, as 
is the fact that this is nearly counterbalanced by the additional directional approximation (with $b = 1$), except near $\eta = 0$. Through use of 
(18.6), the corresponding results for the far precursor decay are dis-
played in Fig. 4 and show, as expected, that (18.1) and (18.3) are of 
little value at a distance of many $\lambda_j$. Also shown is Wetzel's approxi-
mation (18.7), and this is seen to be reasonably good.

These normalized results can be further interpreted within the 
framework of two specific examples (near the archetype) that are con-
considered further in the next Section. In the first example, we consider 
argon, for which $T_j = 182,000^0$K; $X$ is taken as 1.25 and $\Omega_a/\Omega_l$ is assumed 
to be 1; $E_c$ is neglected. The equilibrium shock is specified by choos-
ing $T_1 = 300^0$K, $p_1 = 10^{-3}$ atmos., and $T_j/T_4 = 10$. The equilibrium shock 
relations then give $u_1 = 9340$ m/sec., $M_1 = 28.9$, and $\alpha_4 = 0.720$, $M$ de-
noting Mach number. Thus one computes $(\lambda_j)_1 = 0.788$ cm and the black-body predictions, $\alpha(0;\infty) = 0.0690$, and $\alpha(0) = 0.0672$; about 10% of the photo-
ionization evidently occurs upstream of the discontinuity. Figs. 3 and 
4 can now be interpreted in terms of these values of $(\lambda_j)_1$ and $\alpha(0;\infty)$. 
The threshold or critical level of electron number density that can be 
detected by microwave techniques is about $10^7$ cm$^{-3}$; for the present up-
stream conditions, this corresponds to $\alpha_c = 3.72 \times 10^{-10}$. From Fig. 4, 
one finds that the critical distance from the discontinuity is $x_c = -35$ cm.
In the second example, we consider helium, for which

\[ T_j = 285,000^\circ K \text{; } \chi \text{ is taken as } 0.42 \text{ and } \Omega / \Omega_1 \text{ is taken as } 1; E_c \text{ is again neglected. We choose } T_L = 300^\circ K, \text{ } \rho_L = 10^{-3} \text{ atmos.}, \text{ and } T_j / T_L = 12. \]

From the equilibrium shock relations, one then finds \( u_L = 27,400 \text{ m/sec}, \)

\[ M_L = 29.6, \text{ and } \alpha_L = 0.463. \]

Thus one obtains \( (\lambda_j)_L = 5.73 \text{ cm and the black-body values } \alpha(0; \infty) = 0.00923 \text{ and } \alpha(0) = 0.0109; \text{ comparing these three values to the previous trio, it appears that the photons influence a region seven times thicker here, and produce one seventh of the ions per unit volume. } \]

These values of \( (\lambda_j)_L \text{ and } \alpha(0; \infty) \) can also be used to interpret Figs. 3 and 4. From Fig. 4, one finds \( x_c = -1.7 \text{ m}. \)

XIX. SOLUTIONS OF NORMAL SHOCK STRUCTURE: CASE 2

Numerical calculations of complete shock structure have been carried out for a number of conditions and gases; the calculations have been performed within the framework of Case 2 in order to provide a basis of subsequent comparison for the discussed approximations, which effect great analytical simplifications. These calculations will be reported separately in detail but we have selected two examples that shed light on the black-body emitter approximation just discussed and we shall present the results here; the two examples are those specified at the end of the last section. First we shall summarize the calculation procedure, which is iterative.

The system of equations (16.2)-(16.7) is to be modified for Case 2 by (i) restoring the two 0-terms in accord with (16.8) and (16.1), (ii) replacing (16.7) by the two fluxes (14.1). One begins by tabulating the two integral "kernels" therein as functions of \( |\eta| \) and \( T/T_j \).
[these resemble the integral in (15.1)]; the dependence on $T/T_j$ can be
curve fitted reasonably well. We regard $[a_j^2/(1-a_j)]$ as evaluated by
(8.13). With use of (16.2)-(16.4), $a(\eta)$ and $T(\eta)$ are taken as the de-
pendent variables while (16.5) and (16.6) are taken as the working equa-
tions. For each step in the iterative procedure, $q_j$ in (16.5) and
\[ \int_0^\eta \rho(\eta) q_j \, d\eta \] in (16.6) are known functions of $\eta$ evaluated from (14.1)
from the previous iterative step. Equation (16.6) can then be integrated
from state 1 to state 2 with the aid of the jump conditions (17.4a)-
(17.4e). The procedure is repeated until convergence is obtained. In
these two examples, two steps were needed to achieve convergence and a
third step for verification. The zeroth approximation is taken as the
equilibrium shock configuration, in which the variables are constant over
precursor and tail with a discontinuity between. Clearly this procedure
subordinates radiative effects to some extent and, for the two types of
very strong shocks discussed in the last paragraph of Sect. XV, it is
found not to converge. Straightforward in principle, the procedure is
very laborious, having been carried out by hand to secure maximum
monitoring.

The results for the two examples are presented respectively
in Fig. 5 and Fig. 6, wherein
\[ \tilde{q}_j \equiv -GRT_j^{-1} q_j ; \quad \tilde{q}_{jl} \equiv -GRT_j^{-1} \int_0^\eta d\eta \int_0^\eta \rho(\eta) q_j \, d\eta. \] (19.1)

This normalization is suggested by (16.5) and (16.6); since $GRT_j$ is of
order $(1/2)z \rho^2$, $\tilde{q}_j$ is a measure of the ratio of radiative flux to total
energy flux; since $\alpha = \tilde{q}_{jl}$ without collisional processes, then the separa-
tion of $\alpha$ and $\tilde{q}_{jl}$ in the figures represents the cumulative effect of
the collisional processes and the radiative-collisional interaction process up to the value of $\eta$ considered, a fact which is verified by integrating both sides of (14.6). The ordinate is linear in each quantity plotted. In Fig. 5 the ionization downstream of the discontinuity is dominated by collision and goes to local equilibrium very quickly: there is a sharp spike in temperature, and $\alpha$ and $\rho/\rho_\perp$ rise rapidly. Characteristically of fast collisional reaction, there is a high curvature in $\tilde{q}_j$ and $\tilde{q}_{j1}$ vs. $\eta$ near $\eta = 0$ and these variables decay almost symmetrically to one third (say) within a photon mean free path (i.e. $\eta = 1$) or less. But this outer radiant tail structure over $0 < \eta < 1$ is not also clearly discernible in the case of the variables $T$, $\alpha$, and $(\rho/\rho_\perp)$ because the radiative flux is evidently a small part of the total energy flux. Because these three variables are sensibly constant until a fraction of a mean free path from the discontinuity, we infer from (14.1) that the black-body model should give an acceptable precursor estimate; indeed, $\alpha(0) = 0.0977$ is close to the previous value of $0.0842$.

In Fig. 6 the radiative flux is even a smaller fraction of the total energy flux, but the collisional reaction is slower and the ionization relaxation length is not small compared to a mean free path. The tail is considerably thickened and the curves $\tilde{q}_j$ and $\tilde{q}_{j1}$ are quite asymmetric about $\eta = 0$. This example is more nearly in accord with the estimates of the relative importance of the three characteristic times given in (15.2) and (15.3). The black-body emitter model should give a less satisfactory estimate of the actual value $\alpha(0) = 0.00490$ in Fig. 6, and, in fact, we obtained $0.0109$ in Sect. XVIII.
ACKNOWLEDGEMENTS

In the case of the first writer, this research was partly conducted during an N.S.F. Senior Postdoctoral Fellowship at the Politecnico di Torino, and was subsequently sponsored by the Advanced Research Projects Agency (Ballistic Missile Defense Office) and technically administered by the Fluid Dynamics Branch of the Office of Naval Research under Contract No. NR 061-132. The writers are grateful for the computational assistance of Dott. Maria S. Cugiano and Mr. C. R. Prasad.

APPENDIX

The Rosseland equation, generalized for the nonequilibrium ionization studied, is (see Sect. IX)

\[
\frac{\partial I_v}{\partial s} = n_a \sigma_T \left[ 1 - \frac{\alpha^2}{(1 - \alpha)} \frac{1 - \alpha_E}{\alpha_E^2} \exp(-hv/kT) \right] \left\{ \frac{\alpha^2}{(1 - \alpha)} \frac{1 - \alpha_E}{\alpha_E^2} \frac{2hv^3c^{-2}}{\exp(hv/kT) - \frac{\alpha^2(1 - \alpha_E)}{(1 - \alpha)\alpha_E^2}} - I_v \right\}.
\]
REFERENCES


FIG. 1 FORMALISM OF RADIATIVE TRANSFER
FIG. 2 SCHEMATIC DIAGRAM OF SPECTRAL ABSORPTION COEFFICIENT PER UNIT VOLUME FOR THE MIXTURE
FIG. 3 NEAR-PRECURSOR PHOTOIONIZATION: BLACK-BODY EMITTER MODEL
\[-\eta = -\frac{x}{(\lambda_j)_i}\]

CASE 2
CASE 1
WETZEL'S APPROX.
EXponential

\[\log_{10}\left[\frac{a}{a(0, \theta)}\right]\]

\[T_j/T_4 = 10\]

\[T_j/T_4 = 12\]

FIG. 4 FAR-PRECURSOR PHOTOIONIZATION: BLACK-BODY EMITTER MODEL
$T_1 = 300^\circ K$
$p_1 = 10^{-3}$ atmos.
$T_j/T_4 = 10$
$M_1 = 28.9$

$\tilde{q}_j = 0.111$
$\tilde{q}_{j1} = 0.1003$

$\rho/\rho_1 = 3.94$
$5T \quad T = 1380^\circ K$
$\alpha = 0.0977$

$\rho/\rho_1 = 12.3$
$\alpha_{\text{max}} = 0.770$
$\alpha_4 = 0.720$

$T_4 = 18,200^\circ K$

**FIG. 5 COMPLETE SHOCK STRUCTURE. CASE 2 - FIRST EXAMPLE: ARGON, T_j = 182,000^\circ K**
$T_1 = 3000\, ^\circ\text{K}$
$P_i = 10^{-3}\, \text{atmos}$
$T_j/T_4 = 12$
$M_i = 29.6$

$\rho_4/\rho_1 = 11.5$
$T_4 = 23,800\, ^\circ\text{K}$
$\alpha_4 = 0.463$

$\tilde{q}_j = 5.41 \times 10^3$
$\tilde{q}_{ji} = 4.90 \times 10^3$

$\rho/\rho_1 = 3.99$
$T = 396^\circ$
$\alpha = 4.90 \times 10^{-3}$

$\rho/\rho_1 = 1$

FIG. 6 COMPLETE SHOCK STRUCTURE. CASE 2 - SECOND EXAMPLE: HELIUM, $T_j = 285,000\, ^\circ\text{K}$
DISCUSSION

R. N. Thomas: Zirker and I studied a similar case a couple of years ago.* We did not use your model exactly: Suppose that the gas is at 50,000\(^0\)K at state 2 and at 2,500\(^0\)K at state 1. Let us assume for the purpose of comparison that my gas is not an ion but has two bound levels instead (so that, instead of treating the ionization problem I will treat yours). Then the ratio of the Planck functions \(B\) between \(T_2\) and \(T_1\) is \(10^5\), for an excitation energy between the two levels corresponding to that of He II. But if I actually compute what the ratio of the source functions taking in account the real occupation number corresponding to the degree of ionization, the ratio is ten! Now this is a completely self-consistent problem, where \(\epsilon\) is the ratio of collisional excitation to spontaneous de-excitation. This corresponds quite well to your comment on the very small ionization collision rate.

J. H. Clarke: The ratio of my two \(\tau\)'s is your \(\epsilon\), which I left completely open.

R. N. Thomas: This would be a shock where the temperature reaches 50,000\(^0\)K and decays later on. This is a completely isolated atmosphere and the radiation field arises self-consistently within the shock itself. There is a very small ionization rate and the radiation flux smoothes way out so that ordinarily you have a population ratio that varies by a factor of \(10^5\) across the shock.

J. H. Clarke: If you apply the collisionless photon flux expression (Eq. 7.5 in my paper):

\[ m_a^{-1} \rho u a + \int_{\nu_j}^{\infty} d\nu (h\nu)^{-1} q_\nu = \text{const.} \]

from minus infinity to plus infinity the jump in \( \alpha \), at any discontinuity, must be zero, the jump in \( q_\nu \) across the discontinuity must also be zero; then this expression is true if you write in the subscript 1 and it is true if you write in the subscript 4. But for thermodynamic equilibrium of both the gas and the radiation field, then \( q_\nu \) must vanish at minus infinity and plus infinity. You immediately obtain \( \alpha_1 = \alpha_4 \): no ionization has happened! In other words, the only shock that you might have is the trivial shock. Nothing else.

S. C. Traugott: Is there any hope of explaining how it happens? You can compute this fully dispersed radiative shock. Things do happen and all that you need is radiation; you don't need any collisions. You say that it can't be done?

J. H. Clarke: This is what comes out of the statistical model. The model is borrowed from the astrophysicists. Whether I borrowed it from the right astrophysicists, I don't know. This is true if my statistics are true. And it gives some results that Thomas arrived at in an entirely different way.

R. N. Thomas: Suppose that I have an isolated gas, which is what I tried to put up here. Put a shock in it but don't allow any collisions: you won't get any radiation field. With no radiation field the radiation
is zero. You have that all the way through.

J. H. Clarke: In our case, it is quite all right to neglect collisions in the precursor part. In general, to avoid the complexity of handling collisions, you can group all collisions in a layer reduced to zero thickness, i.e. put your collisions all in one place; the entire shock structure I have analysed is embedded in another structure and embedded in it is another radiative structure mainly due to the lines. I am proposing a layer upon layer approach. I have got three average absorption coefficients or photon mean free paths to consider. And I have in addition for this problem a collision length, to compare with each photon mean free path. The layer upon layer approach (i.e. the asymptotic approach) seems to be where the analytic promise lies.

R. N. Thomas: All your collisional excitation, in essence, comes in the 50,000\degree layer and produces a radiation field that migrates out to the 2,500\degree layer, providing an excitation due to radiation alone. I can neglect the collisions for the 2,500\degree K layer but not for the 50,000\degree K layer.

