Widely tunable on-chip microwave circulator for superconducting quantum circuits

by

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This thesis develops theory for and experimentally demonstrates a new way to break Lorentz reciprocity—the symmetry, in an electrical network, under exchange of source and detector. The approach is based on the sequential application of frequency conversion and delay; as frequency and time are Fourier duals, these operations do not generally commute. We apply this method in the construction of an on-chip superconducting microwave circulator, a critical component for the unidirectional routing of quantum information in superconducting networks. The device requires neither permanent magnets nor microwave control tones, allowing on-chip integration with other superconducting circuits without expensive control hardware. Isolation in the device exceeds 20 dB over a bandwidth of tens of MHz, and its insertion loss is small, reaching as low as 0.9 dB at select operation frequencies. Furthermore, the device is linear with respect to input power for signal powers up to many hundreds of fW (10^3 circulating photons), and the direction of circulation can be dynamically reconfigured. We demonstrate its tunability with operation at a selection of frequencies between 4 and 6 GHz. Given the current status of quantum error-correction and architectures for quantum information processing with superconducting circuits, such scalable non-reciprocal devices will almost certainly be necessary for construction of a superconducting quantum computer intended to be more than a proof-of-principle.

Dedication

For my sister, Ellie, and my parents, Frankie and Bill.

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of others to offer help or advice, and the general sense of camaraderie make it a pleasure to work here.

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List of Symbols

The tables below catalogue the symbols used throughout this thesis. They are organized by the chapters in which the symbols are introduced (and not the chapters in which the symbols appear). For example, k_B is introduced as Boltzmann's constant in Chapter 2 and appears only in the table of symbols for Chapter 2, despite being used in later chapters as well. Although efforts were made to keep notation unambiguous, some symbols are used to represent different quantities in different chapters.

Chapter 1

T_l	transmission of the left mirror in Fig. 1.1
T_r	transmission of the right mirror in Fig. 1.1
a	annihilation operator for the cavity mode in Fig. 1.1
a_{in}	incoming field in Fig. 1.1
a_{out}	outgoing field in Fig. 1.1
	external coupling rate for the cavity mode in Fig. 1.1
ω_c	resonant frequency of the the cavity mode in Fig. 1.1

Chapter 2

k_B	Boltzmann's constant
T	temperature
$\hbar$	Planck's constant divide by 2
ω	frequency scale for a weakly anharmonic microwave resonator

ω_q	transmon qubit frequency (frequency separation between the ground and first excited states of the transmon)
ω_r	frequency of the microwave resonator coupled to the transmon qubit
a	annihilation operator for the microwave resonator coupled to the transmon qubit
$_z$	Pauli z operator for the transmon qubit
	frequency scale for the dispersive coupling between the transmon qubit and the microwave resonator

Chapter 3

$_s$	surface that bounds the network in Fig. 3.1
	mode index for the guided modes in Fig. 3.1
$E_{T;}$	component of the electric field tangential to the waveguide carrying the th guided mode
$H_{T;}$	component of the magnetic field tangential to the waveguide carrying the th guided mode
$e_{T;}$	component of the electric field tangential to the waveguide carrying the th guided mode with transverse dependence
$h_{T;}$	component of the magnetic field tangential to the waveguide carrying the th guided mode with transverse dependence
a	complex amplitude of the wave travelling into the network in the th waveguide
b	complex amplitude of the wave travelling out of the network in the th waveguide
	propagation constant for the th guided mode
	the Kronecker delta
P	the net power entering the network in Fig. 3.1

S	the scattering matrix
a^0	excitation state of a first hypothetical scattering process
a^{00}	excitation state of a second hypothetical scattering process
b^0	output state of a first hypothetical scattering process
b^{00}	output state of a second hypothetical scattering process
E^0	electric field of a first hypothetical scattering process
E^{00}	electric field of a second hypothetical scattering process
H^0	magnetic field of a first hypothetical scattering process
H^{00}	magnetic field of a second hypothetical scattering process
(H)	magnetic permeability tensor
(E)	electric permittivity tensor
$!$	frequency of the harmonic fields
A	a general linear system
B	a second general linear system
h_A	impulse response of A
h_B	impulse response of B
u	initial state of a linear system
y	intermediate state of a linear system
z	final state of a linear system
t, t^0, , ^{0},	various variables for time
h_{BA}	impulse response for the composite-system formed by the cascade of systems A and B
h_{AB}	impulse response for the composite-system formed by the cascade of systems B and A
$!_i$	resonant frequency of the i^{th} mode in a system of parametrically-coupled resonant modes

$_i$	linewidth of the ith mode in a system of parametrically-coupled resonant modes
H_I	interaction Hamiltonian for a system of parametrically coupled modes in the resolved sideband limit
	phase of the beamsplitter interaction between modes i and j
a_i	annihilation operator for the ith mode in a system of parametrically-coupled resonant modes

Chapter 4

! $_p$	frequency of the input microwave field
	frequency of bias signal for the multiplying elements
	phase of a multiplying element's bias signal
	duration of the delay provided by the delay element
l_+	inductance of one pair of inductors in a tunable inductive bridge
l	inductance of a second pair of inductors in a tunable inductive bridge
	imbalance parameter for a tunable inductive bridge
l_0	base inductance parameter for a tunable inductive bridge
c	capacitance of the capacitor in Fig. 4.2b
$_u$	uniform magnetic flux created by an off-chip magnetic coil, used to tune an inductive bridge realized with SQUIDs
$_g$	gradiometric magnetic flux created by an on-chip bias line, used to tune an inductive bridge realized with SQUIDs
Z_0	characteristic impedance of a transmission line
!	angular frequency of an input microwave tone
i	p $\overline{1}$
l	inductance of an N SQUID array threaded by magnetic flux
N	number of SQUIDs in an array with inductance l

	flux threading the SQUIDs in an array with inductance l
$'_0$	reduced flux quantum $\sim=(2e)$
I_0	Josephson junction critical current
	phase of the gradiometric bias flux
$_0$	amplitude of the sinusoidally varying imbalance parameter
	$_u = _0$
	$_g = _0$
$_0$	the flux quantum $2\ '_0 = h=(2e)$
J_n	the n^{th} Bessel function of the first kind
\	phase (of a complex number)
$!_0$	resonant frequency of the resonant delay
I_i	current through port i
$_i$	branch flux across port i
$I_{l;e}$	even current through the left port
$I_{l;o}$	odd current through the left port
$I_{r;e}$	even current through the right port
$I_{r;o}$	odd current through the right port
$_{l;e}$	even branch flux through the left port
$_{l;o}$	odd branch flux through the left port
$_{r;e}$	even branch flux through the right port
$_{r;o}$	odd branch flux through the right port
I_+	co-rotating current through ports q and p
I	counter-rotating current through ports q and p
$_+$	co-rotating branch flux through ports q and p
	counter-rotating branch flux through ports q and p
A, B, C, D	block matrices
Y	admittance matrix

