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THE WILSON EFFECT AND SUNSPOT STRUCTURE

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ABSTRACT

A method is given for interpreting intensity profiles at different heliocentric angles in terms of the attenuation coefficient distribution in the spot.

Qualitative data demonstrating the Wilson effect in a particular spot is analyzed by a simpler method to show that the mean attenuation coefficient in the umbra of a spot is between one and two orders-of-magnitude less than that in the photosphere. A difference of one order-of-magnitude is shown to be consistent with the assumption that the umbra of the spot is in radiative equilibrium.

*First observed by Alexander Wilson at Glasgow in 1774.
1. Introduction

An adequate treatment of the detailed physical structure of a sunspot has long eluded the investigations of astrophysicists. At the recent Symposium on Sunspots in Florence (1964), it was clear that, despite many ingenious attempts, the problem is still without a satisfactory answer. Since the magnetic field in the neighborhood of a spot is widely thought to give rise to the spot, some methods of analysis take a model of the magnetic field as the starting point (see Menzel 1964; Chitre 1963 and others). For reasons discussed below, however, there are difficulties in relating such models to the direct observations.

Others, such as Michard (1953) and Mattig (1953), start with observations of the umbra-photosphere intensity ratio as a function of heliocentric angle $I_u(\theta)/I_{ps}(\theta)$. From these they derive the source function at various optical depths in the umbra using any of the methods for analysing photospheric limb-darkening, and hence, assuming L.T.E., the umbral temperature.

I have already discussed some of the dangers of using methods suitable to the mean photosphere in interpreting granulation data (Wilson 1964a). These arise since horizontal intensity fluctuations strongly indicate the presence of horizontal fluctuations in the absorption properties of the photosphere. The problem is thus one of radiative transfer through an inhomogeneous atmosphere and can only be discussed adequately using appropriate transfer theory (Wilson 1962).

Because of the order-of-magnitude intensity variations across sunspots, these comments apply even more strongly to sunspot analysis. Michard goes on to derive a model in which the umbra is much more transparent than the photosphere, while Mattig finds that the umbra is slightly less transparent (see Bray and Loughhead 1964 for a comparison of their methods). Michard's result casts grave doubts on temperature models derived without using transfer theory adequate to such an atmosphere. Further, the direct contradiction in the opacities derived
from these two models suggests that an independent determination of the absorption properties of a sunspot is fundamental to our understanding of its structure.

This difficulty also weakens the magnetic model approach for, if we are to distinguish between different magnetic models by direct observations, we must know the opacity of the umbra, at least to within an order of magnitude.

In 1958 Loughhead and Bray obtained a series of photos of a small but fairly regular spot as it moved across the solar disc from limb to limb. From these photographs, the reality of the "Wilson effect" in this particular spot is apparent. They were quick to point out that this provides qualitative support for a transparent umbra. They note also that observations of intensity profiles across a spot should, in principle, lead to a detailed knowledge of the attenuation coefficient within a spot, but mention that there are considerable difficulties, both observational and theoretical.

In this paper, I discuss the application to sunspot analysis of methods of handling radiative transfer in inhomogeneous atmospheres. In particular, I use a simplified approach to set an order of magnitude limit to the attenuation coefficient in the umbra of the sunspot which was shown in Loughhead and Bray's photographs. This limit is shown to be consistent with the umbra being in radiative equilibrium.
2. The Methods

The first method uses a solution of the transfer equation which is suitable for atmospheres whose absorbing properties may vary quite generally. Although a three-dimensional solution is available (Wilson 1963), the following two-dimensional form is quite adequate at present. If distributions are assumed for the total intensity \( J(r,z) \) (\( J = \int_{4\pi} \hat{J} d\Omega \), \( r \) is the radial coordinate and \( z \) the vertical coordinate, positive outwards) and \( \epsilon(r,z) \), representing the energy released per unit volume and solid angle from non-radiative sources, the corresponding attenuation coefficient distribution \( \kappa(r,z) \) is given by (Wilson 1962)

\[
\kappa(r,z) = -\frac{2}{3} r \int \left( r_0 \frac{\partial J(r_0,0)}{\partial r} - 8\pi \int_0^z r^1 \epsilon(r^1 z^1) \frac{\partial r^1}{\partial r} \, dz^1 \right)
\]