J. H. Clarke: But where are you going to stop?

R. N. Thomas: You must establish the physical picture first. I have two competing effects: the collisional rate of either excitation or ionization and the diffusion length over which the photons that are produced will go
before they are destroyed. So what I compare all of the time is the rate at which I am absorbing at point A the photons produced somewhere else to the rate at which I am producing photons at point A. If I have a temperature distribution of the type we have been discussing, my rate of collision production is very small in one part of the gas, and it is very possible that incoming photons provide more excitations than do the local collisional excitations in this low temperature region. So it is just what you said: Put them equal to zero someplaces but not everywhere. You must combine your physics and your arithmetic to find out where you can put them equal to zero.

J. D. Teare: Have you compared those Argon profiles with physical reality? All I have seen are results for high Mach number in Argon but the cooling of the hot gases was very rapid near the walls of the shock tube ...

J. H. Clarke: These are only the first examples that we have done. If you lower the density in front by a factor of one half, the first picture that I showed you is completely changed: the radiation and collision processes become again competitive. In terms of normal temperature and pressures, \(10^{-3}\) seems to be the magic density for monatomic gases. The two processes rarely seem to be competitive.

Here, we chose the radiation-convection ratio to come out rather small. In Traugott's language, the radiative density discontinuity is slight; you couldn't see it at a quick look. For the same problem, but with a radiation-
convection ratio more like one, you would have first a layer where the collisional processes go very fast and bring you to ionization equilibrium, followed by a temperature decay, of the type illustrated by Traugott in LTE but with radiative transfer: a fast decrease and then a slow one, layer inside layer. That is what a better example would have brought out.

The shock structure is about one photon mean free path thick. If the collision reaction was slower, the whole structure would thicken. The other point that I didn't make was that I can make my shocks strong enough so that the discontinuity goes away. But it is a ghastly problem mathematically because the iteration method we use does not converge, it is not sophisticated enough. But I know that it is there.

Nothing that I found argues with Heaslet's basic picture. He always has L.T.E., but if you look at it, you can still reason with it and make sense out of it. Physically interpreted, his story seems to be right if you can mentally put in the chemical non-equilibrium aspect.

C. A. Whitney: May I ask how you examine the tendency for the electron temperature to be separated from the ion temperature?

J. H. Clarke: Well, that was the reason for my original apology. For complete exactness, you should let the electron temperature be different from the ion or the atom temperatures and you should also allow for the relaxation of the electrons. I did not do that. Rather, I tried to solve one of these aspects analytically and to understand it. After I understand that part, I will add complications. But before I introduce electronic relaxation, I would rather ignore the ionization problem, and
try to understand how electronic excitation relaxes; then I would put them together. I am not in any hurry; I am trying to understand things. This is the way I think we should proceed.

C. A. Whitney: All I wanted to say is that it does make a difference.

J. H. Clarke: I know that it does.

S. Scala: Apart from such considerations of electron temperature not being equal to ion temperature, what effect would molecular transport phenomena, such as thermal convectivity in the presence of such a large temperature jump, do to the structure?

J. H. Clarke: I compute my \( \mu \)'s, \( k \)'s, and \( D \)'s, and I compare these with the results which I obtained without including them. I can then tell whether they should have been included or not. Now with that spike in the temperature profiles, I am pretty sure that they count. There will be a dispersive effect. But it is not such a central feature that it must be looked into right now. I think that there was a paper in the AIAA Journal recently (by the Avco group, I think) where they showed that sometimes these chemical non-equilibrium problems and presumably also radiative non-equilibrium problems are dispersed by the transport effect and we should do something about them. I fully agree with that, but I don't think it invalidates my analysis.
Radiative Transfer in Plasma Physics

by

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This paper is rather different from the preceding papers which have described how radiative transfer has impinged on their work and the problems that it has raised, whereas I will explain why radiative transfer has not yet impinged on plasma physics and will discuss what might happen in the future. Although plasmas, of course, occur everywhere in the universe, I am going to comment today only on the field of my own work in laboratory plasma physics presently being carried out by those establishments concerned mainly with producing controlled thermonuclear fusion reactions.

The reason for studying the radiation emitted by laboratory plasmas is to be able to say something about the conditions of the plasma; at the same time we hope to learn something about the physics of plasma as well. The plasmas that are produced for research into controlled thermonuclear fusion reactions often emit radiation in lines that are optically thin; and since the interpretation of optically thin lines is so much easier than of optically thick lines, most of the work which has been done in interpreting the radiation has been done for these optically thin lines. Optical thickness has been regarded more as a nuisance and a complication; mostly it has been ignored.

I want to describe briefly the main types of plasmas that are produced; I will divide them quite arbitrarily in terms of their electron densities. Firstly, we have those machines which produce a low electron density plasma, such as the various mirror machines and stellarators. Since I do not want to go into the details of all these machines, it will suffice to say that the
mirror machine is an attempt to retain the plasma by means of magnetic
mirrors at each end of the device. The plasma is introduced by injecting
the charged particles into the magnetic bottle by means of a high energy
gun; this gives a plasma which has a high kinetic energy, although it may
not have a Maxwellian distribution. The electron density is always very
low, perhaps in the range of $10^8$ to $10^{10}$ electrons per cc. The stellarators
which have been built at Princeton are essentially toroidal devices with
large externally impressed magnetic fields. The plasma is heated by Joule
heating. In the stellarators, the electron density goes up to perhaps $10^{11}$
electrons per cc, but this is still rather a low electron density.

One of the troubles with the laboratory experiments is that they are
rather limited in size. The characteristic dimension of a stellarator is
about one foot, and the characteristic dimension of a mirror machine is
perhaps one foot to one yard. With the characteristic low electron densities,
practically all of the lines are optically thin, with the exception of the
radiation emitted by the stellarator at the beginning of its pulse; so that
the complications of radiative transfer can be ignored quite justifiably in
these machines.

A second group of machines produce an intermediate electron density;
they are the toroidal devices like Zeta, which is still working at Harwell
in England. This machine gives an electron density in the range of $10^{13}$ to
about $10^{15}$ electrons per cc, or perhaps a little higher. The heating
mechanism in Zeta is again Joule heating; it has a characteristic dimension
of one yard. If Zeta is filled with hydrogen, the optical depth in Lyman
Alpha will reach one to two thousand in the early part of the pulse. In the
middle of the pulse, which is usually the most interesting and most important,
the optical depth will be in the range of 1 to 10 so that an interesting range of optical depths is really produced which could lead to an experimental verification of the various radiative transfer theories.

In order to get interesting non-LTE effects in lines, the quantity $\varepsilon$, which is the ratio of the rate of collisional de-excitation divided by the combined collisional and radiative de-excitation, must be small. For the Lyman Alpha line in Zeta, $\varepsilon$ is of the order of $10^{-2}$; so that for optical depths up to about 100, we are in quite an interesting region. One might, for example, be able to get self-reversed profiles of the type that are observed in the Lyman Alpha line emitted by the sun and to make some contribution to the understanding of the formation of this line. But, unfortunately there is one great difficulty. A great deal has been said about the instabilities in these machines which prevent the plasma from being contained for more than a few microseconds. One can think of the instability broadly as the plasma reached the wall; but the real problem is that these instabilities are the outward signs of random electromagnetic fields which are generated inside the plasma. These can be measured with probes, and they can have quite large magnitudes. If one considers a charged particle which is in space together with many randomly fluctuating electromagnetic fields, the particle is going to acquire energy from these fields. If one considers the influence of a single component of these random fields, the charged particle will undergo simple harmonic motion; and the profile one gets from the doppler shift of the particle undergoing simple harmonic motion is a sharp bimodal form. There is experimental evidence that this sort of profile exists instantaneously in the plasma producing devices. In order to solve the equation of transfer,
we have to know the absorption profile; and in this case the absorption profile is determined by the statistical properties of these random electromagnetic fields. At the moment we know practically nothing about these fields, so that it is very difficult to see any way in which we can apply the theory of radiative transfer to these devices.

The third group of laboratory plasmas have a high electron density in the region of $10^{16}$ to $10^{19}$ electrons per cc; they are produced by devices like the thetatron and the magnetically driven shock tube. The thetatron has a single turn coil around a tube containing a gas; and if the coil has a high rate of rise of current, it induces fields inside the gas which ionize it. This process needs a high rate of rise of current; $10^{12}$ amps/sec/sec are now commonly achieved. With these high electron densities the plasma is in LTE, and the solution of the equation of transfer is then quite simple and does not present a great deal of interest. Under this assumption, the equation has, however, been utilized, particularly by Griem and Kolb and their colleagues, to calculate the populations of excited levels and hence to obtain oscillator strengths for the transitions.

This is a brief resume of what research in controlled thermonuclear fusion is producing in the way of plasmas. What is likely to come in the future? After today's discussion of shocks, it will not come as any great surprise when I suggest that shocks are also important in these plasma producing devices. The thetatron, which produces a very high rate of rise of current and a consequent rapid change of the physical properties of the gas, usually produces magnetohydrodynamic shocks which converge toward the center. These have been studied very successfully, both experimentally and theoretically. The theory involves all the processes of ionization,
plus all the fluid magnetohydrodynamic forces which are involved in this transient situation. It is not clear at present just what contribution radiative transfer makes to this type of shock. Possibly resonance radiation such as Lyman Alpha can precede the shock, causing excitation of the gas, which may then be ionized more easily by the shock when it arrives.

A second study, which I am sure is still a long way off, is concerned with the random electromagnetic fields in the plasma. I have talked of these as instabilities, although the name does not really mean very much. One can describe it as magnetohydrodynamic turbulence in the sense that the gas is moving in a random fashion and one has all the complications of electrical and magnetic forces included. We have got to tackle the problem of radiative transfer in plasmas containing this magnetohydrodynamic turbulence, since the power lost by radiation will be a vital factor in working out the energy balance of any successfully controlled thermonuclear fusion device.

**Discussion**

Q. Could you talk about another kind of plasma; the partially ionized gas produced by electric arcs, giving temperatures very much lower than the fusion devices, say 20,000 or 30,000° K? There are perhaps many more people working on devices of this type.

A. I am not too familiar with these devices, but they all have high electron densities where local thermodynamic equilibrium is justifiably assumed. If one has a plasma in LTE, admittedly one has to use the equation of transfer to calculate the radiation emitted by the plasma; but this is a simple calculation.
Q. Perhaps Dr. Lick would like to comment about this.

Dr. Lick: The main problem is to calculate the properties of helium and argon at these temperatures.

Q. I did not understand your comment about turbulence. You either have radiation which is affecting the structure of the turbulence, or the radiation just cools it in some way without changing it. I suspect that in the case of turbulence the size of any structure you would be interested in would be small.

A. I am not sure your statement is true. The only observations of the autocorrelation function of the fluctuations I know of, which we have done on Zeta, indicated that the characteristic size of the turbulence is something like 10 cm. This indicates that the scale length in the turbulence and in the radiative transfer are of the same order of magnitude, and it is certainly not clear to me what will happen under these circumstances.

Q. I have some questions on cyclotron radiation. How does this compete in general with the line emission in terms of the energy loss, and is there a non-trivial transfer problem?

A. The transfer problem for cyclotron radiation is extremely complicated. The precise calculation appears to be very difficult, and even in a small low-density device one would immediately run into radiative transfer problems. But I do not think there is a significant energy loss. This is simply an experimental observation; if cyclotron radiation makes a contribution, it will be mainly at radio wavelengths. The emission at these wavelengths has been measured, although it has never been explained satisfactorily. One can see that the energy
loss at these wavelengths is quite small. One has to remember that the energy of a photon at radio wavelengths is very small, and it is competing with some of the lines of highly-ionized ions which are carrying perhaps 60 eV with each photon. It is these resonance lines of impurities like N V and C VI that are really doing the damage.
APPLICATIONS OF A RADIATION-HYDRODYNAMICS METHOD TO
SOME PROBLEMS IN ASTROPHYSICS

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Recent interest in astrophysical problems in which radiative and
hydrodynamic processes occur concurrently has led to a need for theoretical
methods which simultaneously take account of the disciplines of radiative
transport and fluid motion. The study of pulsating stars, fireballs
resulting from nuclear explosions, thermal ablation of re-entry vehicles,
and laboratory shock-tube experiments are but a few examples of physical
problems for which the methods of radiation-hydrodynamics are applicable.

The fundamental approach to simultaneous treatment of radiative
transport in conjunction with fluid flow involves modification of the equations
of hydrodynamics to include radiative mechanisms in the equation which
expresses the conservation of energy for a sample of material.

This discussion is oriented toward the use of numerical techniques
since real equations of state, non-gray opacities, and shocks are essential
to the problems to be treated.

The fundamental equations for a spherically symmetric one dimensional
system in a Lagrangian frame of reference may be written as:

\[ \frac{\partial m}{\partial t} = - 4\pi R^2 \frac{\partial}{\partial m} \left( P + Q \right) - \frac{G M R}{R^2} \]  \hspace{1cm} \text{Conservation of Momentum (1)}

\[ U = \frac{\partial R}{\partial t} \]  \hspace{1cm} \text{Definition of Velocity (2)}

\[ V = 4\pi R^2 \frac{\partial R}{\partial m} \]  \hspace{1cm} \text{Conservation of Mass (3)}
\[ Q = \frac{c(\rho_0 \Delta R)^2}{V} \left( \frac{\partial V}{\partial t} \right)^2 \text{ on compression} \]

\[ Q = 0 \quad \text{otherwise} \]

\[ \frac{\partial E}{\partial t} + (P + Q) \frac{\partial V}{\partial t} + \frac{4\pi \alpha (R^2 F)}{3m} = E \quad \text{Conservation of Energy} \]

where

\[ U = \text{fluid velocity} \]
\[ t = \text{time} \]
\[ R = \text{radius} \]
\[ P = \text{pressure} \]
\[ Q = \text{artificial viscosity} \]
\[ m = \text{mass} \]
\[ c = \text{an arbitrary constant} \]
\[ \rho_0 = \text{initial density} \]
\[ V = \text{specific volume} \]
\[ F = \text{Radiative net flux at } R \text{ (derived from the integral expression in Eq. (6))} \]
\[ E_m = \text{energy generation by sub-atomic processes} \]
\[ G = \text{gravitational constant} \]
\[ M_R = \text{mass interior to radius } R \]
\[ \overline{\text{over}} = \text{time averaged quantity.} \]

These are used in conjunction with suitable equations expressing the thermodynamic and radiative properties of the equilibrium fluid.