S_o	odd scattering matrix
S_e	even scattering matrix
U	unitary (change-of-basis) matrix that transforms between numbered ports and left/right, even/odd ports

Chapter 5

$_r$	relative permittivity of the silicon wafer in the NIST Nb trilayer process
T_c	transition temperature of a superconductor
L	inductance of a superconducting loop
B_0	magnitude of a magnetic field
w	width of a microstrip
e	electron charge
Q	quality factor
l	length of a resistor parallel to the direction of current flow
$_c$	\clean" coherence length
l_n	mean free path
v_F	Fermi velocity
$_d$	\dirty" coherence length
I_n	critical current of an SNS tunnel junction
R_n	room-temperature resistance of an SNS junction
l	inelastic scattering length
E_J	Josephson energy of an SNS junction
R	low-temperature resistance of an SNS junction
	frequency of the on-chip gradiometric flux lines
I_{ind}	current induced in the microwave lines by the RF bias signals
M_A	mutual inductance between a bias line and a large circuit loop
M_a	mutual inductance between a bias line a SQUID loop

A	area of a large circuit loop
a	area of a SQUID loop
E	EMF around a large circuit loop
I_g	current amplitude in the on-chip bias line
p	participation ratio for the large circuit loop
Z_A	impedance of the large circuit loop
I_s	critical current of a SQUID
B_g	magnetic field created by the shielded, on-chip bias line
r	distance from the shielded bias line
	separation between the inner and outer lines in the shielded configuration
$_g^{edge}$	amplitude of gradiometic flux where the "edge" feature appears

Chapter 6

$\tilde{}_u$	effective uniform magnetic flux used for the predictions in Fig. 6.2
R	reflection coefficient
T	transmission coefficient
g	frequency splitting of the circulator's resonant modes
S_I^{AN}	sideband noise power caused by amplitude fluctuations of the gradiometric flux
S_I^{PN}	sideband noise power caused by phase fluctuations of the gradiometric flux
I_{1dB}	signal current in the device at its 1 dB compression point
S_I	Johnson noise current spectral density
n	photons of added noise

Chapter 7

Z_m	impedance transformer in Fig. 7.1
T	modulation period 2 =

Appendix A

I_i	current in the ith element of Fig. A.1
V_i	voltage in the ith element of Fig. A.1
V_b	vector of branch voltages
V_c	vector of chord voltages
I_b	vector of branch currents
I_c	vector of chord currents
r	resistance of the resistors in Fig. A.1
l_1	inductance of the first kind of inductors in Fig. A.1
l_2	inductance of the second kind of inductors in Fig. A.1
c	capacitance of the capacitors in Fig. A.1
F	matrix relating chord currents to branch currents through Kirchoff's current law
Z_b	matrix of branch impedances
Z_c	matrix of chord impedances
s	Laplace variable
s_i	ith eigenfrequency of the circuit

Appendix C

T_n	Noise temperature of the input lines

Appendix D

R_{bal}	measured reflection of the circulator when all bridges are balanced
G	gain of the reflection measurement chain
$_{bal}$	reflection coefficient of the circulator when all bridges are balanced
S_{bal}	scattering matrix of the circulator when all bridges are balanced

l	effective inductance of one arm of an inductive bridge in the circulator's balanced (unbiased) state
R_{op}	measured reflection during operation
$_{op}$	reflection coefficient during operation
	duration of the circulator's resonant delay
$_{d}$	electrical delay of the measurement chain

Chapter 1

Introduction

The idea to use quantum systems for computation was first proposed in the 1980s by notable scientists such as Feynman [1, 2] and Deutsch [3, 4]. The notion gained major traction in 1994 when Peter Shor showed how such a machine could efficiently factor composite integers [5]. (The RSA encryption scheme [6]|a widely utilized encryption method for commerce and other forms of sensitive communication|is founded on the difficulty of this task.) Factoring large composite integers is but one example from a list of difficult problems that could be solved by these efforts: others include efficient search algorithms [7], quantum chemistry and catalyst design [8, 9, 10], and communication whose security is guaranteed by physical laws [11]. For these reasons, efforts to create analog and digital quantum computers, quantum simulators, and quantum annealers span many realms of physics.

Across all of the proposed platforms for constructing a quantum computer, one of the key challenges is engineering a quantum system which is both well isolated from its environment (to protect it from dissipation and decoherence) and rapidly measurable (and controllable). This challenge is not unique to problems specific to quantum computing; rather, its roots extend into the theory of quantum measurements. According to Heisenberg, measurements of a system are perturbing, as illustrated famously (and controversially) by his microscope thought experiment [12].

The problem is also present in purely classical systems. Heuristically, oscillators that interact weakly with their environment respond slowly to control fields, and oscillators that respond quickly to control fields rapidly dissipate energy into their environment. This is no conspiracy: from the perspective of the oscillator, the experimentalist is a part of the environment.

To illustrate this issue more quantitatively, consider a system formed by a single electromagnetic mode in a Fabry-Perot cavity, where the left mirror is perfectly reflective (vanishing transmission) and the right mirror has a transmission T_r (Fig. 1.1). The quantum Langevin equations

Figure 1.1: A single electromagnetic mode in a Fabry-Perot cavity, characterized by the annihilation operator a. The left mirror is perfectly reflective, and the right mirror has transmission T_r, which couples the cavity mode to an incoming field a_{in} and an outgoing field a_{out}.

that describe the time-evolution of the cavity field a and the outgoing field a_{out} are [13]

$$\dot{a} = -i\omega_c a - \frac{\kappa}{2} a + \sqrt{\kappa}\, a_{in} \tag{1.1}$$

$$a_{out} = \sqrt{\kappa}\, a - a_{in}. \tag{1.2}$$

Here, ω_c is the frequency of the cavity mode, and κ is a rate determined by T_r which describes the coupling between the cavity and the incoming and outgoing fields a_{in} and a_{out}.

In this example, the cavity mode is the system of interest, which an experimentalist may hope to study via measurements of the outgoing field a_{out}. The prospects for that aspiration can be understood by examination of two limiting cases, in which the transmission of the right mirror is small or large.

When T_r approaches 0, the coupling rate κ also vanishes. This is beneficial from the perspective of isolating the system of interest from its environment: the dynamics of the cavity mode, given in Eq. (1.1), are now uncoupled from the external field a_{in}. Unfortunately, the experimentalist has also forfeited the ability to infer the state of the cavity from measurements of the outgoing field a_{out} (Eq. (1.2)).