(1)

where \( (r_0,0) \) is the point at which the curve

\[
dr^1 / \frac{\partial J}{\partial r} = dz^1 / \frac{\partial J}{\partial z}
\]

through the point \((r,z)\) cuts the \( r \)-axis. With this solution for \( \kappa \) the directional intensity at the surface \( I(r_1,0) \) may be found from (see Wilson (1964(a)))

\[
I(r_1,0) = \int_0^\infty \exp \left\{ - \int_0^s \kappa(r,z) \, dt \right\} \kappa(r,z) S(r,z) \, ds,
\]

(2)

where \( r = r_1 - s \sin \Theta, z = -s \cos \Theta \), and the source function \( S(r,z) \) is given by \( S = \frac{\epsilon}{\kappa} + J/4\pi \).

If observed intensity profiles across a sunspot at various heliocentric angles are available, models for \( J(r,z) \) and \( \epsilon(r,z) \) may be tested by comparing calculated profiles with observation. In particular there are grounds for supposing
that the umbra of a sunspot is in radiative equilibrium. If this is the case
\( \varepsilon(0, z) = 0 \) and only a model for \( J(r, z) \) is to be determined.

Recently Loughhead and Gray have re-examined their 1958 photographs to see
if intensity profiles could be obtained by photometric methods. They report however
that, owing to the small size of this particular spot, any numerical measurements
would be of insufficient accuracy. Therefore, since the numerical details of
the above method are rather tedious, its detailed application should await the
appearance of suitable intensity profiles.
3. The Observational Data

Although suitable profiles are not available, Loughhead's and Bray's photographs do effectively demonstrate the reality of the Wilson effect for this particular spot. While this effect is conventionally described in terms of the relative widths of the penumbra on either side of the spot as it approaches the limb, it is not unreasonable to interpret this effect in terms of an asymmetry in the intensity profile of the spot at large heliocentric angles. Reference to the isophotal contours which they made of a large spot near the center of the disc (Bray and Loughhead 1962) indicates that in an idealized (circular) sunspot, the intensity profile at \( \theta = 0 \) from the photosphere to photosphere across the spot might well be as shown in Figure 1. If no Wilson effect is observed in a spot the profile at \( \theta = 75^\circ \) (say) might be as shown in Figure 1(a). The intensity minimum is of course lower and is shown displaced by parallax, but the profile would be symmetrical. Unless the Wilson effect is a localized property of the penumbral filaments, independent of the overall structure of the umbra-penumbra, such a spot could not exhibit a Wilson effect. Only if the intensity profile at large \( \theta \) has a marked asymmetry, as shown in 1(b), would we expect to find the effect in limb to limb photographs. If we may infer from the Lounghhead and Bray photographs that the intensity profiles for this spot are qualitatively as shown in 1(b) rather than 1(a) we may use a simplified approach to set limits to the order-of-magnitude of the absorption coefficient in the spot.
4. The Simplified Method

This method is based on one used in a recent investigation of photospheric granulation data (Wilson 1964(b)). A two-dimensional continuum model of the source function is postulated and, using the Planck function and the absorption properties of the negative hydrogen ion and the neutral hydrogen atom, a corresponding attenuation coefficient distribution is calculated. Hence using Equation (2) the directional intensity is found in the usual way.

It is relatively easy to set up a source function model which is suitable to a sunspot. Consider for example

\[ S(r, z) = 0.004946 e^{-0.01795z} + 0.446 
- (0.0049 e^{-qz} + 0.36) e^{-kr^2} \] (3)

As \( r \to \infty \) this model tends to the mean photospheric model (normalized by dividing by the directional intensity at the center of the disc) which was used in the granulation calculations. At \( r = 0, z = 0 \) the parameters have been chosen so that the umbra photosphere ratio is approximately 0.2. The parameter \( k \) determines the structure size and, for the sunspot in question which has an umbral radius of approximately 5 sec. of arc, \( k \) has the value 4 x 10^{-8}. This is determined by identifying the umbra-penumbra boundary with the point of inflection of \( e^{-kr^2} \). The parameter \( q \), which is still to be chosen, determines the extent of the inhomogeneity below the surface. Following the granulation calculations, the surface is arbitrarily taken to be at optical depth \( \tau = 0.03 \) and the same corrections are made for absorption above this optical depth (see Wilson (1964 (b))).