The basic partial differential equations are replaced by a set of centered finite difference equations which are advanced explicitly in time according to a variety of time-step criteria, except for the integro-differential energy equation which is advanced implicitly by simultaneous
iteration of the radiative and hydrodynamic properties throughout the configuration.

A more complete discussion of the basic equations and methods of solution is given in Ref. 1.

The present work is an extension of earlier work by Brode\(^2\) who used a radiative diffusion approximation in the formulation of the radiative term of the energy equations. Improved methods were required to take account of the spectrally selective (non-gray) character of the absorption coefficients and to provide a better representation in regions near the outer boundary of the problem from which the observable radiation originates.

In the present formulation, the physical system is represented by a system of concentric spherical shells of material. For brevity, consider a system, such as a cepheid, in which the zones near the center can be chosen so as to be optically thick. Then the diffusion approximation, which is independent of geometry, will be valid near the center. In the outer regions, the zones can be chosen geometrically thin enough so that they may be considered as nearly plane-parallel. It is shown\(^1\) that use of a representative ray treatment, in lieu of the full angular dependence of the intensity field, yields an acceptable numerical approximation for the spectral flux. (Numerically, this simply constitutes replacement of the exponential integral in the exact solution for the plane-parallel geometry with a simple exponential function.) The selection of a particular representative ray is made by choosing the inclination to the radial direction in such a way that the diffusion approximation is identically reproduced at large optical depths when used with the formulation which follows. The resultant representative inclination \((\cos \theta = 2/3)\) has been shown to be of sufficient accuracy for all optical depths.\(^1\)
The equation of transfer along an inclined ray traversing a plane-parallel-like slab can be written:

\[
I_v(\theta, 0) = I_v(\theta, \Delta t_v) e^{-\Delta t_v / \cos \theta} + \int_0^{t_v} B_v(t_v') e^{-t_v' / \cos \theta} \frac{dt_v'}{\cos \theta} \tag{6}
\]

where the definitions are the standard ones and \( t_v \) is the inclined spectral optical thickness of the slab. Eq. (6) is an exact expression for the intensity of monochromatic radiation along one direction in space. By replacing \( \cos \theta \) by the numerical value of 2/3, a similar equation is obtained for the monochromatic flux. Integration over frequency can be carried out to obtain the relatively simple equation for \( F_{\Delta \nu} \), the flux emergent from one side of a slab in the frequency band of width \( \Delta \nu \):

\[
F_0 = F_{\Delta \nu} Z + B_0(1 - A) + \frac{2}{3} \left( \frac{\partial B}{\partial \tau} \right)_0 W \tag{7}
\]

emergent flux transmitted flux emitted flux

where the source function \( B_v \) has been expanded in a Taylor series about the surface of the slab (higher order terms have been neglected) and the following spectral averages have been defined for the band width \( \Delta \nu \):

\[
A = \frac{\int_{\Delta \nu} B_v e^{-3/2 \mu_v \Delta R} dv}{\int_{\Delta \nu} B_v dv} \tag{8}
\]

\[
Z = \frac{\int_{\Delta \nu} F_{\Delta \tau} e^{-3/2 \mu_v \Delta R} dv}{\int_{\Delta \nu} F_{\Delta \tau} dv} \tag{9}
\]
\[
W = \frac{\int_{\nu}^{\nu + \Delta \nu} \left( \frac{\partial B}{\partial \tau} \right) \, \text{d}\nu}{\int_{\nu}^{\nu + \Delta \nu} \left( \frac{\partial B}{\partial \tau} \right) \, \text{d}\nu} 
\]

\[1 - e^{-3/2 \Delta \tau} - \frac{3}{2} \Delta \tau \, e^{-3/2 \Delta \tau} \text{d}\nu \]

(10)

and \( \Delta \tau \) is the normal spectral optical depth of the slab. \( \Delta \tau \) is obtained from \( A = e^{-3/2 \Delta \tau} \).

The spectral fluxes \( F_0 \), as calculated from (7), can be simply added to obtain the total flux which emerges from the slab. In any particular problem the appropriate \( \Delta \nu \) intervals should be chosen after a careful study of the properties of the spectral absorption coefficient. In particular, Eq. (9) requires an average value of the spectral flux. The spectral intervals must therefore be narrow enough so that this average value can be meaningfully defined.

The three functions \( A \), \( Z \), and \( W \) are in turn functions of slab temperature, density, and thickness. Since only a finite number of slab conditions can occur in a given problem, the \( A \), \( Z \), and \( W \) functions can be calculated in advance for specific ranges of values of \( \rho \), \( T \) and \( \Delta R \). Interpolation formulas can then be fitted to these values for use in the actual radiation hydrodynamics computation. Such a procedure need be carried out only once for each fluid. The \( A \), \( Z \), and \( W \) functions have step-function like characteristics, and in many regions of the \( \rho-T-\Delta R \) space, may be determined by inspection of the absorption coefficient data. It is to be noted that these spectral averaging functions are somewhat analogous to the better known Rosseland, Planck, and Chandrasekhar spectral means (which are point-wise properties of the fluid), but that they are slab
type averages involving three parameters while the point-wise averages only involved $\rho$ and $T$.

This method of radiation hydrodynamics computation was first tested in 1962 as a relaxation technique for determining stellar models. The basic equations are identical to the equations of static stellar structure except for the acceleration term of Eq. (1). A simple ideal gas equation of state including ionization was developed following the Stromgren technique of grouping the metals. The absorption coefficient data of Vitense$^3$ were used to calculate the $Z$, $A$, and $W$ functions. Professor L. G. Henyey of the Astronomy Department at the University of California at Berkeley provided a stable stellar model known as Darwin 404B. The Darwin model had 25 concentric shells each having constant physical properties throughout each shell. A system of 100 zones was used in the test computation, each group of four zones having properties identical to one of the Darwin 404B zones. The test code was thus being used as a sort of an interpolation formula. The computation was allowed to progress until the velocity of each shell boundary, which was induced originally by the initial conditions, reached a negligible value.

The star was found to relax to an equilibrium solution near the center in a time of the order of seconds, while the relaxation time for the atmosphere was many orders of magnitude larger. The code was thus modified to effectively delete the inner zones as they came to equilibrium, and thus increased the allowable time steps while decreasing the number of zones in use.

The test code reproduced the $\rho$ and $T$ profiles of Darwin 404B with surprising accuracy, except for the outer atmosphere which was not completed due to the extensive computer time required.
Darwin 404B was a hot star in which hydrogen was ionized at the surface. A test run was then made on the Sun. Two interesting results were obtained:

1. The code insists on a high temperature (Vitense, 1958) atmosphere, even though the initial conditions corresponded more closely to a low temperature model (Minnaert, 1953).

2. The code builds something that resembles the a chromosphere. Material is cycled outward and the density drops while the material is still warm (6–7000° K), leaving the material unable to radiate. Of course, the code comes nowhere near duplicating the complicated physics of the chromosphere, but the result is nevertheless thought provoking.

The main interest in the code is, of course, an attack on the cepheid pulsation problem. At first it was envisioned that the code would be used to treat an atmosphere that was being driven from below by an artificial piston. However, it was soon discovered that such artificial driving mechanisms were entirely unnecessary. It proves to be impossible to start a stellar model with an exact equilibrium solution. Even the smallest transient introduced in loading the numerical data will start an oscillation. If the surface gravity of the star is high enough, the oscillations will damp out. For a narrow range of intermediate surface gravities, an apparently nearly reproducible cycle is established, while for very small surface gravities the star continues to flow outward without apparent reproducible periodicity. It was also found that the appropriate range of surface gravities could be deduced directly from Arp's extensive radical velocity measurements.

The cepheid pulsation problem is still, however, far from an acceptable solution. Almost any reasonable cepheid model will pulsate, but
an arbitrary choice of model cannot be expected to provide both a reproducible cycle and observables resembling real cepheids. The family of possible models is a many parameter family, and the computational time required to reach a reproducible cycle can be quite extensive. Some 40 to 50 models in the range mass 5 to 8 solar masses were attempted, and the results were indicative rather than conclusive. The models were, unfortunately, found to depend on the details of the giant-like core (in particular its chemical composition) and upon convective transport in the atmosphere.

While a convective theory exists for "static" atmospheres, even though it is not in a very satisfactory state, there appears to be no method at all available for treating convection in non-stable atmospheres. One of the things that can be done with a computer, however, is to test the numerical effect of a convective transport even though a proper physical description is not available. These numerical experiments, while not conclusive, indicated that a small convective flux seems indeed to be necessary to build a mass 8 cepheid that had a radius anywhere near the values expected from observations. In addition, one interpretation of the results would seem to indicate that cepheid masses in the 5 to 8 range are somewhat smaller than the values deduced from observations, i.e., the luminosity to mass ratio is higher. It is hoped that further work will be done in this area as more realistic giant models are now becoming available.

The most extensive and exhaustive use of the code has been in the area of fireball histories following nuclear detonations. Only a limited account is available in the unclassified literature. It should be mentioned that on account of its radiation the ball of hot air which results after a nuclear detonation is a ready-made astrophysical laboratory. One can observe, at close range, many of the phenomena seen in stars and other astronomical
sources, and it is possible to cross check the results of radiometric observations by other types of measurements.

In the period 1950-1958, more than 3000 radiation measurements were made which included many which were similar to the measurements made by astronomers. The theoretical models available at that time could not explain the observed radiation in a quantitative fashion nor did they predict many of the radiative effects so well displayed by Life Magazine.

The radiation-hydrodynamics code referred to herein was first applied to fireball models in late 1962. Calculation of the Z, A and W functions for this application involved very extensive and detailed calculation of the molecular and atomic absorption coefficients of air. The calculation succeeded in reproducing the experimental results in a quantitative fashion over very wide ranges of initial conditions.

A further application of the code has been in the study of radiation-hydrodynamics phenomena for idealized hot-air spheres. When originally used for this application, temperature spikes, apparently due to computational instabilities, were found under conditions involving strong shocks and high opacities. Further investigation showed that these temperature profiles had been predicted by Heaslet and Baldwin.

Attempts to further investigate details of the shock profiles using the code showed the results to be limited by the artificial viscosity technique which was employed.
References


GAS DYNAMICS OF STELLAR ATMOSPHERES

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I. The Dynamical Superposition of Radiation and Gas

The early papers on the theory of stellar atmospheres, written about forty years ago, employed a standard set of four physical assumptions.

The absorption coefficient of the atmospheric gas was considered to be independent of wavelength (gray), and the transfer equation was integrated over wavelength.

The gas was assumed to be in a state of local thermodynamic equilibrium, so that its emission coefficient was given by the product of the Planck function and the absorption coefficient.

With these important steps, the problem of specifying the radiation field was separated from that of the matter field. The radiative transfer equation was replaced by a linear integral equation whose solution was obtained by numerous methods.

Further, the atmosphere was assumed to be completely quiescent, and governed by the laws of radiative equilibrium and hydrostatic equilibrium. This pair of assumptions reduced the equations determining the structure of the atmosphere to an ordinary, first-order differential equation which was solved by quadrature.

There are several ways of relaxing the restriction to radiative equilibrium. One could, for example, imagine the existence of a statistically-steady state of convective transport in a certain region of the stellar atmosphere. In fact, it is fairly obvious from simple
calculations that most stellar atmospheres must have zones which are unstable against convective transport. Some 50 years ago a mixing-length theory was developed for convection, and there has been very little progress since that time. I shall not discuss the concept of fully developed turbulent convection, which is really required to examine the structure of the convection zones properly, because I can say very little worthwhile that wasn't said 30 years ago.

So the sense in which I wish to drop radiative equilibrium and hydrostatic equilibrium is by inserting coherent fields of gas motion in the stellar atmosphere. These fields quite generally have the appearance of mixed gravitational and acoustic waves.

During the discussion yesterday, there were really two points of view which became evident. In the first place the gasdynamic problem that was considered was controlled essentially by momentum considerations, and the energy equation stated that energy is trapped within the gas. That is, the adiabatic equation was generally employed. On the other hand, it was also pointed out that the mean structure of stellar atmospheres is essentially controlled by radiative transfer of energy, and it was suggested the momentum transfer was not significant. The point which I must make explicit at the outset is that for a discussion of the gasdynamics of stellar atmospheres one must recognize that radiative transfer will significantly influence the gas flow.

That is, we are really dealing with a mixture of two kinds of particles: photons and atoms. These particles obey essentially the same sort of transport equations, and these transport equations can, of course, in turn be derived from the Boltzmann equation. There are two differences between these types of particles which should be pointed out. In the first
place, photons do not obey the number conservation law that particles do. That is, one must insert sink and source terms in the photon transport equation. The second difference is that the photon free path is considerably greater than a particle free path in a stellar atmosphere.

Remembering that the definition of a stellar atmosphere is that part of a star which we can see, it is clear that radiative transfer will introduce a coupling throughout the atmosphere which is not at present an ordinary gasdynamic problem. Even in the presence of shock fronts, photons will tie the atmosphere together and will tie together portions of gasdynamic fields which would otherwise have been disconnected.

A further point is that the photon velocity can be treated as infinite, and we may neglect the time for transport of photons from one part of the atmosphere to another. This may be expressed by the inequality \( c/a \gg 1 \), where \( c \) is the velocity of light and \( a \) is a characteristic velocity for gas in the atmosphere, namely the acoustic velocity.