In the opposite limit where T_r is large, may become non-negligible with respect to the cavity frequency ω_c. The outgoing field now contains considerable information about the state of the cavity mode. The cost of this encoding, however, is that the time-evolution of the cavity mode is now subject to the "slings and arrows" of the environment, in the form of the incoming field a_{in}.

The link between the environment's effect on the cavity mode (through a_{in}) and the cavity mode's radiation into the environment (through a_{out}) is an instance of the fluctuation-dissipation theorem [14, 15]. The right mirror's transmission creates dissipation in the cavity mode (which may be measured and recorded by an experimentalist) but also causes fluctuations in the mode, driven by the incoming field.

As an aside, we note that in this example, measurement of the cavity state requires that it decays into a propagating mode. This is not a general requirement for measurement of a quantum system. For example, quantum non-demolition measurements [16, 17] allow measurement without dissipation [18]. Measurements are not necessarily accompanied by dissipation; they are accompanied by disturbance.

Returning to the measurement of the cavity mode, the examined limiting cases of large and small suggest that our measurement dilemma cannot be resolved by optimization of the cavity's parameters. The solution is external, and involves directing the outgoing field a_{out} to the measurement apparatus, while ensuring that the incoming field a_{in} is supplied by a very cold bath. In a classical analysis, this strategy can protect the cavity mode from its environment, while still allowing an experimentalist to study the state of the mode. Quantum mechanically, vacuum fluctuations persist even at zero temperature. But the influence of these vacuum fluctuations on the cavity's dynamics are small compared to, for example, the black-body radiation of the measurement apparatus.

In the vacuum of free-space, however, Maxwell's equations make this kind of directional signal routing impossible. Practically, an interaction between electromagnetic waves and matter is required to separate signals based on their direction of propagation. Devices which do so are called non-reciprocal, in reference to their violation of Lorentz reciprocity [19]. In the context of quantum networks, non-reciprocal devices serve a variety of purposes beyond the fundamental one described above. For example, they enable protocols for the generation of remote entanglement [20, 21, 22] and proposals for the construction of rich quantum many-body systems [23] that rely on directional signal routing, and allow for the use of quantum-limited reflection amplifiers [24, 25]. Such applications impose a strict set of design constraints: for quantum measurements, the ideal non-reciprocal device has no loss and no added noise. These requirements make familiar non-reciprocal devices like diodes and op-amps unsuitable for use with quantum systems.

More broadly, enforcing the unidirectional flow of energy and information is a critical signal processing primitive in a variety of networks. A common use in telecommunication networks, for instance, is antennae duplexing, in which the incoming and outgoing signals from an antenna are separated, allowing for simultaneous transmission and reception. Given their utility, it should come as no surprise that commercial non-reciprocal devices have been developed across the electromagnetic spectrum, and are widely used in a variety of experimental contexts. For example, in experiments on quantum superconducting circuits—one of the most promising platforms for the development of a quantum computer—non-reciprocal devices are ubiquitous. State-of-the-art commercial non-reciprocal devices, however, are constructed with large permanent magnets and gyrotropic media, precluding their miniaturization for on-chip applications, as well as their construction with superconducting materials.

The optimal architecture for a quantum computer built with superconducting circuits is very much still an open research question. The surface code, though, is one of the leading contenders, and would require over a hundred million physical qubits to accomplish the task [26][1]. As each of these qubits must be simultaneously shielded from its environment and controlled in hardware, the need for a scalable non-reciprocal technology is pressing.

To address that need we have developed an alternative method for routing propagating electromagnetic fields with minimal loss or added noise, which uses no permanent magnets and can be

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[1] This estimate is for a computer that could execute a practical quantum computation—in this case, Shor's algorithm on an integer with 600 digits—in approximately one day.

integrated on-chip with superconducting circuits in a scalable way. In this thesis we describe the design, construction, and measurement of a non-reciprocal device which is based on this method.

Design goals for the circuit were set with a superconducting qubit application in mind, and in the context of modular quantum networks. But the device will also find use in other low-temperature microwave networks, especially ones which require high measurement efficiency. Examples include rapid single flux quantum logic [27]; microwave kinetic inductance detectors [28], transition edge sensors [29] and other astronomical detector arrays; dark-matter searches [30]; and tests of Lorentz invariance with high-quality microwave resonators [31].

The structure of the thesis is as follows: Ch. 2 gives a brief overview of quantum information processing, and efforts to process quantum information with superconducting circuits, broadly referred to as circuit quantum electrodynamics. The assumptions and consequences of the Lorentz reciprocity theorem are discussed in Ch. 3, along with other proposals and demonstrations for creating non-reciprocal devices. Ch. 4 describes the theory of operation for our device, and Ch. 5 describes its realization as a superconducting circuit, as well as relevant layout considerations. Experimental results are summarized in Ch. 6. Finally, conclusions and an outlook for the work are presented in Ch. 7.

Chapter 2

Quantum information processing

The enormous computational power of a quantum processor has attracted interest from a variety of subfields within physics and other disciplines. In the atomic domain, work is underway to store quantum bits of information (qubits) in the electronic states of trapped ions and Rydberg atoms. Notable solid-state platforms include nitrogen (and other) vacancy centers, in which information is encoded in the composite spin of the vacancy; semiconductor quantum dots, where the qubit is formed by the singlet state and the projection-less triplet state of two confined electron spins; and superconducting circuits, where information is stored in the ground or first excited state of an anharmonic microwave circuit. Many other platforms are also being investigated (optical, NMR, etc); Refs. [32, 33] provide a high-level overview of these approaches. Even within the experimental systems noted above, there exist a variety of ways to encode a bit of quantum information. [1] More exotic proposals also exist, in which the braiding of topological excitations with non-Abelian statistics provides the basis for computation [37]. While the listed platforms and physical realizations of a qubit are intended to be representative of the field's breadth, they are far from exhaustive.

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[1] For example, some researchers in the field of superconducting circuits use microwave resonators as the memory elements, and construct the computational basis $\{|0\rangle = |\alpha\rangle + |-\alpha\rangle; |1\rangle = |i\alpha\rangle + |-i\alpha\rangle\}$ with even superpositions of coherent states $|\alpha\rangle$—so-called cat codes [34], named for the cat states [35] that comprise them. Other resonator-based schemes also exist; Ref. [36] provides an overview and comparison of several prominent codes.

2.1 The road to quantum computation

Across all of these platforms, however, the ultimate goal of fault-tolerant quantum computation is shared. Fig. 2.1, from Ref. [38], illustrates one conception of the path to this goal.