In determining the corresponding attenuation coefficient distribution it is obvious that at \( r \to \infty \) it should be identical with the mean photospheric attenuation coefficient as previously calculated. However the assumptions made in this calculation (L.T.E., hydrostatic equilibrium, and the constancy of gas pressure in
the horizontal directions), while not unreasonable in the mean photosphere, are much less reliable in the umbra of a sunspot. In fact, the object of this analysis is to determine the attenuation coefficient in the umbra independently of such assumptions since, as Mattig and Michard have demonstrated, slightly different assumptions can lead to very different umbral models.

Therefore a very simple model is chosen for the attenuation coefficient

\[ \kappa(rz) = \kappa_0(z)(1 - \alpha e^{-kr^2}) \]  

Here \( \kappa_0(z) \) corresponds to the mean photosphere and \( \alpha \) is the parameter which determines the umbra-photosphere ratio. The model requires that this ratio is independent of depth which is undoubtedly not the case. However, the data is so meager that a two parameter model is quite unjustified and this model is adequate to determine the order of magnitude of the ratio.
5. The Results

For various values of $\alpha$ and $\eta$, $I(r, \theta)$ has been evaluated by the numerical quadrature of the integral in Equation (2). This has been carried out in the Basser Computing Laboratory of the Department of Physics in the University of Sydney. For each value of $\alpha$, a value of $\eta$ is chosen so that $I(0,0)/I(\infty,0) = 0.24$ (approx.) which is a typical value of the umbra-photosphere intensity ratio given by various observers (although in larger spots it has recently been observed to be lower). Typical distributions at $\Theta = 0^\circ$, $60^\circ$, $76^\circ$ are shown in Figures 2-6 for $\alpha = -0.05, + 0.5, 0.8, 0.9, 0.99$. The asymmetries of the profiles at $\Theta = 75^\circ$ are measured roughly by plotting the median point of each distribution at various intensities. By drawing a straight line through these points and measuring its gradient $m$, an arbitrary measure of the asymmetry is obtained. In Figure 7, $1/m$ is plotted against $\alpha$.

It is clear from these distributions that for $\alpha = 0.05$ (corresponding to an umbra which is slightly more opaque than the photosphere) there is no observable asymmetry. Even for $\alpha = 0.5$ (the umbra is approximately twice as transparent as the photosphere) the asymmetry $1/m$ is hardly measurable. Thus Mattig's model, yielding a slightly more opaque umbra, is inconsistent with these observations. Only for $\alpha = 0.9$ is $1/m$ significantly different from zero, and for an asymmetry to be observed as clearly as in Loughhead's and Bray's photographs one might expect a value of $1/m$ between 0.2 and 0.3. Thus $\alpha$ should be in the range $0.9 < \alpha < 0.99$, and the umbra photosphere attenuation coefficient ratio for this spot is found to be of the order or less than 0.1.
6. Umbra–Photosphere Intensity Ratio

Another quantity which may be compared with observations, is the umbra–photosphere intensity ratio as a function of heliocentric angle. Bray and Loughhead (1964) point out that many observations, both in integrated light and at selected wavelengths, indicate that this ratio is nearly independent of $\theta$. This is in agreement with a calculation of Minnaert and Wanders (1932), which assumes that the umbra is in radiative equilibrium. In Figure 8, the ratio $I_u(\theta)/I_{ps}(\theta)$ derived from the above calculations is plotted against $\theta$ for the values of $\alpha$ shown. Observational points noted by deJager (1959) are also shown and it is clear that these are not of sufficient accuracy to distinguish between the models.
7. Radiative Equilibrium

Both the observations of umbra-photosphere intensity ratio as a function of $\theta$, and the expressed inhibition of photospheric convection by a strong magnetic field lend weight to the suggestion that the umbra of a spot is in radiative equilibrium. Since, in Equation (1) above, $\varepsilon$ effectively measures departures from radiative equilibrium, the method described in Section 2 is well suited to test this suggestion when observed profiles are available.