An alternative statement is that the photon flight-distance, during a characteristic time for the atmosphere, is much greater than the characteristic length for the atmosphere. We make this explicit in the following way. Let \( H \) be the scale height of the equivalent isothermal atmosphere, defined by the relations,

\[
H = -\frac{1}{\frac{d\ln p}{dx}} = \frac{a^2}{\gamma g}.
\]

We may take \( H/a \) as a characteristic time for the matter of the atmosphere, and restate our inequality as \( cH/a \gg H \). But this is, in fact, equivalent to the earlier expression. So we have learned nothing new.
But let me compare this acoustic characteristic time \( \tau_A = H/a \) with a thermodynamic characteristic time which I shall call \( \tau_T \). With the definition

\[
\tau_T = \frac{c_p \rho^T}{16 \tau T^4} \frac{1}{\kappa \rho}
\]

\( \tau_T \) is essentially the time required for a gas element in the atmosphere, treated as though it were optically thin, to radiate away its thermal energy under the assumption that it absorbs no energy from its surroundings. In other words, this would be the time required for the atmosphere to cool if the star underneath were suddenly turned off. To obtain quantitative estimate of the relation between these characteristic times, consider the solar atmosphere, for which the characteristic acoustic time is on the order of 20 seconds, since \( H = 100 \text{ km} \) and \( a = 5 \text{ km/sec} \). (If one considers larger stars with lower surface gravities, the acoustic time may be increased by as much as several orders of magnitude.) The characteristic thermal time varies quite significantly with height in the solar atmosphere, principally because the opacity decreases rapidly with increasing height. The value \( \tau_T = 10 \text{ sec} \) is characteristic of low photospheric levels and \( \tau_T \) is probably on the order of 10 minutes in the higher levels corresponding to the low chromosphere. In the corona, \( \tau_T \) becomes very large and the thermal conduction becomes the dominant mode of energy transfer.

Thus the thermal and acoustic characteristic times are very similar in the photospheric levels, and acoustic phenomena must be significantly influenced by radiative transfer. The ratio of these two times \( \tau_T/\tau_A \) is, in fact, the Boltzmann number divided by \( \kappa \rho H \). However, since \( \kappa \rho H \) is on the order of unity (corresponding to the fact that we see to optical depth
unity), we find the following approximate equality

\[
\frac{\tau_T}{\tau_a} = \frac{N_{Bo}}{\kappa \rho H} \sim 1
\]

II. Three Substantial Difficulties of Treating the Gas

Dynamics of a Stellar Atmosphere

The problem which is at once the most persistent, the most subtle, and the most damaging if improperly treated is that of boundary conditions. Those of you who work in the laboratory have walls surrounding you, and the boxes with which you deal have solid walls themselves. The unfortunate fact is that astrophysical walls are nothing more than the gravity field, and this is a very soft kind of wall. The conditions which we apply to the differential equations are, in other words, asymptotic conditions. We can specify the behavior of the radiation field way off there, or of the velocity way in there, but we can't tell you that at this particular level in the atmosphere the velocity will have this particular value.

The second problem is that the classical solutions, based on the standard set of four restrictive assumptions, neglecting gasdynamic phenomena, should not be used as a mean state about which to construct a perturbation theory. The effects which we are investigating will significantly influence the mean state of the system, and the problem is highly non-linear.

Finally, there arises the question of specifying the heat transfer terms in the energy equation. That is, how does one evaluate the radiative terms coupling the various portions of the atmosphere for which the continuum
radiation dominates the energy transfer, it may be legitimate to use the assumption of local thermodynamic equilibrium. This will facilitate evaluation of the radiative exchange. There will be, however, certain dynamical problems in which radiation transfer in the line spectrum is significant and for which local thermodynamic equilibrium cannot be assumed. In these cases, the static problem is difficult enough, and there has been very little progress in treating dynamical problems.

III. Stellar Chromospheres

I shall now consider a sequence of problems in the general area of radiation gasdynamics, starting with perhaps the most generally important one, although the one about which we can say the least. This is the general question of the production of stellar chromospheres.

I think it is quite clear that stellar chromospheres must be essentially ubiquitous, since I cannot imagine constructing a star without its producing its own chromosphere. The point is that, since there is bound to be a region in which convection takes place in the outer layers of the star, and since there is a negative outward density gradient, the fluctuations generated in the convection zone will amplify as they propagate outward and carry a certain amount of acoustic energy out into the upper atmosphere. While there will be damping effects due to radiative transfer, this field of fluctuations will probably carry sufficient energy, and in some cases sufficient momentum also, to heat and levitate the outer layers. So from a theoretical standpoint one expects stellar chromospheres to be quite a general phenomena.
There is also fairly substantial *observational* evidence that stellar chromospheres are, in fact, quite common.

But the theory of the structure of chromospheres is still in a very rudimentary state, and the actual construction of a model chromosphere is still a long way away. I think that substantial progress will be intimately tied to our understanding of three particular aspects of this problem.

First, we need a much clearer picture of the kinematic state of the solar photosphere. Since we must use the spectrum lines as diagnostic tools, our investigation of the velocity field will be contingent on a theory for the formation of spectrum lines in a moving atmosphere. Progress in this direction is persistent, if slow, and we should, before long, have a reasonably reliable picture of the velocity structure of the photosphere. But until that is done, I think it would be very difficult to estimate quantitatively the heat or momentum flow up through the photosphere.

Secondly, I think our intuition (or at least mine) concerning the effect of magnetic fields in the solar photosphere and chromosphere is very weak. We need to strengthen our intuition by carrying out relevant numerical or analytical experiments on the influence of magnetic fields.

Finally, I would reiterate the crucial role played by the radiation-exchange terms in the energy equation, and draw your attention to the work of Parker, and others since then, concerning the thermal instability in the chromospheric layers. While it has been quite generally assumed that the intense activity in the chromosphere arises from fluctuations propagated upward from below, we should bear in mind that, through thermal instability, a significant portion of this activity may be generated locally in the chromosphere. That is, there may, in fact, be a direct conversion from thermal to mechanical energy within the chromospheric layers themselves.
IV. The Cepheid Variables

Pulsating stars, or Cepheids, are isolated stars undergoing what appears to be spherically symmetric oscillations. In a rather crude sense, one of these stars may be considered as behaving like a spherical organ pipe in which boundary conditions of zero displacement and zero pressure apply at the center and the surface, respectively. This is essentially an ordinary organ pipe with a solid bottom and an open end. The role of gravitation, of course, distinguishes the organ pipe from the pulsating star, since the purely acoustic wave in the organ pipe is replaced by a mixed acoustic and gravitational wave in the star. That is, the potential energy associated with the stellar oscillation appears in the form of work done against pressure forces. The existence of discrete spectrum of eigen-frequencies for the organ pipe has a precise analog in the stellar case, and one can isolate the normal modes of small amplitude pulsation.

Thus we may construct another characteristic time for each star, namely the pulsation period. In the case of the sun this period is slightly less than one hour. On the other hand, the characteristic time for the solar atmosphere is on the order of minutes; so the pulsation has a period which is considerably greater than the characteristic time of the atmosphere itself. Therefore, we are not dealing with high-frequency waves generated in the interior and passing through a small density gradient; quite the contrary, we are dealing with waves which are very long compared to the local characteristic distances. In a linear approximation, one would expect the atmosphere to rise and fall more or less synchronously with the interior.

Although it is never rigorously correct to separate a continuous mechanical system into separate components, considerable progress has been
achieved by attacking separately three aspects of the Cepheid problem. First is the determination of the period of pulsation for a given stellar model with assumption of infinitesimal, spherically-symmetric displacements. According to the full, linear theory these periods should be complex, because energy exchange within the pulsating star will produce either damping or enhancement of the amplitude. However, the imaginary part of the period is exceedingly small, so that an adiabatic calculation will give very reliable values for the real part of the frequency.

The second, and of course more interesting, problem is the source of the instability which leads to the pulsation. Fortunately, several approximations may be used in investigating this problem. In the first place the equations of motion may be linearized, since the amplitude of pulsation is very small in those layers of the star providing the seat of the instability. (Of course, in principle any instability analysis can be based on the linearized equations, since one can always imagine starting from an infinitesimal initial displacement.) In the second place, the radiative transfer equation can be replaced quite accurately by a diffusion equation because the photon free path is exceedingly small in the relevant stellar layers.

It is quite clear now that the seat of the pulsation is not near the center of the star but is confined to the outer 20 per cent of the radius of the star. That is, the pulsation is not related to the variations of nuclear burning in the deep interior, and in fact if the nuclear processes were turned off, the star would slowly cool over a period of tens of thousands of years, and it would continue to oscillate until the star had evolved out of the unstable configuration.
The immediate cause of the pulsation appears to be a "heat valve" originally proposed by Eddington. A peripheral zone of high specific heat alternatively traps and releases thermal energy in the course of a pulsation, and the star becomes a heat engine. It converts a bit of the thermal energy flowing out from the interior of a star into mechanical energy of oscillation.

The third aspect of the Cepheid phenomenon concerns the atmospheric motions. We now take the motions of the interior as given and ask how they will influence the atmosphere itself, assuming that these motions will not react back on the interior. We now ask two related questions about the atmospheric motions. First, what is the overall velocity pattern in the atmosphere which is produced by the interior pulsation. Second, since the resulting atmospheric velocities are supersonic, one next seeks the detailed structure of the shock fronts that develop. These questions are then combined in an attempt to identify elements of this structure with the observed spectrum variations.

In treating the overall flow pattern, we used the method of characteristics, and we treated the boundary conditions at the bottom of the atmosphere by the simplified picture of a spherical piston oscillating sinusoidally. We conceive of these calculations as comprising the following scheme of successive approximations. We initially carried out a numerical integration by the method of characteristics, treating the gas flow as isothermal. We would take the resulting flow pattern as input for the next stage and compute the temperature field and the radiation field with the correct energy equation. Then we would examine the influence of this resulting radiation field on the flow pattern itself. This still appears to be a feasible, and fairly rapidly converging process, although we have
not carried it beyond the first stage. Of course, digital computer programs can incorporate all of the equations simultaneously, and, in the last analysis, they will provide the definitive calculations. On the other hand, a great deal of physical insight can be obtained by the method of successive approximations.

In our initial calculations, we postulated a piston oscillating at some level with a period which is long compared to the acoustic period of the atmosphere. The piston velocity was a sizable fraction of the local acoustic velocity. From this piston motion, shocks developed and propagated upward with an essentially constant velocity. Of course, the initial shock that went out after we had just started the integration was one which accelerated, but the interaction between the flow which had occurred earlier and the new shocks was such that later shock trajectories became very nearly straight. The particle paths corresponding to this flow somewhat resembled those of ping pong balls bouncing on a table. Each time a shock passed up through the atmosphere the gas was given up upward impulse, and it then underwent very nearly constant downward acceleration under the influence of gravity and the local pressure gradient.

Figure 1 shows an idealization of the model in which the shock trajectories are taken as straight and the particle paths are parabolic, corresponding to a constant downward acceleration between shock impulses. (This diagram is adapted from Annales d'Astrophysique, 19, 142, (1956).) Distance and time are indicated on this diagram in convenient dimensionless units. If we imagine now the construction of an observed velocity curve from this schematic diagram, we are encouraged by recognizing a number of details corresponding to actual observations.
If, first, we adopt the simplest assumption, that the observed velocity curve corresponds to a single, fixed particle, then clearly the observed curve will resemble a saw-tooth with portions of constant slope separated by discontinuous jumps. If we complicate the picture just slightly by recognizing that an absorption line actually is a tracer of a finite mass depth in the atmosphere, we recognize that the velocity width of an absorption line will be determined by the atmospheric velocity gradient over the region of formation of the line. Therefore, as the shock front emerges through the photospheric layers, we would expect to observe, at least temporarily, a splitting of the absorption lines corresponding to the velocity discontinuity in the atmosphere. Furthermore, even in the absence of the shock itself, we would anticipate that the lines should be broadened quite a bit by the velocity dispersion within the atmosphere.

The solid line of Figure 2 shows the velocity curves anticipated for a single particle. The case in which the absorption lines correspond to a finite mass-depth in the atmosphere is indicated by the dashed extensions. Figure 3 contains the velocity curve of W Virginis obtained from the work of Sanford and Apt. This curve represents the velocity shifts obtained from measured displacements of the absorption lines in the spectrum of this star. According to the sign convention adopted here, negative velocity indicates motion toward the observer or away from the center of the star. The qualitative similarity between this curve and the schematic curves shown earlier is quite clear. In particular the fairly long and nearly-constant downward acceleration is shown. The doubling of the absorption lines persists for a fraction of the cycle which is quantitatively consistent with the model. Strong emission lines of hydrogen appear just
before the emergence of the new set of absorption lines. I will comment on these a bit later.

Figure 4 shows the velocity curve of a rather different kind of variable star, with a much shorter period and higher temperature than W Virginis. This star, BW Vulpeculae, shows a flow pattern which is quite puzzling. There is short-lived set of absorption lines which is apparently produced by material at rest with respect to the center of the star. It is still not clear whether this material has been suddenly brought to rest by a violent interaction between gasdynamic waves moving in opposite directions or whether this is stationary material of the deeper photosphere, momentarily visible. The consideration which may be relevant here is that these shocks are propagating out into a fairly steep density gradient, and the question of the stability of the flow pattern arises. Since the radius of curvature of the atmosphere is very large compared to the thickness of the atmosphere itself, we may treat the problem by assuming plane-parallel geometry in the undisturbed atmosphere. Shocks propagating in a homogeneous gas are very stable against distortions from plane parallel geometry, but one can well imagine that, because of the density gradient, there may be a tendency for the shock to develop dimples. That is, waves may appear on the shock front as it propagates through convective fluctuations in the atmosphere. These may grow as the shock propagates upward into the regions of lower density. Therefore, it is not inconceivable that this rather odd velocity curve shown by BW Vulpeculae is connected with the shattering of the shock front and the production of jets in the atmosphere.

I would like next to consider the details of the flow pattern in the immediate neighborhood of the shock front. Figure 5 shows an example of the data available pertaining to this region of the atmosphere of W Virginis.
It displays the profiles of the $\text{H}_\alpha$ line of the Balmer series in hydrogen. There is strong emission when the shock is emerging through the photospheric layers. This extremely intense line has a rather complicated profile which is strongly influenced by Stark broadening and Doppler shifts associated with the velocity gradients in the atmosphere. As yet there has been no detailed calculation providing an interpretation of this profile, although the shock model provides insight into various of its features.