Figure 2.1: One conception for the path to fault-tolerant quantum computation, divided into seven steps. The y-axis of the graphic illustrates the relative complexity of each stage—advancing along the path requires mastery of the previous steps. The x-axis, time, indicates that active efforts to improve at each stage are ongoing. After [38].

The first step involves operations (also known as gates) on single physical qubits. This is followed by algorithms (long sequences of gates) on multiple physical qubits. Completing these initial two steps requires satisfaction of the first five DiVincenzo criteria [39] (a list of five necessary conditions for constructing a quantum computer, plus two additional conditions for quantum communication). Following this is a more advanced stage, requiring quantum non-demolition (QND) measurements of qubits and advanced control. This stage enables, for example, QND measurements of error syndromes like parity, and the stabilization of arbitrary quantum states. Demonstrations of this third stage have been made with trapped-ions [40], Rydberg atoms [41], Nitrogen vacancy centers [42], and superconducting qubits [43, 44, 45].

The fourth step is the creation of a logical memory with a lifetime that exceeds the physical qubits which comprise it. This step, which entails encoding information redundantly among multiple physical qubits and actively correcting for errors in that encoding, is critical, as it allows the preservation of quantum information for timescales long enough to enable meaningful computation. Two groups have demonstrated this stage with superconducting circuits [46, 47].

The final two stages|operations on single logical qubits and algorithms on multiple logical qubits|mirror the initial two stages of the progression, but for logical qubits in place of physical qubits. One caveat for these steps: they must be completed without degrading the lifetime of the logical qubits. The first demonstrations of operations on single logical qubits [48, 49] were made this year with superconducting circuits. As of yet, no experimental platforms have demonstrated the sixth stage.

Naturally, the different experimental platforms have their specific affordances and drawbacks. The purpose of Fig. 2.1 is not to compare progress among subfields, but rather to emphasize the commonalities among the different approaches, highlight the progress that the field as a whole has made, and motivate superconducting circuits as a viable platform for quantum information processing. Indeed its promise as a subfield has attracted interest outside the academic realm: a variety of private enterprises (IBM, Google, Rigetti Computing, DWAVE Systems, etc.) are now actively engaged in the construction of a superconducting quantum computer.

2.2 Quantum error correction and hardware architectures

As efforts in industry and academic settings look forward and plan their ascent to the upper echelon's of Fig. (2.1), their hardware requirements will depend strongly on the architectures they employ, especially in the context of error correction. A detailed review of these proposals is beyond the scope of this thesis, but a brief overview is helpful in estimating the hardware resources required for a fault-tolerant quantum computer.

It is now widely accepted that error correction will occupy a majority of the resources of a fault-tolerant quantum computer [38]. One approach to error-correction involves the use of stabilizer codes [50, 51, 52], in which a logical qubit is formed by redundantly encoding information in a register of entangled physical qubits. The Steane code, for example, can be implemented with seven physical qubits in its register [51]. Under the assumption that error rates are sufficiently low such that at most one error occurs during each pass of the code, errors are detected by measuring collective properties of the register (the eponymous \stabilizer operations"). Additional gates can then be applied to correct the error, if needed.

The assumption that errors occur singly is, however, quite stringent. For the range of currently conceivable error rates, its satisfaction requires that the protocol be nested, such that each element in the error correcting register is itself a logical qubit, stabilized by its own register, which could itself be composed of logical qubits, and so on. As the stabilizer codes are also fairly operation intensive|that is, they drastically increase the number of gates required to implement an algorithm|stabilizer codes are recognized as a feasible but challenging way to perform error correction.

A more recent but related approach is known as the surface code, in which identical physical qubits are arranged in a rectangular array, which forms a surface [53, 26]. Connecting groups of four nearby qubits in a particular way allows rapid measurements of their parity to detect errors. The advantages of this approach are its flat hierarchy, which can be easily scaled once demonstrations of a unit cell are made, and its relatively relaxed requirements for single-qubit error rates (with respect to the stabilizer codes, for example). The drawbacks, however, are that a very large number of qubits (current estimates are on the order of 10^8 or 10^9 physical qubits for a modest quantum computation [26]) are required to reach a fault-tolerant level with the protocol, and that these benefits do not become substantial in systems with less than hundreds or thousands of physical qubits [38]. Nonetheless, efforts to demonstrate proofs-of-principle are already underway at Google [46] and IBM [54]. Ref. [55] gives a good overview and summary of the technical challenges relevant to this architecture.

A third strategy, which may be called the modular approach, is based on networks of nested modules, which rely heavily on hierarchy. In this scheme, information is stored in memory qubits which interact with the greater network via intermediate communication qubits. Computation is performed via manipulations of the communication qubits, which distributes entanglement across the network. In some sense, this strategy represents the antithesis of the surface code approach: its affordances are the capability to test and iterate individual modules and processes, and possibly a lower requirement of physical qubits, at the cost of increased architectural and process complexity.

Ultimately, as with the choice of experimental platforms, selection of the optimal architecture for fault-tolerant quantum computing is an open question. But while the precise structure is yet to be determined, it seems clear that the necessary hardware resources will be immense.

2.3 Circuit quantum electrodynamics

To provide a more concrete notion of these hardware needs, we specialize now to a discussion of quantum information processing with superconducting circuits. Detailed reviews on this subject may be found elsewhere [56, 57]. Here we provide only an informal introduction, intended to give context for the design of a scalable circulator for superconducting quantum circuits.

The qubits in this platform are usually formed by slightly anharmonic electromagnetic oscillators. At sufficiently low temperatures, $k_B T < \hbar\omega$, the quantized energy levels in these circuits are resolvable. Here, k_B is Boltzmann's constant, T is the temperature of the environment, $\hbar$ is Plank's constant over 2π, and ω is a frequency scale for the anharmonic oscillator. Modern dilution refrigerators allow for experiments at temperatures in the tens of mK, which makes the energy available from the thermal bath about ten times less than a photon at 5 GHz. Hence, the oscillators are designed to have resonant frequencies in the microwave range, typically between 1 and 20 GHz.

Like atoms, superconducting qubits come in a variety of forms. Most, though, can be modelled as a network of capacitors, inductors, and Josephson junctions. The Josephson junction [58] provides a dissipationless nonlinearity to the circuit, which breaks the harmonic spacing of its energy levels and allows individual transitions to be addressed with coherent microwave tones. A qubit can then be created by restricting attention to a pair of energy eigenstates (typically, the two lowest energy levels).