Even at this stage, however, we can make a rough test of the suggestion with the results derived above for the umbra photosphere ratio of attenuation coefficient $\kappa_u/\kappa_{ps}$. Although it is tedious to evaluate (1) over a range of values of $r$, it is easy to evaluate it at $r = 0$ for various $z$, particularly when, in radiative equilibrium, $\varepsilon = 0$ and $S = J/4\pi$. Thus $\kappa_u/\kappa_{ps}$ for radiative equilibrium can be compared with the order-of-magnitude result derived from the Wilson effect observations. This is shown in Table 1 in the case $\alpha = 0.9$ for which $q = 0.01778$, and $\alpha = 0.99$, $q = 0.01791$.

<table>
<thead>
<tr>
<th>$-z$(km)</th>
<th>0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.9$ i.e. $\kappa_u/\kappa_{ps} = 0.1$, $q = 0.01778$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_u/\kappa_{ps}$ (R.E.)</td>
<td>0.043, 0.076, 0.121, 0.208, 0.306, 0.331, 0.314</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 0.99$ i.e. $\kappa_u/\kappa_{ps} = 0.01$, $q = 0.01791$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_u/\kappa_{ps}$ (R.E.)</td>
<td>0.023, 0.038, 0.054, 0.084, 0.115, 0.119, 0.080</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

It is immediately clear that the ratio $\kappa_u/\kappa_{ps}$ (R.E.) is not independent of depth. It can be said however that, while the radiative equilibrium calculations are inconsistent with an average value $\kappa_u/\kappa_{ps} = 0.01$ (the second case), they are not inconsistent with the average value $\kappa_u/\kappa_{ps} = 0.1$. 
8. Conclusion

From these results it is tempting to conclude that $\frac{\kappa_u}{\kappa_{ps}}$ lies in a range, $0.05 < \frac{\kappa_u}{\kappa_{ps}} < 0.15$. However, in view of the qualitative nature of the observations and the uncertainty of the physical conditions, I feel that to specify figures at this stage is misleading.

On the basis of the Loughhead and Bray photgraphs we can conclude that, for this spot, the attenuation coefficient in the umbra is between one and two orders-of-magnitude less than that in the photosphere, and that one order-of-magnitude is not inconsistent with radiative equilibrium in the umbra.

Perhaps it has been of greater importance to demonstrate how intensity profiles across sunspots at different heliocentric angles can be used to analyze their structure, and thereby encourage observers in the admittedly difficult task of obtaining these profiles.

This work has been carried out partly at the University of Sydney and partly at the Joint Institute of Laboratory Astrophysics, Boulder, Colorado, where the author is the grateful recipient of a visiting fellowship.
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LEGENDS FOR FIGURES

Figure 1(a) Intensity profiles at \( \Theta = 0^\circ, 75^\circ \) across an idealized sunspot exhibiting no Wilson effect.

(b) Similar profiles for a spot showing a strong Wilson effect.

Figure 2 Calculated intensity profiles at \( \Theta = 0^\circ, 60^\circ, 75^\circ \) for the case \( \alpha = -0.05 \quad q = 0.0168 \).

Figure 3 As for 2 with \( \alpha = 0.5 \), \( q = 0.0174 \).

Figure 4 As for 2 with \( \alpha = 0.8 \), \( q = 0.01767 \).

Figure 5 As for 2 with \( \alpha = 0.9 \), \( q = 0.01778 \).

Figure 6 As for 2 with \( \alpha = 0.99 \), \( q = 0.01791 \).

Figure 7 Asymmetries at \( \Theta = 75^\circ \) measured by \( 1/m \) are plotted against \( \alpha \). The estimated uncertainties are shown.

Figure 8 Umbra photosphere intensity ratio plotted against \( \Theta \) for values of \( \alpha \) shown. Points \( \times \) are observations given by deJager.
Figure 4.

\[ i^* (r, \theta) \]

\[ r (\text{km}) \]

\( \theta = 0^\circ \)
\( \theta = 60^\circ \)
\( \theta = 75^\circ \)
Figure 7.
\frac{I_u(\theta)}{I_{ps}(\theta)}

\theta^\circ

Figure 8.