In order to attack this problem we have performed some shock-structure calculations rather similar to those reported by Professor Clarke at this meeting, although we have confined ourselves to a gas of pure hydrogen. We have not made a quantitative comparison with his work, but there is a clear qualitative similarity. In particular, we also find a precursor flow. That is, radiation escaping from the shock and penetrating into the preshock gas, heats and ionizes the gas before the arrival of a shock. In our case the significant radiation is that in the Lyman continuum, and the rather high opacity to this radiation assures that the precursor region is fairly thin in a stellar atmosphere. It will also be optically thin in most visible radiations, and it will have a very slight velocity shift. Physically, the important point is that the flow ahead of the shock is supersonic with respect to the shock front so that radiation which it absorbs can increase the gas pressure, but it will only very slightly alter the velocity of the gas. According to our calculations, the effect is less than 0.5 km/sec, and this is quite unobservable. For shocks of moderate strength, that is with velocities greater than 20 and 30 km/sec, the degree of ionization in back of the shock increases significantly, through radiative and collisional ionizations. This approach to a higher degree of ionization takes place over distances which are of moderate optical thickness. It is important
to note that we are dealing here really with at least two radiation continua, those of the Balmer and of the Lyman radiation. There are, therefore, at least three distinct characteristic-lengths associated with this ionization region; namely, a collisional-ionization length (the distance an atom will move with the flow before becoming collisionally ionized), the mean free path of the Balmer photons, and the mean free path of the Lyman photons.

Our calculations have verified a familiar fact that, despite radiative exchange of energy between various elements of the gas and a rather significant change in the degree of ionization, the pressure profile in back of the shock front is very nearly flat. This results from the fact that the flow in back of the shock is subsonic. Therefore, if energy is removed from the gas, for example, and the temperature thereby decreased, the conservation of momentum and mass leads to a rise in density and only a very slight change in the pressure.

In treating these problems of shock structure we have made a very significant simplification by eliminating from the differential equations those terms representing partial differentiation with respect to the time. That is, we have explicitly assumed that the flow can be treated as a steady flow, despite the fact that we are clearly dealing with an unsteady problem as the shock propagates out through the atmosphere. The rationale for this simplification is quite simple and fairly convincing, I think. We may phrase the argument in terms of a ratio of characteristic distances or, alternatively, a ratio of characteristic times.

Consider a ratio of distances first. One relevant distance is the atmospheric scale height, since we may safely assume that the shock will not alter significantly until it has propagated a sizable fraction of a
scale height. The other relevant distance is the thickness of the reaction zone immediately in back of the shock front. Our calculations indicate that, although this reaction zone is not necessarily optically thin, it is under most circumstances very thin geometrically. Therefore, an argument in terms of relative lengths states that gradients of the flow parameters produced by gradients in the undisturbed atmosphere will be relatively small and may be neglected.

An alternative statement in terms of characteristic times, would be that elements of the gas are carried through the reaction zone in a time which is very short compared to the time required for the shock to propagate one scale height. This follows from the fact that the velocity of gas flow relative to the shock front is not very different from the velocity of propagation of the shock front itself. Therefore, the approximation of steady flow is not a bad one in this context.

V. Oscillations of the Solar Atmosphere

Let me now set aside the problem of stellar pulsation and turn to a problem closer to home which may be treated in the acoustic approximation.

It has been known for over half a century that superimposed on the mean state of a solar photosphere is a gentle field of fluctuations. The brightness fluctuations, which were in fact observed first, have a characteristic horizontal length of about a thousand kilometers. They are therefore large when measured in units of the vertical scale height of the atmosphere (about 100 km). The intensity fluctuations are about 5 to 10 per cent in the white-light continuum, although quantitative studies of brightness pattern are very
drastically hindered by the earth's atmosphere. The velocity fluctuations
have an amplitude of about one km/sec and a characteristic scale of several
thousand kilometers in the horizontal direction. It is, of course, very
difficult to infer quantitative characteristics of the vertical extend of
these fluctuations.

In addition to their intrinsic interest, these fluctuations attract
attention for two further reasons. In the first place, these fluctuations
represent a response of the solar photosphere to various sorts of driving
forces imposed on it by the underlying convection zone. Therefore, an
analysis of the velocity and temperature variations in the photosphere allows
us to prove, in a sense, the convection zone.

In addition to looking downward into the convection zone, we can
look upward and ask what the field of fluctuations in the photosphere implies
for chromospheric and coronal structure. That is, since the chromosphere
and corona represent the results of departures from radiative equilibrium
in the outer solar atmosphere, they are intimately connected to the dynamical
state of the photosphere, or more specifically, to the flux of mechanical
energy passing up through the photosphere. Time-correlation studies of the
brightness fluctuations in the solar granulation have been performed by many
investigators. These studies have consistently indicated a monotonic decrease
in the correlation function with a characteristic time between 5 and 10
minutes. (See Figure 6.) The similarity of this time to the acoustic time
and the thermal time of the solar atmosphere is not surprising if one admits
that these parameters have any physical significance. Examinations of the
correlation between brightness variation and velocity shift measurable on
single spectrograms of the solar surface have not been very conclusive, but
they do indicate that the instantaneous spatial correlation between brightness
and velocity is weak. In all probability the weakness of this correlation results from the fact that the observations themselves refer to rather different layers in the solar atmosphere.

Time-correlation studies of the velocity field have, within the past several years, indicated that the pattern is a very strongly oscillatory one. That is, a plot of the velocity at a particular point on the solar disc as a function of time shows a very striking oscillation with a period of about 5 minutes (see Figure 7 for an example). The horizontal coherence of these oscillations extends about 5 to 10 thousand km, a distance which is significantly greater than the horizontal scale of the brightness fluctuations. The rather striking purity of the frequency spectrum of the oscillations is well shown in Figure 8 which contains histograms of the distribution of observed periods. Spectrum lines which are produced high in the atmosphere show a shorter mean period than lines produced low in the atmosphere.

Thus there is a very real qualitative distinction between the temporal behavior of the brightness fluctuations as observed in the continuum and the velocity fluctuations as determined by measurements of the shifts of spectrum lines. The velocity field is oscillatory, with the oscillations lasting frequently for several periods; while the brightness field seems to consist of impulses with a subsequent decay which appears to be monotonic.

The frequency spectrum of the velocity fluctuations is now quite well known, and one naturally seeks the specific cause of its observed form. Among the possible factors influencing this form there are four which I shall isolate for discussion now. In order to make the comments more explicit let me write $W(P)$ for the observed power spectrum of the velocity fluctuations
determined from the time correlation studies of velocity, I write the following schematic equation:

\[ W(P) = \frac{S(P) F(P)}{L(P) D(P)} \]

I shall discuss the various factors separately, commencing with \( S \) which is the power spectrum of the forcing function, that is the field of fluctuations in the convection zone. Although very little is know about \( S \), it does appear that it is a fairly flat function of \( P \) containing significant contributions over a factor of at least two of three in the period. This is a considerably broader spectrum, therefore, than shown by \( W \), so we must look to the remaining three factors.

Schematically, \( F \) represents the efficiency of energy and momentum transfer between the fluctuations of the convection zone and the fluctuating field of the photosphere itself. Stated in slightly different terms, it represents the filtering effect of the interface between the convection zone and the photosphere. In yet another way, it may be called the response function of the photosphere to a unit impulse from the underlying zone.

\( L \) represents the leakage of acoustic energy from the photospheric oscillations due to wave propagation out of the photosphere. That is, once the oscillations are established, they will generate sound waves which will carry energy both upward and downward away from the photospheric levels.

Finally, \( D \) represents the rate of the damping of the oscillations through thermal radiation. This function does not vary strongly with period.

The nature of \( F \) and \( L \) may be seen by examining Lamb's original discussion of this problem, or from the more recent work of Noyes and Leighton. For short periods the response function, \( F \), is nearly a constant, but it
drops off rapidly for periods greater than 5 minutes becoming very small for longer periods. The leakage rate, \( L \), on the other hand, is fairly large for short periods and becomes very small for periods greater than 5 minutes. As has been pointed out by Noyes and Leighton, the ratio \( F/L \) for the steady state solution has a resonance denominator and becomes very large in the neighborhood of five minutes. Quite clearly, the high peak in the power spectrum of the observed oscillations is in the neighborhood of five minutes results from this resonance denominator in \( F/L \).

The resonance behavior of the function \( F/L \) results from the interaction between gravitational potential and ordinary compressional potential within the atmospheric waves. In other words, these waves are a mixture of gravity waves and acoustic waves, and the relative importance of the two modes will be a function of the impressed frequency.

Although a consideration of waves in which the motion is restricted to the vertical direction provides a great deal of insight into the nature of the fluctuations in the solar photosphere, a rather distinct class of waves is introduced when this restriction is relaxed. When horizontal as well as vertical motions are allowed, a wave mode is possible in which the potential energy is entirely associated with work done against gravitational force. The density of a chosen mass element is independent of time during the oscillations, and this wave is similar to surface waves on an incompressible liquid. A baby in the bathtub will excite gravitational oscillations in which compressibility effects are negligible. This type of wave is evidently quite efficiently excited in the solar atmosphere by convective motions. Whittaker has pointed out that this type of wave may be able to penetrate the chromosphere and carry considerable energy into the solar corona. A recent investigation
by Uchida has examined the influence of a gradient in the temperature of
the undisturbed atmosphere. He obtains a differential equation in the wave
amplitude which is reminiscent of the Schrödinger equation for a particle
in a potential well. In fact, with an appropriate temperature distribution,
standing gravitational waves are possible which are effectively trapped
within the atmosphere. That is, their amplitude decreased exponentially
in both directions from a layer of maximum amplitude. These waves are
analogous to interior waves in the ocean which may be considered as waves
on a surface between layers of different density.

VI. The Acoustics of a Radiating Medium

In this concluding section I wish to draw attention to three
investigations which have a direct bearing on the interpretation of the
oscillations of the solar atmosphere. In the first of these investigations
Vincenti and Baldwin considered the propagation of small plane disturbance
in a radiating gas. They confined their attention to solutions that were
periodic in time, and solved the resulting integro-differential equation
for the velocity potential by a substitute kernel method. Lick has extended
this work by considering a more general initial-value problem.

As an example of the physical problem which these authors studied
consider the following idealized configuration. A homogeneous, absorbing
gas extends infinitely far from a plain black wall. There is no gravitational
field, and the wall and gas have the same temperature initially. Each
element of the gas receives radiation from other elements of the gas and it
also receives radiation from the wall. The absorption coefficient is
considered to be independent of wavelength, although not necessarily
independent of temperature. If, at \( t = 0 \), a time variation is imposed
on the temperature of the wall, the heat balance of every element of the gas
will be perturbed. This perturbation will initially propagate outward with
the velocity of light and with an amplitude decreasing exponentially with
distance from the wall. The temperature of the gas, and also its pressure,
will be perturbed by this disturbance of the heat balance, and the pressure
perturbation will initiate velocity and density perturbations. The initial
temperature perturbation will further alter the heat balance by modifying
the emission and absorption terms throughout the gas. A particularly
interesting aspect of the resulting field of motions is that two characteristic
velocities enter, namely the velocity of propagation of an acoustic pulse
and the velocity of propagation of light. (We have already seen that in the
stellar atmosphere case the velocity of light may be effectively taken as
infinite, however.)

A quantitative application of the results of these investigations
to the solar atmosphere is not possible, because the gravitational field
and the resulting inhomogeneity in the solar atmosphere will have a profound
influence on the wave field. However, if we characterize the solar atmosphere
by certain mean parameters and insert these into the theory for the homogeneous
medium, two qualitative results emerge. In the first place, the velocity
of acoustic propagation in the solar atmosphere will probably be essentially
the adiabatic value. Second, acoustic waves will be heavily damped by
radiative effects in the solar atmosphere. On the other hand, variations
in the radiation field can very effectively initiate acoustic disturbances.
More explicitly, we must expect that the temperature variations associated
with convective motions in the low photosphere will introduce fluctuations
in the radiation field which, in turn, will excite acoustical and
gravitational waves in the upper photosphere.

The third investigation I wish to mention gives an explicit example
of this initiation of an acoustic wave by a purely thermal variation. Robert
Noyes has extended the investigation of Lamb's problem of an atmosphere driven
from below by various types of perturbations. Noyes has investigated numerically
and analytically the influence of radiation transfer, and has shown that
these waves are very heavily damped in the low photosphere. In one of his
calculations, Noyes considered a hot wall to be stationary at the bottom of
the atmosphere and imposed a rapid time variation on the temperature of this
wall. He then solved the propagation equations in their finite difference
form and showed that an acoustic wave of moderate amplitude was induced by
this thermal variation. Noyes' work is particularly intriguing since it
indicates that a significant portion of the observed oscillations in the
solar photosphere may be excited by fluctuations in the thermal radiation
field associated with the deeper convective layers. He has, in fact,
carried out a detailed comparison between the results of his calculations
and observations of the sun, and finds a very encouraging agreement between
the two.
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Legend for Figures

Figure 1 Idealization of shock paths and particle paths in the pulsating atmosphere of W Virginis (cf. Ann. d'Ap., 19, 142 (1956)). H = atmospheric scale height; a = sonic velocity.

Figure 2 Schematic velocity curve for W Virginis derived from idealized model of atmospheric pulsation. Positive velocity is measured downward into the star and away from the observer. H = atmospheric scale height; a = sonic velocity.

Figure 3 The velocity curve of W Virginis, as derived from the work of Sanford and Abt, compared with the light curve.

Figure 4 Schematic velocity curve of BW Vulpeculae, after Struve. Thickness of lines indicates intensity of line components.

Figure 5 Emission line profile for H$\alpha$ taken from the work of Abt (Ap. J. Suppl., 1, 63 (1954)).

Figure 6 Time-correlation function for brightness fluctuations of the solar disc (granulation). The curve is a simple exponential. Taken from J. Bahng and Schwarzschild, Ap. J., 134, 312 (1961).

Figure 7 Velocity as a function of time for discrete points at the center of the solar disc.

(a) Measures in the $b_2$ spectrum line for 15 successive points 1000 km apart.

(b) Seven independent points in $b_2$ (solid curves and crosses) and Ti 5173.75 (dashed curves and dots). Taken from J. Evans and R. Michard, Ap. J., 136, 493 (1962).

Figure 8 The distribution of periods of vertical atmospheric oscillation as measured in the $b_2$ and $\lambda 5173.75$ spectrum lines of the sun. Taken from J. Evans and R. Michard, Ap. J., 136, 493 (1962).
Figure 6
Figure 8
DISCUSSION

R. W. Hillendahl: The numerical solutions of the Cepheid problem which I was discussing yesterday gives us a shock behavior of the type illustrated in your Fig. 4 (B. W. Vulpeculae). But we can actually take a variety of star models on our computer; I think the Cepheids correspond to a narrow range and to a certain type of instability only.