Readout of such a qubit, in the circuit quantum electrodynamics (cQED) scheme [59], is accomplished by dispersively coupling the qubit to a microwave cavity.[2] In the dispersive limit, the cavity's resonant frequency is dressed by the state of the qubit, and a transmission measurement of the cavity can therefore indicate the state of the qubit in a QND manner [60, 61].

For a transmon[3] in the dispersive limit, the effective Hamiltonian that governs the qubit-cavity system is given by [62]

$$\begin{aligned} H &= \hbar\omega_q\sigma_z + \hbar(\omega_r + \chi\sigma_z)\hat{a}^\dagger\hat{a}; \\ &= \hbar\left(\omega_q + \chi\hat{a}^\dagger\hat{a}\right)\sigma_z + \hbar\omega_r\hat{a}^\dagger\hat{a}. \end{aligned} \tag{2.1}$$

Here ω_q is the qubit frequency (the difference between the ground and first excited states of the transmon), ω_r is the resonator's frequency, $\hat{a}$ is the annihilation operator for the cavity, σ_z is the Pauli z operator for the qubit, and χ is the dispersive shift—a parameter that characterizes the coupling between the qubit and the cavity. In the first line of Eq. (2.1), the Hamiltonian is factored to make explicit the qubit-state-dependent dressing of the cavity's resonant frequency.

The same effect, however, also dresses the qubit frequency by an amount proportional to the photonic occupation in the cavity, as shown in the second line of Eq. (2.1). This allows the qubit to act as a photon counter/detector [22], and reveals the importance of controlling the photonic occupation in the cavity: a "hot" cavity, occupied by a high-temperature thermal state, will contain a broad distribution of Fock states, effectively broadening the qubit transition. This effect is so pronounced that superconducting qubits have become the most sensitive thermometers in these experiments. Measurable reductions in qubit coherence are observed with average cavity-photon occupations of 10^{-3} [63].[4]

In this light, one can see that the need to enforce the directional propagation of signals in cQED measurements arises directly from the Hamiltonian in Eq. (2.1). To probe the state of the qubit, electromagnetic waves departing the resonant cavity must be directed to a microwave

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[2] Typically a distributed resonator formed, for example, by a quarter-wave transmission line, or a literal (three-dimensional) metal cavity.

[3] One example, chosen from a selection of different superconducting qubits (see, for example, Sec. 2 in Ref. [57]).

[4] As an aside, it may be noted that in some implementations this scheme is inverted, and a microwave resonator serves as the memory element, while a qubit such as the transmon provides a dispersive nonlinearity that allows individual transitions in the resonator to be addressed [64]. This is advantageous as coherence times in microwave cavities currently exceed those in superconducting qubits [65].

receiver. At the same time, however, electromagnetic waves propagating toward the cavity should originate from a cold bath—in practice, a 50 Ohm resistor to ground, well-thermalized at the base of a dilution refrigerator—to avoid dephasing the qubit.

Experimentally, this uni-directional signal flow is enforced by non-reciprocal circuit elements known as circulators and isolators. Isolators are impedance-matched two port devices which transmit a signal incident on one of their ports, but absorb signals incident on their other port. Circulators are impedance-matched n-port devices with three or more ports ($n \geq 3$), which transmit signals incident on port $m \leq n$ to port $1 + (m \text{ modulo } n)$: they “circulate” an incident signal, directing it out of one of the two adjacent ports.

Fig. 2.2 depicts this process for an $n = 3$ port circulator. In Fig. 2.2a, a signal incident on the circulator's first port is transmitted to its second port. In Fig. 2.2b, a signal incident on port 2 is routed to port 3. Fig. 2.2c shows a signal incident on port 3 directed to port 1. A circulator may be configured as an isolator by terminating all but two adjacent ports in 50 Ohms.

Figure 2.2: Schematic displaying the way in which an $n = 3$ port counter-clockwise circulator (circle with blue counter-clockwise arrow) routes incident fields. (a) A field incident on the circulator's first port is transmitted out the second port. (b) A field incident on the second port is routed out the third port. (c) A field incident on the third port is routed out the first port.

To illustrate how circulators and isolators are utilized in the cQED architecture, Fig. 2.3 shows a simplified experimental schematic for readout of a superconducting qubit coupled to a resonant cavity. To reduce the thermal population of states in the qubit-cavity system, the devices are mounted at the base of a dilution refrigerator. In the readout procedure, a signal generator at room temperature first creates a microwave tone which propagates to the base of the refrigerator and impinges on the weakly-coupled port of the resonant cavity (depicted as a Fabry-Perot cavity in Fig. 2.3). The tone then traverses the cavity, and a portion of its power exits out the second, strongly-coupled cavity port. Here the tone is transmitted through an isolator and directed by a circulator to a “quantum” reflection amplifier, where the signal is reflected with gain of approximately 20 dB. Here, “quantum” is used to mean an amplifier with added noise approximately equal to half a photon [66]—for example, a Josephson parametric amplifier [24] or a Josephson parametric converter [25]. Next, the circulator routes the tone through a second isolator, is amplified a second time by a high-electron mobility transistor (HEMT) amplifier, and finally propagates up the fridge for further room-temperature amplification and digitization. In various schemes, the phase or amplitude of the transmitted tone can be made to encode the state of the qubit, via the interaction described in Eq. (2.1).

Figure 2.3: Simplified experimental schematic illustrating readout of a superconducting qubit in a dilution refrigerator. Signal filtering and attenuation, and control lines of the reflection amplifier, are omitted for clarity. The superconducting qubit (green cross with capacitive paddles) is situated inside a microwave resonant cavity (illustrated graphically by the green Fabry-Perot resonator) with two ports. A measurement of the transmission through the cavity can encode the state of the qubit. Circulators and isolators are used to direct the signal that exits the cavity out to a detector, while ensuring that the reverse-propagating signals originate from a cold-bath (and are not, for example, carrying the Johnson noise of the HEMT amplifier, or the strong pump tones in the reflection amplifier).

In this measurement, the signal routing capabilities of circulators are used in several ways, as evidenced by the multiple circulators employed in the readout of a single qubit. The first circulator, configured as an isolator, directs reflected signals and the strong pump tones powering a quantum reflection amplifier into a cold bath (serving as an entropy dump), while supplying the strongly coupled port of the qubit/cavity system with Johnson noise from this cold bath. The second circulator separates the incoming and outgoing signals for a quantum reflection amplifier, transforming that one-port device into a usable amplifier. A third circulator, again configured as an isolator, directs the Johnson noise emitted by the HEMT amplifier into an entropy dump, and replaces it with the Johnson noise from a cold bath.