S. Scala: In Figs. 7 and 8 you show a periodicity of about 300 sec. Why this period?

C. A. Whitney: The periodicity is caused by the dispersion of the atmosphere. Those frequencies around 300 sec. correspond to the value of \( \frac{4\pi h}{a} \) and are effectively trapped where they are produced. The high frequency component can propagate upward because the high frequency wave sees the atmosphere as essentially a homogeneous gas. The low frequency components are not effectively produced; one always has an impedance matching problem for very low frequencies.

So one puts in a whole range of frequencies and this is what is left behind.
INVERSION BY SLABS OF VARYING THICKNESS

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The advent of the space age has given an impetus to the role of radiative transfer in atmospheric physics. All of a sudden the meteorological satellite gave the atmospheric physicists a space platform, a new environment to probe radiometrically the upper atmosphere. The exciting thing, therefore, is to see to what extent one can then infer the structure of the atmosphere, having access only to the electromagnetic radiation that is impinging on the vehicle. It is a real challenge to see just what one can infer from radiometry alone. Now in one respect, as compared to other radiative transfer fields, the atmospheric physicist has an enormous advantage. That is in the accessibility of his environment to independent measurement by other means. This provides a check, of course, on any radiative transfer formalism that is developed. It is impossible to over-estimate this desirable feature because there is nothing like the cold water of real data, that must be explained to sift the relevant from the trivial. The satellite problem affords this opportunity. In terms of the conditions set down yesterday by the host, Professor Thomas, radiative transfer in the atmosphere is simple. We assume a plane parallel one-dimensional geometry, collisions do predominate, local thermodynamic equilibrium does hold, at least up to a height of around 50 km. In fact, the problems as he outlined them seemed so simple, that I wonder why I have been so dense in understanding them. This line of reasoning if logically pursued, is very destructive to the ego, so I will drop it.
As an example of radiative transfer applied to atmospheric physics let us consider the inversion problem. Simply stated the problem is this: Given the upwelling radiative flux from an atmosphere viewed externally at a number of viewing angles and/or frequencies, deduce the vertical variation of the state of the atmosphere. Since the emergent radiation is given from the transfer equation as an integral transform of the vertical state, the problem becomes one of inverting the integral equation to recover the profile from the observations. The idea dates all the way back to the remote scientific past when astrophysicists correctly interpreted the solar limb darkening as due to the fact that the temperature in the photosphere was decreasing outward. The limb-darkening implying that as one looks toward the edge of the solar disc where the eye is intercepting photons from the cooler outer layers, the intensity drops off. The inference is strong that, given a precise knowledge of the variation from center to limb, one could, in principle, infer something about the vertical thermal structure in the atmosphere. This idea of applying it to the terrestrial atmosphere was made in 1956, the year before Sputnik. In 1959, Lewis Kaplan proposed that rather than looking from a satellite downward and scanning over Nadir angle that instead one look downward at the atmosphere and scan over frequency. A logical candidate for this scan is the 15 μ carbon dioxide band which is quite intense. Obviously, if one looks at the Q-branch in the center of the band one doesn't see very far down. Backing away from band center, one sees all the way down to the ground. This then is the inversion problem.
Inversion by Slabs of VaryingThickness

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26 March 1964

In the inversion problem an algorithm is sought for the inference of atmospheric vertical thermal structure from remotely-sensed radiometric observations. To accomplish this we propose a new inversion method, the variable slab technique, and demonstrate its application by two illustrative examples.

All inversion procedures attempt to recover the thermal profile from observations of the upwelling intensity $I(\kappa/\mu)$ at various directions and/or frequencies. Transfer theory specifies the temperature dependence on depth as the solution of a linear integral equation

$$I(\kappa/\mu) = -\int_{0}^{\infty} B(u) \frac{\partial \mathcal{Y}}{\partial u} du,$$  \hspace{1cm} (1)

where $B$ is the Planck intensity considered here an implicit function of absorber depth $u$, and $\mathcal{Y}$ is the kernel transmittance averaged over a narrow frequency interval

$$\mathcal{Y}(\kappa u/\mu) = \frac{1}{\Delta u} \int_{\Delta u} e^{-\pi u/\mu} du,$$  \hspace{1cm} (2)

with $\kappa$, the monochromatic absorption coefficient.

For our purposes a simplified gray, plane-parallel, fixed-frequency model suffices in which the intensity is scanned over nadir angle $\theta = \cos^{-1} \mu$. Under these conditions the intensity is a Laplace transform of the indicial function

$$I(1/\mu) = \int_{0}^{\infty} B(\tau) e^{-\tau/\mu} d\tau/\mu = \frac{b(1/\mu)}{\mu},$$  \hspace{1cm} (3)

where we have transformed to the new variable, optical depth $\tau = \kappa u$.

Conventionally the temperature profile is approximated by an appropriate series expansion

$$B(\tau) = \sum a_i F_i(\tau).$$  \hspace{1cm} (4)

The series need not be orthogonal but should converge to the exact solution.

The substitution of this expansion into equation (3) identifies the intensity with the Laplace transforms of the indicial approximation

$$\mu I(1/\mu) = \sum a_i F_i(1/\mu).$$  \hspace{1cm} (6)

Intensity observations at $n$ discrete directions enable one to determine $n$ coefficients of the temperature expansion, equation (4), as the solution of the linear simultaneous equation set

$$\mu I(1/\mu) = \sum a_i f_i a_{i}, \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (6)

A variety of different function classes have been used in inversion attempts. Examples are power series (King, unpublished), exponential functions (King, 1964), and various orthogonal sets such as Legendre, Chebyshev, or Laguerre polynomials (Yamamoto, 1961). All these
expansions share a common defect rendering them unsuitable for the inversion procedure. By choosing a particular finite polynomial expansion we restrict the form of the thermal profile.

Standing in contrast to this analytic procedure in which the temperature is broken down into components, is the synthetic method which approximates the profile by isothermal slabs. This is expressed by expanding the lapse rate in a sum of delta functions

$$\frac{dB(\tau)}{d\tau} = \sum (\Delta B)_j \delta(\tau - \tau_j).$$

(7)

Proceeding as before, the substitution of this slab approximation into the transfer equation yields the equation set

$$I(1/\mu_i) - B(0) = \int_0^\infty \frac{dB(\tau)}{d\tau} e^{\tau/\mu_i} d\tau = \sum (\Delta B)_j e^{-\tau_j/\mu_i}.$$

(8)

We have not specified the slab boundaries \(\tau_j\). Herefore these positions have been assigned in advance, usually at significant levels in the atmosphere where lapse-rate discontinuities are anticipated (Kaplan, 1959; Wark, 1961).

Once again a knowledge of the intensity profile at the \(n\) directions \(\mu_i\) leads to a linear equation set determining the slab temperatures at \(n\) preset intervals

$$I(1/\mu_i) - B(0) = \sum (\Delta B)_j \exp(-\tau_j/\mu_i).$$

(9)

As we shall see this synthetic method is extremely sensitive to the choice of slab boundaries. As with the analytic procedure, the same criticism holds. The choice of \(\tau_j\) is critical, forcing in advance a particular structure on the slab profile.

We propose, therefore, a variable or floating slab method which determines uniquely the slab strengths and thicknesses for a given intensity profile. Consider equation (9). With the substitution \(x_j = \exp(-\tau_j)\) we succeed to a set of nonlinear simultaneous equations each of degree \(1/\mu_i\)

$$I(1/\mu_i) - B(0) = \sum (\Delta B)_j x_j^{1/\mu_i}.$$  

(10)

By choosing the sequence of viewing directions

$$\frac{1}{\mu_i} = i = 0, 1, 2, \ldots, 2n-1,$$

(11)

we obtain the equation set of successively higher degree

$$\alpha_i = \sum a_i x_j^{1/\mu_i}, \quad i = 0, 1, 2, \ldots, 2n-1,$$

(12)

where we have written \(\alpha_i = I(i) - B(0)\) and \(a_i\) for \((\Delta B)_j\).

The equation set arises in the construction of quadrature formulas of the Gaussian type (Lanczos, 1956; Kopal, 1961). Despite its nonlinearity the set is soluble uniquely by an elegant algorithm given, for example, by Chandrasekhar (1950) which consists of three steps. First, \(n\) auxiliary constants \(c_i\) are determined from the linear equation set

$$\alpha_i + \sum_{i=0}^{n-1} c_i \alpha_{i} = 0, \quad i = 0, 1, \ldots, n-1.$$  

(13)

The slab boundaries \(\tau_j = \ln x_j\) are then obtained as the \(n\) roots of the equation

$$x^n + \sum_{i=0}^{n-1} c_i x^i = 0.$$  

(14)

The knowledge of the roots enables one to determine the weights \(a_j\) from the first \(n\) equations of the set (12).

The solution admits the interpretation: Given \(n\) isothermal slabs, the \(a_j\) and \(\tau_j = \ln x_j\) are the unique choice of slab weights and thicknesses fitting the \(2n\) intensity observations.

Fig. 1 displays three and five slab atmospheres inferred from intensity values of the following model atmosphere

$$\frac{I(1/\mu_i)}{\mu_i + 1} = \frac{1}{\mu_i} = \begin{cases} 0, 1, \ldots, 5 \\ 0, 1, \ldots, 9 \end{cases}$$

(15)

The solid curve is the exact solution obtained directly by inversion

$$B(\tau) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mu I(1/\mu) e^{\tau/\mu} d(1/\mu)$$

$$= 1 - e^{-\tau}.$$  

(16)

The slab approximation is impressive. Note that the slab thickness is smallest in the region of greatest slope, thus minimizing the "cornering" error.

For comparison the same ten intensity values were used to infer the ten weights of slabs bounded at the ten present intervals

$$\tau_j = 0.1, 0.2, \ldots, 1.0.$$  

(17)

The thermal structure inferred by solving the equation set (9) for this model is grossly unrealistic (see Table I).

The superiority of the floating over the fixed slab method can be understood by its relation to quadrature formulas. The Gaussian quadrature method achieves more accuracy than the preset Newton-Cotes intervals by allowing the integrand thicknesses to vary. Similarly in our inverse problem we have the additional degree of freedom in the determination of the slab boundaries as the unique solution of equation (14).

A second, more complicated atmospheric thermal
adding a constant slope to the lapse-rate which is subsequently subtracted after the inversion operation.

In a forthcoming paper “Meteorological Inferences from Satellite Radiometry, II” this floating slab method will be applied to thermal inferences of synthetic atmospheric models. An extension of the formalism to treat arbitrary band transmittance kernels is planned. The implications of the technique for the error analysis of raw radiometric data will be discussed.

Acknowledgments. Discussions with Dr. E. T. Florance were helpful in the formal aspects of the problem. All computations were performed by Mr. Richard Harrison.

This research was supported by the NASA Goddard Space Flight Center under Contract NASS-3352.

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DISCUSSION

J. Jefferies. I have been tackling essentially the same problem* in the context of the inversion of the limb darkening equation

$$I(\mu) = \int_0^\infty S(t) \exp{-t/\mu} \, dt/\mu \quad (1)$$

which relates the intensity $I$ of radiation emerging at an angle $\cos^{-1}\mu$ to the normal to a semi infinite plane parallel atmosphere to the distribution of the source function $S$ with optical depth $\tau$. For given values of $I(\mu)$ we wish to determine $S(\tau)$. The procedure used for inverting this integral equation is described in detail in a forthcoming JILA report by Hummer and me so I shall not give details here. In outline it consists firstly in replacing the integral by a quadrature sum; since this is well known to yield an extremely poorly conditioned system however, it is also required that the unknowns $S(t_1)$ should represent as smooth a distribution as possible consistent with the requirement that they also satisfy equation (1). There are many ways in which this 'smoothness' could be required; the one chosen was essentially that which minimized the quantity $\sum_i [S(t_i)]^2$.

The procedure has been tested using a stringent case where $S$ consisted of three linear segments. This form was inserted into equation (1) and the corresponding intensities $I(\mu)$ were then determined for a set of values of $\mu$. These were then used in the inversion technique to try to recapture the segmented source function. Results were remarkably good and are illustrated in the JILA Report cited already. Tests with more simple

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functions \( S(t) \)--such as those used by King in his trials--gave an even better reproduction.

Dr. King is under a misapprehension. My procedure does not involve smoothing the data; it simply determines the smoothest set of the unknowns \( S(t_1) \) which are consistent with the observed data. Thus it is not smooth data but a smooth source function that is required.

King. But the source function was discontinuous.

Jefferies. No, the slope was discontinuous in the actual example used but that is irrelevant. The point is this: Even were equation (1) theoretically precise, \( I(\mu) \) is known in practice only within certain limits imposed by the observational accuracy. The difficulty in solving these Fredholm integral equations of the first kind arises because solutions \( S(t) \) corresponding to almost identical data show in general almost no relationship to one another. This is well known to anyone who has tried to solve the quadrature system based on equation (1), as is the fact that the solutions \( S(t) \) fluctuate wildly in \( \tau \).

In practice, therefore, one must introduce an auxiliary condition on \( S(t) \) to select a sensible solution--i.e. one which is not critically dependent on non-significant changes in the data. The auxiliary condition adopted will depend on the problem. In the limb darkening case it has been customary (and marginally defensible) to expand \( S(t) \) in a series with unknown coefficients which are determined from the data. If one is not prepared to force an \textit{a priori} form on \( S(t) \) then some (smoothness) requirement on \( S(t) \) must be introduced. This is as much a part of King's procedure as any other.
A. E. Fuhs: There is a criteria that you can form to tell how sensitive your results are going to be to errors in measurements; you take your exponential and form a Grammian determinant: the lowest eigen value will tell you how much error you can expect in your profile for a given error in your measurements; this a kind of figure of merit.

J.I.F. King: This is an obvious next step. Our NASA contract is precisely on this problem of error analysis. I don't think that it is going to be too easy to do because you see this is a non-linear problem and it is very hard to follow through. One part of the algorithm is getting roots of a \( n \)th degree equation. The error analysis for the linear case is much easier than for this non-linear case.

S. C. Traugott: What are the possibilities of this technique of measuring temperature profiles in other atmospheres?

J.I.F. King: I think they are very good. I would even go further. It seems to me that from ground base measurements you could determine the temperature profile of the ozone in our own atmosphere. If you have a station such as Flagstaff, which is above the water vapor, you are seeing primarily ozone.

S. C. Traugott: Could you for instance fly a probe in orbit around Mars and then infer temperature profiles for the Martian atmosphere?