It should be emphasized that the schematic in Fig. 2.3 illustrates the need for three circulators in a relatively simple single-qubit experiment. Experiments with multiple qubits, or additional requirements for directional signal routing, will employ more circulators. For example, a recent demonstration of the generation of remote entanglement between two superconducting qubit/cavities, using a third qubit/cavity as a detector, employed seven circulators [22]. Even with multiplexed readout and broadband (several GHz) circulators, a surface code quantum computer with modest capabilities [26] could still easily require millions of circulators.

Chapter 3

Lorentz reciprocity

First observed in optical systems, where it is referred to as Helmholtz reciprocity, Stokes-Helmholtz reciprocity, or the principle of reversibility, Lorentz reciprocity can be summarized colloquially as \if I can see you, you can see me" [67, 68]. In a more general sense, Lorentz reciprocity is the electromagnetic version of a symmetry common to a variety of physical systems, which implies that the response at point a due to an impulse at b is identical to the response at b due to an impulse at a. The prevalence of reciprocity theorems stems from the invariance of many systems under time-reversal [69].

As in the electromagnetic case, breaking reciprocity can be of fundamental scientific interest and great technological use. Efforts to generate non-reciprocity are therefore also underway in, for example, acoustical and mechanical systems [70, 71]. Connections between non-reciprocity and the one-way propagation of edge-states in topological systems have also been observed, and are driving research in topological acoustics [72, 73], metamaterials [74], and non-reciprocal devices based on the quantum Hall effect [75, 76, 77].

In the electromagnetic context, Lorentz reciprocity is essentially a statement about the time-reversibility of Maxwell's equations [19], which occurs under certain assumptions: in a framework where electromagnetic fields propagate in guided modes into and out of a bounded network at ports, Lorentz reciprocity implies that the scattering between a pair of ports is invariant upon exchange of the source port and the detection port [78, 79]. This statement can be proven and expressed succinctly in a scattering matrix formalism, which we now introduce, following Ref. [80].

3.1 The scattering matrix

Consider a bounded region of space, enclosed by the surface$_s$, which contains a linear electrical network with time-independent components (Fig. 3.1). Energy can be exchanged between the network and the environment (the region outside $_s$) only through lossless and reciprocal waveguides, which we refer to as ports. (We assume the surface is sufficiently large such that radiative modes may be neglected, and shaped such that each waveguide is normal to$_s$).

Figure 3.1: Model of an electrical network as a region of space bounded by the surface$_s$. Electromagnetic signals can propagate into and out of the network in guided modes, referred to as ports. After [80].

For each waveguide, we imagine a coordinate system with the z-axis oriented along the waveguide, and directed into $_s$. These waveguides carry guided modes which are eigensolutions of Maxwell's equations, propagating in the $+\hat{z}$ and $\hat{z}$ directions. The components of the electromagnetic fields tangential to $_s$ have the form

$$\begin{aligned} E_{T;}(x;y;z) &= a\ e^{i\ \ z} + b\ e^{i}\ \ e_{T;}(x;y); \\ H_{T;}(x;y;z) &= a\ e^{i\ \ z}\ \ b\ e^{i}\ \ h_{T;}(x;y): \end{aligned} \tag{3.1}$$

Here is the mode index, a and b are the complex amplitudes of the waves travelling into and out of the network, and is the propagation constant. The modes are normalized such that

$$\int_{s} (e \quad h)\, dA = 2 \quad ; \tag{3.2}$$

where indicates complex conjugation and is the Kronecker delta.

Eq. (3.2) has two interesting consequences. First, at the surface $_s$ the tangential components of the fields have the simple form

$$\begin{aligned} E_T &= \sum (a + b)\, e_{T;} \; ; \\ H_T &= \sum (a \quad b)\, h_{T;} \; ; \end{aligned} \tag{3.3}$$

which is useful in deriving one of the principal consequences of the Lorentz reciprocity theorem for networks.

Second, the power P carried into the network is just the difference of the wave power entering the surface and the wave power exiting the surface:

$$\begin{aligned} P &= \sum |a|^2 \quad |b|^2 \\ &= a^{\,t} a \quad b^{\,t} b : \end{aligned} \tag{3.4}$$

Here in the second line we've introduced a vector notation a and b for the complex amplitudes a and b , and superscript t indicates vector transposition.

Our assumption that $_s$ encloses a passive and linear network ensures that the vector b is completely determined by a. If this were not the case, the network could scatter an excitation a into distinct outputs b and b^0. The linearity of the network implies that a null excitation (0 = a a) could therefore scatter into b b^0. Eq. (3.4) shows that the power carried out of the network in this event is non-zero, in violation of the assumption that $_s$ encloses a passive network.

As the network is assumed to be linear, there is thus a unique matrix S which maps a to b:

$$Sa = b : \tag{3.5}$$

The matrix element S is the ratio of the outgoing field at port to the incident field at port . The matrix S is known as the scattering matrix, for its description of how the network bounded by $_s$ scatters incident fields.

3.2 The Lorentz reciprocity theorem

Consider two scattering processes, with excitation states a', a'', output states b', b'', and their corresponding fields E', E'' and H', H''. With no electromagnetic sources, the time-harmonic form of Maxwell's equations for the fields of the first scattering process is

$$\nabla \times E' = -i\omega\,\mu(H')H', \tag{3.6}$$

$$\nabla \times H' = i\omega\,\varepsilon(E')E'. \tag{3.7}$$

Here $\mu(H)$ and $\varepsilon(E)$ are the magnetic permeability and electrical permittivity tensors of the media in Ω_s. Taking the inner product $[\cdot]$ of Eq. (3.6) with H'' and Eq. (3.7) with E'' and summing them yields

$$H'' \cdot \nabla \times E' + E'' \cdot \nabla \times H' = i\omega\left[E'' \cdot \varepsilon(E')E' - H'' \cdot \mu(H')H'\right]. \tag{3.8}$$

Repeating this procedure with the second scattering process yields the same result, with interchanged primes. When the two equations are subtracted, one obtains

$$\nabla \cdot \left(E' \times H'' - E'' \times H'\right) = i\omega\left[E'' \cdot \varepsilon(E')E' - E' \cdot \varepsilon(E'')E'' - H'' \cdot \mu(H')H' + H' \cdot \mu(H'')H''\right]. \tag{3.9}$$

A vector identity for the divergence of the cross-product of two vectors was used to simplify the left-hand side of this expression. It may be further simplified to have a vanishing right-hand side, if the two following conditions are met:

(1) ε and μ are symmetric tensors: $\varepsilon = \varepsilon^t$ and $\mu = \mu^t$.

(2) ε and μ describe linear media.