J.I.F. King: J.P.L. is constructing a spectrometer at 4.3 microns to look at the carbon dioxide band on Mars. The idea is exactly the same as for the Barnes instrument I have discussed and it is eventually going to be flown on a Mars fly-by mission. So the same technique is available right
now. I talked at the AMS meeting a couple of weeks ago to one of the people who is working on that. They are pretty advanced in the instrument stage and this inversion technique is available.

M.A. Heaslet: It seems to be that in sailing to India you have discovered America! You have given rules to invert any Laplace transform.

J.I.F. King: Any linear integral transform, yes.
RADIATIVE TRANSFER IN THE SEA

by

Rudolph W. Preisendorfer
University of California, San Diego

Introduction

My purpose this afternoon is to present a brief sketch of the state of radiative transfer theory as it is currently used in the study of natural waters such as the seas, lakes, rivers and harbors of the world.

As you know, modern radiative transfer theory originated at the turn of this century principally in the studies by Schuster and Schwarzschild of the transfer of light through the sun's atmosphere. Subsequently its tenets found applications to the atmosphere of the earth and to industrial problems of the manufacture of fine pigments and papers. It has only been recently, within the past fifteen years or so, that the principles of radiative transfer have been fully developed and applied to the problems of hydrologic optics. I might observe that in the process of applying the original astrophysical formulations to terrestrial settings, many conceptual gaps and weaknesses in the theoretical superstructure of the theory were diagnosed and remedied.
This is certainly not to be taken as a reflection on the work of the originators of the theory, but rather as an illustration of the rule that accessibility to the object described by a theory increases the rate and depth of experimental and observational feedback to that theory. This rule applies with double intensity to radiative transfer phenomena (such as scattering, absorption, emission) in the sea wherein such phenomena arise and unfold within the very small space of several meters as compared to the hundreds or thousands of meters required for the same phenomena to evolve in the atmosphere. Furthermore, the delicate radiative transfer models of the early astrophysicists were virtually crushed in the task of describing real light fields in real seas and lakes with their highly anisotropic scattering functions, their severe stratifications and turbulent boundaries. Therefore what I shall do this afternoon is to outline two particularly rugged, yet simple models of the light field which describe the light fields in natural hydrosols and also which provide a rigorous conceptual scheme within which we may frame the optical properties of natural hydrosols. I shall be quite brief and let my slides do most of my talking. If you wish to go further into the theoretical foundations of the subject and the ramifications of its principles, I refer you to my monograph concerned with Radiative Transfer on Discrete Spaces or to my forthcoming work on Hydrologic Optics.

The Two Main Problems of Hydrologic Optics

Fig. 1 describes schematically the first of two main
problems of radiative transfer in the sea. The sun pours about
1400 watts per square meter of radiant flux of all wavelengths into the
top of the atmosphere. Of this initial amount, about 1200 watts per
square meter percolate down to the air-water boundary on clear dry
days. About 1000 watts per square meter penetrate the boundary to go on
to initiate and sustain the submarine light field. **Direct Problem:**
given the absorbing and scattering properties \( (\alpha , \sigma ) \) of a particular
body of water, to predict the radiance \( N(z, \xi) \) in each direction \( \xi \)
at each depth \( z \) below the surface. The **Inverse Problem** is: given
the radiance \( N(z, \xi) \) for each pair \((z, \xi)\), to determine the optical
properties \( (\alpha , \sigma ) \) of the medium.

**Two Approaches to the Problems**

Figures 2, 3 depict two general theoretical approaches to
solve the Direct and Inverse problems. Fig. 2 summarizes the local
approach. This uses the equation of transfer. The light field's net
rate of gain is kept track of at each point. Given: \( \alpha , \sigma \) throughout
the medium, \( M \), and \( N \) on the boundary, the equation is to be solved for
\( N \) at each \( p \) in \( M \). In the global approach, the medium \( M \) is
conceived of as a whole whose various parts such as \( A, C \) interact one
with the other. Given the scattering operators \( S_A \) and \( S_C \) for the
parts \( A \) and \( C \), one can compute the internal radiance field \( N \) at
points on the interface of \( A \) and \( C \). Fig. 4 depicts a special case of
Fig. 3. Here the medium has a plane parallel structure. The five
equations in this setting completely define the light field at any
depth $\tau$, given the incident light field $N^0$ at the upper boundary and the reflectance and transmittance operators for the sub-slabs A and C. $r_1, r_2$ are reflectance operators for the boundaries.

The Two Flow Model

The first of the two models I wish to bring to your attention is the two-flow model for irradiance. Let $H(z,+)$ denote the irradiance, i.e., the watts/m$^2$ in the upward (+) and downward (-) directions across a horizontal plane at depth $z$ in an arbitrarily stratified natural hydrosol. These may be measured by flat plate collectors as shown in Fig. 5. Companion radiometric quantities are the scalar irradiances $h(z,+)$ measurement of the three main radiometric concepts $H$, $h$, $N$.

Fig. 6 shows how the three types of radiometric concepts $H$, $h$, $N$ can be used to operationally define the basic inherent optical properties of the seas, lakes, or other natural hydorsols. These are given in (2)-(5). Eq. (6)-(10) represent the quantities in (2)-(4) in terms of radiance, which is the most basic of all radiometric concepts.

Fig. 7 depicts a list of basic apparent optical properties. $R(z,-)$ is the reflectance of the hydrosol at depth $z$ for downward (-) flux. The six $k$-functions give the depth rate of decay of the indicated radiometric concepts. The current status of the theory is such that the relatively small differences between these various rates are a source of much scientific information on natural light
fields in the seas and lakes.

The whole theory of this model swings about the exact two-flow equations (1) in Fig. 8. A representative list of interrelations among the inherent and apparent optical properties yielded up by (1) in Fig. 8 is given in the remainder of the figure. Further discussion of these may be found in Reference 3.

The two-flow model allows a systematic classification of the optical properties of the sea. This is shown in Fig. 9. Within the context of the two-flow equations (1) of Fig. 8, the two main problems of Hydrologic Optics have been completely solved in both the local and global approaches to them.

A Simple Radiance Model

Finally I should like to describe a simple model for the radiance field which I have found quite useful in resolving many practical questions about underwater visibility and light field predictions. This is shown in Fig. 10. \( N_r(z,\theta,\phi) \) is the observed apparent radiance of an object at a distance of \( r \) meters from the observer where the observer is at depth \( z \) and where the line of sight is fixed in direction by \( \theta, \phi \). The model is built on the assumption that the path function \( N_\ast \) (which gives radiance generated by scattering per unit path length) varies with depth \( z \) as shown. When this is used in the integral form of the equation of transfer, shown in the figure, the model is forthcoming after a simple integration. This model is to the practicing optical oceanographer as Ohm's law is to an electrical engineer: a simple law useful for
everyday work which can be pushed to fantastic limits before its fidelity is strained. Fig. 11 compares its predictions (solid curves) with measured radiances (circles) in three extreme directions under sunny conditions in a typical lake environment (reference 5). More detail on this model may be found in reference 4.
References


Captions for Figures

Fig. 1  Problems of Geophysical Optics, that is, radiative transfer applied to geophysical settings.

Fig. 2  Local Approach to radiative transfer problems.

Fig. 3  Global Approach to radiative transfer problems.

Fig. 4  Special Case of Global Approach: plane-parallel atmosphere.

Fig. 5  (no caption)

Fig. 6  (no caption)

Fig. 7  (no caption)

Fig. 8  (no caption)

Fig. 9  (no caption)

Fig. 10 A Simple Radiance Model.

Fig. 11 Experimental Verification of Simple Radiance Model.
Figure 1
\[
\frac{dN(p,\xi)}{dr} = -\alpha(p)N(p,\xi) + \int N(p,\xi')\sigma(p,\xi',\xi)d\Omega(\xi') + N_\eta(p,\xi)
\]

- Rate of Change
- Rate of Loss
- Rate of Gain

Figure 2
\[ N_+ (b) = N(c) S_A(a, b) + \int_B N_-(b') S_C(b', b) \]

\[ N_- (b) = N(a) S_A(a, b') + \int_B N_+(b') S_A(b', b) \]
\[ \tau = 0 \]

\[ \tau = \tau_1 \]

(1) \( N_-(\tau_1) = N^0 T(0, \tau_1) \quad ; \quad N_+(0) = N^0 R(0, \tau_1) \)

(2) \( N_+(\tau) = N_+(\tau_1) T(\tau_1, \tau) + N_-(\tau) R(\tau, \tau_1) \)

(3) \( N_-(\tau) = N_-(0) T(0, \tau) + N_+(\tau) R(\tau, 0) \)

(4) \( N_+(\tau_1) = N_-(\tau_1) r_2 \)

(5) \( N_-(0) = N_+(0) r_1 + N^0 \)

Figure 4
TWO-FLOW ANALYSIS OF LIGHT FIELD

Shielded Spherical Collectors
\[ h(Z, -) \quad h(Z, +) \]

Plane Collectors
\[ H(Z, -) \quad H(Z, +) \]

Radiance Meter
\[ N(Z, \xi) \]

Figure 5
OPERATIONAL DEFINITIONS OF INHERENT OPTICAL PROPERTIES

\[
\frac{dN(Z,\xi)}{dr} = \xi \cdot \nabla N(Z,\xi) = -\alpha(Z)N(Z,\xi) + N_*(Z,\xi)
\]  
(1)

\[
\alpha(Z) = \frac{N_*(Z,\xi)}{N(Z,\xi)} - \frac{1}{N(Z,\xi)} \frac{dN(Z,\xi)}{dr}
\]  
(2)

\[
\sigma(Z',\xi',\xi) = \frac{\Delta N_*(Z,\xi)}{N_0(Z,\xi')\Delta \Omega_0}
\]  
(3)

\[
S(Z) = \frac{h_*(Z)}{h(Z)}
\]  
(4)

\[
a(Z) = \frac{-1}{h(Z)} \frac{d\bar{H}(Z,+)}{dZ}
\]  
(5)

\[
N_*(Z,\xi) = \int_{\Omega} N(Z,\xi') \sigma(Z,\xi';\xi) d\Omega(\xi')
\]  
(6)

\[
h(Z) = \int_{\Omega} N(Z,\xi) d\Omega(\xi)
\]  
(7)

\[
h_*(Z) = \int_{\Omega} N_*(Z,\xi) d\Omega(\xi)
\]  
(8)

\[
H(Z,\pm) = \int_{\Omega_{\pm}} |\xi \cdot n| N(Z,\xi) d\Omega(\xi)
\]  
(9)

\[
\bar{H}(Z,+) = H(Z,+) - H(Z,-)
\]  
(10)
OPERATIONAL DEFINITIONS OF APPARENT OPTICAL PROPERTIES

\[
R(Z, \pm) = \frac{H(Z, \mp)}{H(Z, \pm)} \quad (1)
\]

\[
K(Z, \xi) = \frac{-1}{N(Z, \xi)} \frac{dN(Z, \xi)}{dZ} \quad (2)
\]

\[
K(Z, \pm) = \frac{-1}{H(Z, \pm)} \frac{dH(Z, \pm)}{dZ} \quad (3)
\]

\[
k(Z, \pm) = \frac{-1}{h(Z, \pm)} \frac{dh(Z, \pm)}{dZ} \quad (4)
\]

\[
k(Z) = \frac{-1}{h(Z)} \frac{dh(Z)}{dZ} \quad (5)
\]

\[
D(Z, \pm) = \frac{h(Z, \pm)}{H(Z, \pm)} \quad (6)
\]

HYBRID PROPERTIES

\[
\alpha(Z, \pm) = \alpha(Z) D(Z, \pm) \quad (7)
\]

\[
s(Z, \pm) = s(Z) D(Z, \pm) \quad (8)
\]

\[
a(Z, \pm) = a(Z) D(Z, \pm) \quad (9)
\]

\[
f(Z, \pm) = \frac{1}{H(Z, \pm)} \int_{H_{\pm}} \left[ \int_{H_{\pm}} N(Z, \xi', \xi) \sigma(Z, \xi', \xi) d\Omega(\xi') \right] d\Omega(\xi) \quad (10)
\]

\[
b(Z, \pm) = \frac{1}{H(Z, \pm)} \int_{H_{\pm}} \left[ \int_{H_{\pm}} N(Z, \xi', \xi) \sigma(Z, \xi', \xi) d\Omega(\xi') \right] d\Omega(\xi) \quad (11)
\]
SOME THEORETICAL RELATIONS AMONG
THE VARIOUS OPTICAL PROPERTIES

\[ \mp \frac{dH(Z, \pm)}{dZ} = - \left[ a(Z, \pm) + b(Z, \pm) \right] H(Z, \pm) + b(Z, \mp) H(Z, \mp) \]  \hspace{1cm} (1)

\[ R(Z, -) = \frac{K(Z, -) - a(Z, -)}{K(Z, +) + a(Z, +)} \]  \hspace{1cm} (2)

\[ l = \frac{b(Z, -)}{K(Z, -) - a(Z, -)} - \frac{b(Z, +)}{K(Z, +) + a(Z, +)} \]  \hspace{1cm} (3)

\[ a(Z) \leq \frac{K(Z, -)}{D(Z, -)} \leq a(Z) \]  \hspace{1cm} (4)

\[ \frac{dH(Z, -)}{dZ} \leq \frac{dH(Z, +)}{dZ} \]  \hspace{1cm} (5)

\[ \frac{dR(Z, -)}{dZ} = R(Z, -) \left[ K(Z, -) - K(Z, +) \right] \]  \hspace{1cm} (6)

\[ K(Z, -) = \left[ a(Z) - \frac{h(Z, -)}{h(Z, -)} \right] D(Z, -) \]  \hspace{1cm} (7)

Figure 8
CLASSIFICATION OF OPTICAL PROPERTIES
OF THE SEA

\[ N(Z, \xi) \]

**INHERENT PROPERTIES**
- \( \alpha(Z) \)
- \( \sigma(Z, \xi, \xi') \)
- \( \alpha(Z) \)

**APPARENT PROPERTIES**
- \( R(Z, \pm) \)
- \( K(Z, \xi) \)
- \( D(Z, \pm) \)

**HYBRID PROPERTIES**
- \( f(Z) \)
- \( b(Z) \)
- \( k(Z, \pm) \)
- \( K(Z, \pm) \)
- \( s(Z) \)
- \( k(Z) \)

Figure 9
\[ N_r(z, \theta, \phi) = N_0(z_t, \theta, \phi) e^{-ar} + \frac{N_*(z, \theta, \phi)}{(a+k \cos \theta)} \left[ 1 - e^{-(a+k \cos \theta)r} \right] \]

\[ N_*(z, \theta, \phi) = N_*(0, \theta, \phi) e^{-kz} \]

\[ N_r = N_0 e^{-ar} + \int_0^r e^{a(r'-r)} N_* \, dr' \]

Figure 10
DISCUSSION

E. U. Condon: What are the radiative properties of water? Do they change with the location?

R. W. Preisendorfer: In the wave length of interest to us (around 4800 Å) a typical attenuation length \( \frac{1}{K} \) is of the order of the 28 meters for clear "blue" ocean water. Forty per cent of this is due to absorption and 60 per cent is scattering. In all other spectral regions, absorption is much larger and overwhelmingly predominant in very pure water. A very complete presentation of such properties was given by S. Q. Duntley in the Journal of the Optical Society of America, 53, No. 2, pp. 214–233, February 1963. The variable contamination in different locations can reduce this length by orders of magnitude.

J. Dave: I am quite interested by the similarities between your theories and results and the ones we use in atmospheric radiation transfer. In particular, your Fig. 11 exhibits a maximum upward intensity at a certain depth, when the sun is not at the zenith. This is an effect we also obtain.

R. W. Preisendorfer: The features of the problem, as long as its anisotropic scattering, will be qualitatively the same. This shift of the maximum intensity toward these lower optical depths is observed. The simple radiation model that I have discussed predicts the maximum point of that curve, by just setting a derivative of the radiance as a function of depth equal to zero and solving.

Also, we have measured at great depths the radiance distribution, which is normally a function of depth. As you go deeper into the water,
the shape of the radiance distribution becomes independent of the incoming radiance pattern; it depends only on the volume scattering function $\sigma$ of that particular water. This is the asymptotic radiance theorem; there are about four proofs of it.

**E. U. Condon:** I must imagine that you get big effects connected with how rough the sea is?

**R. W. Preisendorfer:** Yes, we do. We have been able to do some theoretical interpretation of this, on a statistical basis. There is an article on this by C. Cox and W. Munk of La Jolla, "Some Problems in Optical Oceanography" in the Sears Foundation Journal of Marine Research, 1955, pp. 63-78.
Following a power failure on the University Campus on the second afternoon of the Workshop, we were unable to record the last presentation by Dr. Viskanta. Fortunately, Dr. Viskanta agreed to send us a summary of his talk; similarly, Dr. Condon accepted to give us the substance of the contribution he made during the discussion following Dr. Viskanta's presentation. These two documents follow.

R.G.
HEAT TRANSFER BY SIMULTANEOUS CONDUCTION 
AND RADIATION IN TRANSPARENT MEDIA 

by 

R. Viskanta 
School of Mechanical Engineering 
Purdue University 

Introduction 

At elevated temperatures radiation is an important or the only 
mode of energy transfer. Heat transfer by radiation alone or by simultaneous 
conduction and radiation is of considerable practical importance, for example, 
in glass, oxides, refractories, ceramics and cermets, porous and fibrous 
materials and many others. Interest in energy transfer in media semi-
transparent to visible and infrared radiation has been motivated by technolo-
logical progress which has increased the demand for engineering systems 
capable of withstanding higher and higher temperatures. By transparent media 
we refer here to: 1) homogeneous solids such as glass in which absorption 
of radiation predominates, and 2) porous materials such as particles and 
fibers dispersed in non-radiating gases in which scattering or radiation 
predominates. 

Heat transfer in glass, for example, has probably received most 
attention. Notable contributions toward gaining a more complete understanding 
of energy transfer in semi-transparent materials has been made by glass 
technologists. Genzel, Czerny, Geffcken and others in Germany, and Gardon 
in this country have contributed substantially to this end. An extensive 
review of the literature is given in a paper by Gardon.¹
Before proceeding to discuss some specific problems it is desirable to point out what type of information is needed, for example, by an engineer. The engineer is interested in predicting: 1) the temperature distribution in the material as a function of position of time, and 2) either the radiative or the conductive plus radiative energy fluxes at the boundary of the system. There are, however, important differences between problems encountered in astrophysics and engineering. First, the boundaries in astrophysical problems are generally absent, but they are an essential part of any engineering system. Second, the parallel layer or the semi-infinite slab is the most commonly encountered geometry in astrophysics, while in engineering other geometries such as cylindrical or spherical are quite common.

**Mathematical Formulation of the Problem**

The general approach of the solution of the heat transfer problems can be most easily demonstrated by considering the one-dimensional slab of finite thickness. In order to reduce the mathematical complexity, the following assumptions are made:

1. The medium is isotropic and homogeneous,
2. The medium is in local thermodynamic equilibrium,
3. The heat transfer is steady and one-dimensional,
4. The scattering is isotropic and incoherent,
5. The radiation is unpolarized, and
6. The faces of the slab are bounded by opaque surfaces.

The mathematical details can be found elsewhere$^{2,3}$ and will be omitted here. The basic equations can be written as:
Conservation equation of (total) energy:

\[ \frac{d}{dy} \left( k \frac{dT}{dy} \right) = \frac{dF}{dy} = 4 \pi \int_0^\infty \kappa_\nu \left[ n^2_\nu B_\nu (T) - J_\nu (y) \right] dv \]  \hspace{1cm} (1)

Radiation flux:

\[ F = 2 \pi \int_0^1 I_{1\nu} (\mu) \exp \left( -\tau / \mu \right) d\mu - \int_0^1 I_{2\nu} (\mu) \exp \left[ - (\tau_0 - \tau) / \mu \right] d\mu \]

\[ + \int_0^\tau S(t) E_2 (\tau - t) dt - \int_0^{\tau_0} S_\nu (t) E_2 (t - \tau) dt \]  \hspace{1cm} (2)

Average intensity:

\[ J_\nu = \frac{1}{4 \pi} \int_{\Omega = 4 \pi} I_\nu d\Omega = \frac{1}{2} \left[ \int_0^1 I_{1\nu} (\mu) \exp \left( -\tau / \mu \right) d\mu \right. \]

\[ \hspace{1cm} + \int_0^1 I_{2\nu} (\mu) \exp \left[ - (\tau_0 - \tau) / \mu \right] d\mu + \int \cdots \]  \hspace{1cm} (3)

Source function:

\[ S_\nu (t) = (1 - \omega_0) n^2_\nu B_\nu (T) + \omega_0 J_\nu \]  \hspace{1cm} (4)

In these equations, \( k \) is the thermal conductivity, \( n_\nu \) is the spectral index of refraction, \( \tau_0 \) is the optical thickness of the slab, and the rest of the symbols are standard in the astrophysics literature.

The two intensities \( I_{1\nu} (\mu) \) and \( I_{2\nu} (\mu) \) leaving respectively the boundaries 1 and 2 of the slab depend on the type of surfaces enclosing the medium. We shall discuss here the two limiting cases:

1. The transparent solid is faced with an opaque material having diffuse emission and reflection characteristics, i.e., the emission and reflection of radiation from the boundaries does not depend on direction, and
2. The transparent solid is faced with an opaque material having specular reflection and arbitrary emission characteristics. Thus for diffuse boundaries, we have

\[
I_{1\nu} = \varepsilon_{1\nu} n^2 \nu B_{\nu}(T_1) + 2\rho_{1\nu} \left[ I_{2\nu} E_3(\tau_0) + \int_0^{\tau_0} S_{\nu}(\tau)E_2(\tau) \, d\tau \right] 
\]

(5)

\[
I_{2\nu} = \varepsilon_{2\nu} n^2 \nu B_{\nu}(T_2) + 2\rho_{2\nu} \left[ I_{1\nu} E_3(\tau_0) + \int_0^{\tau_0} S_{\nu}(\tau)E_2(\tau_0 - \tau) \, d\tau \right] 
\]

(6)

and for specularly reflecting boundaries, we can write

\[
I_{1\nu}(\mu) = \varepsilon_{1\nu}(\mu) n^2 \nu B_{\nu}(T_1) + \rho_{1\nu}(\mu) \left[ I_{2\nu}(\mu) e^{-\tau_0/\mu} + \int_0^{\tau_0} S_{\nu}(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu} \right] 
\]

(7)

\[
I_{2\nu}(\mu) = \varepsilon_{2\nu}(\mu) n^2 \nu B_{\nu}(T_2) + \rho_{2\nu}(\mu) \left[ I_{1\nu}(\mu) e^{-\tau_0/\mu} + \int_0^{\tau_0} S_{\nu}(\tau) e^{-\tau_0 - \tau)/\mu} \frac{d\tau}{\mu} \right] 
\]

(8)

In the equations \(\varepsilon\) is the emittance and \(\rho\) is the reflectance of the surfaces bounding the medium. Solution of Eqs. (5) and (6) or of (7) and (8) would yield the desired intensities at the boundaries for the two cases. The boundary conditions for the temperature can be chosen of the form:

\[
T = T_1 \quad \text{at} \quad y = 0 
\]

\[
T = T_2 \quad \text{at} \quad y = L 
\]

(9)

Equations (1), (3) and (9) must be solved simultaneously to determine both \(T\) and \(J_{\nu}\). Once these have been found, the radiation and
conduction fluxes can be readily evaluated.

The general case of variable $\kappa_\nu$, $\omega_\nu$, $\varepsilon_\nu$ and $\rho_\nu$ introduces considerable computational difficulties. Further, the lack of reliable radiation property data and large number of independent parameters require additional simplifications. Generally it is assumed that the transparent media is gray, and the spectral absorption and scattering coefficients are replaced by appropriate mean values. The mean absorption coefficients introduced by the astrophysicists$^4$ are not appropriate$^5$, however, in heat transfer problems where conduction and bounding surfaces are of necessity present.

**Concluding Remarks**

A theoretical discourse such as this would not be complete without brief reference to some of the many points that remain to be cleared up. Although the basic phenomena involved is radiant heat transfer are well understood, mathematical formulation of the problem is quite complex, so that solutions have, as yet, been obtained for the gray case and only for the simplest geometries. More realistic problems could be formulated, including non-diffuse emission and reflection from bounding surfaces, and non-isotropic scattering if mathematical techniques were available for solving systems of non-linear integral and integro-differential equations. The presently available techniques for solving numerically Eqs. (1) and (3) are too time-consuming even on fast digital computers. Thus,
1. There is a need for simple but physically realistic models for formulating radiation transfer problems in order to reduce the mathematical complexity particularly for the non-gray case, and

2. From the viewpoint of the user of the information that has been generated even some of these "solutions" are too complicated, and search must continue for simpler representations to specific problems.
References


E. U. Condon: Perhaps it is of interest to add a few comments on the role of radiative heat transfer in practical glass-making. In industrial production, glass is melted in large tanks from 2 to 6 meters deep. Heat is supplied by radiation from gas or fuel oil flames above the surface of the glass. Heat is lost by conduction through the refractory blocks that make up the side walls and bottom of the tank.

The slow process here is the fining or gradual elimination of gas bubbles and the general homogenization of the melt which occurs as a result of slow convection currents in the tank. In a continuous melting tank, raw batch is put in at one end and molten glass is taken out at the other end through one or more forehearth, at such a rate that the average dwell-time of the glass in the tank is some 2 to 5 days, depending on the quality sought.

Radiative heat transfer due to the spectral component in the range \( \nu \) to \( \nu + d\nu \), depends on the density of radiation, \( n^3 \rho(\nu,T) d\nu \), in that range where \( \rho(\nu,T) \) is the Planck function and \( n \) is the refractive index, and on the mean distance \( A^{-1}(\nu) \) travelled before absorption (where \( A(\nu) \) is the absorption coefficient in reciprocal length units). The contribution to effective conductivity from this frequency band is

\[
\frac{1}{3} \left( \frac{c}{n} \right) A^{-1}(\nu) \cdot n^3 \rho(\nu,T) d\nu
\]

so the total effective radiative conductivity is

\[
k = \left( \frac{c}{3} \right) \int_0^\infty A^{-1}(\nu) n^2(\nu) \rho(\nu,T) d\nu.
\]

In glass melting we deal with temperatures of the order 1300 to 1600 \(^\circ\)K, at which the bulk of the radiation has a wave length in the neighborhood of 2.5 \( \mu \). \( k \) is a rapidly increasing function of temperature,
not only because $\rho(\nu, T)$ increases rapidly, but also because most glasses are essentially opaque for $\lambda > 3\mu$, and quite transparent for $\lambda < 3\mu$, and the fraction of the total radiation in this transparent region also increases with temperature. In practice $A^{-1}(\nu)$ is essentially zero for $\lambda > 3\mu$.

Probably the most sensational effect that confronts this glassmaker is the strong dependence of $A^{-1}(\nu)$ in the important frequency region on the state of oxidation of the small amount of iron impurity which is always present in commercial glass. The ferrous ion ($Fe^{++}$) has a strong absorption band at $\lambda = 1.1\mu$, which the ferric state.

This effect is so great that if the conditions in the factory are such that the iron content of the melt largely goes over into the ferrous state, the radiative heat transfer rate to the bottom layer of the tank is insufficient to compensate for its conductive losses, and the bottom part freezes up, in which case there is nothing that can be done but to shut down, let the tank cool off, mine out the glass with an air hammer, reload, and start over. A little matter of whether 0.1% Fe is ferrous instead of ferric can easily cost $100,000 or so if the situation gets this far out of control.

It is therefore essential to keep the iron in the ferric state in normal production. A further complication is the fact that the ferrous-ferric situation (rarely, an "equilibrium") is affected by the other polyvalent ions such as Sb and As, which are also present. The strong ferrous band at $1.1\mu$ is deliberately produced in the glasses that are intended for use as radiative heat filters, as for protection of workers in steel mills, or for absorbing the heat from the arc lamps in motion picture projectors to protect the film. For such glasses the melting technology must be quite different from that of ordinary ferric glasses because of
the great difference in their radiative transfer properties.

An interesting discussion of some of these points is to be found in Chap. 7 of W. A. Weyl, "Colorised Glasses" (Society of Glass Technology, Sheffield, 1951.)