To see this, consider one of the terms on the right-hand side:

$$\begin{aligned} E'' \quad (E')E' = E'' \quad E' & \\ = \; & E'' \quad E'^{\,t} \\ = \; & E'' \, E'^{\,t} \\ = \; & E'^{\,t} \, {}^{t}E'' \\ = \; & E' \quad E'' \\ = E' \quad & E'' \, E''. \end{aligned} \tag{3.10}$$

Here we've used the linearity of in the first line and the the fact that scalars are invariant under transposition in the second line. The second to last line follows from the symmetry of . A magnetic term can be similarly commuted, to obtain the Lorentz reciprocity theorem:

$$r \quad E' \quad H'' \quad E'' \quad H' = 0: \tag{3.11}$$

Eq. (3.11) gains a clear physical interpretation when it is expressed in the scattering matrix formalism. Integrating it over the volume enclosed by $_s$ and applying the divergence theorem yields

$$\int_{s} E' \quad H'' \quad E'' \quad H' \quad dA = 0: \tag{3.12}$$

When the modal expansions in Eq. (3.4) are substituted into Eq. (3.12), the minus sign ensures that the cross-terms add constructively, while the other terms add destructively:

$$\begin{aligned} 0 \;&=\; 2 \sum_{;} \; b' a'' \quad a' b'' \; \int_{s} e \quad h \;\; dA\,; \\ &=\; b'^{\,t} a'' \quad a'^{\,t} b''; \\ &=\; a'^{\,t} \;\; S^{t} \quad S \;\; a'' \end{aligned} \tag{3.13}$$

Here we've used the orthogonality condition in Eq. (3.2) and the definition of the scattering matrix in Eq. (3.5). As a' and a'' are arbitrary, the scattering matrix must be symmetric:

$$S = S^{t}: \tag{3.14}$$

This symmetry ensures that the scattering between a pair of ports is invariant under exchange of the source port and the detection port: $S = S$.

3.3 Methods for breaking Lorentz reciprocity

Concepts for non-reciprocal devices may be broadly divided into categories, based upon which assumption of the Lorentz reciprocity theorem they violate.

3.3.1 Magneto-optic devices

Virtually all the circulators and isolators currently used in cQED experiments are constructed with magneto-optic materials. A common example of this is the Faraday-effect [81], in which left and right circularly polarized fields propagating in a gyrotropic material (which is biased by a magnetic field parallel to the propagation direction) have different group velocities. Careful choice of the propagation length and or magnetic field strength therefore allows the polarization of the field to be rotated by the desired angle.

Critically, the polarization's sense of rotation depends only on the orientation of the magnetic field, and not on the direction of propagation. Fig. 3.2a and Fig. 3.2b illustrate this process. The effect, from the perspective of an observer receiving the propagating field, is therefore a propagation-direction-dependent rotation of the polarization. The observer's perspective is relevant here as this represents the perspective of a detector receiving the propagating field, and Lorentz reciprocity is an invariance upon exchange of source and detector.

In optical applications, this phenomenon is combined with a pair of polarizers to create an isolator. In the microwave domain, it is commonly used to couple signals (non-reciprocally) between orthogonally-polarized waveguides, realizing a circulator. Ref. [83] provides a detailed description of ferrite-junction circulators that operate via the Faraday effect.

The Faraday effect may be contrasted with a similar (and reciprocal) phenomenon in optically active materials [81], in which the chirality of the crystal structure rotates the polarization of the propagating field. This causes the polarization vector's sense of rotation to reverse when the

Figure 3.2: Faraday rotation and a seemingly similar but distinct phenomenon observed in optically active media. (a) An electromagnetic wave propagating from left to right through a Faraday-active medium in the presence of an applied on-axis magnetic field B . From the perspective of an observer (or a detector) receiving the field, the polarization of the wave rotates clockwise by an angle . (b) If the signal is reflected and propagates back through the medium (from right to left), the observer perceives a counter-clockwise rotation of the wave's polarization by the same angle. Transmission is non-reciprocal. (c) A wave propagating left to right through an optically active medium, where a material chirality (depicted as a helix) determines the sense of rotation. An observer receiving the wave perceives a clockwise rotation of the polarization. (d) Reflection of the wave and right to left propagation results in the same clockwise rotation of the polarization, and reciprocal transmission. After [82].

propagation direction is inverted, in the same way that pushing or pulling on a right-handed screw will induce clockwise or counter-clockwise rotation. From the perspective of an observer receiving the propagating field, however, the rotation-direction is independent of the propagation direction, as depicted in Fig. 3.2c and Fig. 3.2d.

Magneto-optic devices are unconstrained by reciprocity because the permeability tensor is not symmetric in the presence of a static magnetic field. This leads to a non-vanishing right-hand side of Eq. (3.9), and a breakdown of the reciprocity theorem.

With respect to the design needs of superconducting qubit applications, magneto-optic non-reciprocal devices are attractive in that they may have broad operation bandwidths, low (but not vanishing) insertion loss on the order of several tenths of a dB, and they are passive devices. Their drawbacks include their roughly 10 cubic cm volumes, and their reliance on large permanent magnets, which makes them difficult to miniaturize (despite many efforts [84, 85, 86, 87, 88, 89]) and integrate with superconducting circuits, and precludes the elimination of their insertion loss by replacing their metallic components with superconductors.

3.3.2 Nonlinear devices

Nonlinear devices can break reciprocity because the intensity-dependence of their material properties prevents the cancellation of terms like

$$E'' \quad (E')E' \quad E' \quad (E'')E''. \tag{3.15}$$

As in the case of magneto-optic devices, this leads to a non-vanishing right-hand side of Eq. (3.9).

In practice, this is leveraged by introducing some kind of spatial asymmetry into the design of the device. A simple example of this is a finite-length rectangular waveguide which is filled by distinct dielectrics on its two ends [90]. The device is reciprocal, but careful choice of the dielectrics and the relative fractions of the waveguide that they fill makes the spatial profile of the electric field dependent on the propagation direction of an incident signal. By positioning a nonlinear resonator at a location within the waveguide where the intensity of the electric field is strong when a signal is incident on one port of the waveguide but weak when a signal is incident on its other port, a propagation-direction-dependent loss-channel can be introduced to the system—usually by arranging the frequency-pulling of the nonlinear resonator to move the resonator in or out of resonance with the incident signal. Different implementations of this general scheme have resulted in a variety of realizations for optical diodes constructed with silicon [91, 92, 93, 90].

The advantages of this approach include ease of miniaturization, and compatibility with superconducting circuits and fabrication methods. The disadvantages are an inherent restriction on linearity (such schemes only work when higher-power incident fields or pump tones activate a material nonlinearity), and typically narrow operation bandwidths.

3.3.3 Active devices

Given the intimate connection between reciprocity and time-reversal, it seems intuitive that active devices—that is, devices with parameters that vary in time—could break reciprocity. This can be seen explicitly with a simple example of two general cascaded linear systems A and B, which we describe following Ref. [94], and depict in Fig. 3.3. The impulse responses of A and B are denoted h_A and h_B, and u, z, and y describe the input, intermediate, and output states of the system.

Figure 3.3: A cascade of two linear systems. The impulse response of the composite system is non-reciprocal when the impulse responses of the individual systems are time-dependent. After [94].

The output y can be written in terms of the input u and the impulse responses h_A and h_B:

$$y(t) = \int_{-\infty}^{\infty} h_B(t;\tau)z(\tau)d\tau \tag{3.16}$$

$$z(t) = \int_{-\infty}^{\infty} h_A(t;\tau)u(\tau)d\tau\,. \tag{3.17}$$

Eliminating z in Eq. (3.16) yields

$$y(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h_B(t;\tau')h_A(\tau';\tau)d\tau'\right]u(\tau)d\tau\,; \tag{3.18}$$

and the bracketed term in that expression may be identified as the impulse response h_{BA} of the composite system BA.

When A and B are time-invariant systems, their impulse responses depend only on the difference of their arguments,

$$h_{BA} = \int_{-\infty}^{\infty} h_B(t-\tau')h_A(\tau'-\tau)d\tau'\,. \tag{3.19}$$

Changing integration variables to $t' = t -$ [illegible] and $\tau' =$ [illegible], Eq. (3.19) becomes

$$h_{BA} = \int_{-\infty}^{\infty} h_B(t' - \tau') h_A(\tau') d\tau'. \tag{3.20}$$

which is a convolution. In this form (or from the convolution theorem), it is clear that the system is reciprocal: $h_{BA} = h_{AB}$. For the time-dependent case, though, h_{BA} is not in general equal to h_{AB}.

The dependence of reciprocity on time-invariance is reflected in the arguments of Sec. 3.1 and Sec. 3.2, which respectively assume passive networks and harmonic fields. Efforts in the 1960s extended these arguments to time-dependent systems, and showed that reciprocity also places symmetry and time-invariance constraints on the scattering matrices of time-variable networks [95]. Since then, a variety of schemes have been developed for generating non-reciprocity with actively modulated circuits. We do not attempt a complete catalogue of them here, and instead limit discussion to two particular approaches.

3.3.3.1 Parametric coupling of resonant modes in the resolved sideband limit

The first of these involves the parametric coupling of resonant modes in the resolved sideband limit, which is a widely used strategy in radio frequency [96, 97], superconducting microwave [82, 98, 99, 100, 101], electroacoustic [102], electromechanical [103, 104, 105], and optomechanical [106, 107] devices. Excepting the circulator discussed in this thesis, most state-of-the-art non-reciprocal circuits for superconducting circuits are implemented with this approach.

An excellent conceptual description of this strategy is given in Ref. [108], which provides a graph-theoretical description for the Langevin equations of parametrically coupled resonant systems. Rather than reproduce that treatment here, we attempt a succinct and general summary of the concept.

Consider a system of two or more resonant modes (frequencies $\{\omega_i\}$ and linewidths $\{\gamma_i\}$) in the resolved sideband limit. By this, we mean the frequencies in the set $\{\omega_i\}$ are separated by many times their respective linewidths, and that the linewidths of each resonator are narrower than all the resonant frequencies. Now imagine a parametric interaction, which allows variation of a system parameter to shift the frequencies of the resonant modes $\{\omega_i\}$ by approximately their linewidths $\Delta\omega_i \approx \kappa_i$. Varying this parameter at the difference frequency $\omega_i - \omega_j$ between a pair of modes induces a beam-splitter interaction, allowing excitations to be exchanged between the two resonant systems:

$$H_I \propto e^{i\phi} a_i a_j^\dagger + e^{-i\phi} a_i^\dagger a_j. \tag{3.21}$$

Here, a_i is the annihilation operator of the i^{th} resonant mode, and ϕ is the phase of the parametric interaction. In practice, this phase can be adjusted by tuning the phase of the electrical signal which is controlling the parametric interaction.

Examination of Eq. (3.21) shows that the phase of the beamsplitter interaction depends on the direction in which excitations are exchanged. For example, if the phase $\phi = \pi/2$, excitations which originate in mode i and are transferred to mode j will be π out of phase with excitations which originate in mode j and are transferred to mode i. If desired, a frequency diplexer can then be used to route the different resonant modes out separate physical ports of the device.

There is a complication, though, in generating non-reciprocity in this way: the phase reference of the parametric interaction is arbitrary. Said another way, a gauge may always be chosen in which the phase ϕ is zero, making the beamsplitter interaction reciprocal. This difficulty may be resolved with the introduction of another parametric interaction between modes i and j (or to a third mode k). Experimentally, this can be accomplished by cascading a duplicate of the original device in series with the original [101]. The second parametric interaction also has a phase, and while its phase reference is also arbitrary, the difference between the phases of the two parametric interactions is a gauge-invariant.

A description of this approach can also be couched in the language of geometric phases created by synthetic magnetic fields [109, 110, 111, 102, 99, 107]. In that framework, the coupled resonant modes are viewed as nodes of a lattice, with a hopping interaction determined by the system's parametric modulation. The phase of the parametric modulation sets a Peierls's phase [112], and a Berry curvature (synthetic vector potential) is quantified by the total Peierls's phase acquired in the circumnavigation of one plaquette of the lattice.

The affordances of using parametrically coupled modes to generate non-reciprocity are the relative simplicity of the circuit designs, and a wide versatility in function: in addition to realizing elementary non-reciprocal devices such as isolators [103] and circulators [99, 100], the parametric interaction can also be used to drive parametric down-conversion, enabling processes like directional amplification [98, 99, 100, 113]. This flexibility, and the ability to dynamically switch between the various modes of operation, has led some groups to refer to these devices as \field-programmable," in reference to multi-function field-programmable gate arrays [100]. The major drawback inherent to this method is its reliance on resonant modes in the resolved sideband limit, which constrain the device's bandwidth. A second drawback, in some situations, is the frequency-conversion which accompanies some implementations of this approach [99, 100].

3.3.3.2 Frequency conversion and delay

A second approach for the generation of non-reciprocity with active devices is enabled via successive translations in frequency and time, or in other words, a combination of frequency conversion and delay. As frequency and time are Fourier duals, successive translations in these quantities do not, in general, commute. Signals traversing a network in which a series of these translations are made may encounter them in a different time-ordering, depending on the port of the network at which they are arrive. Such a dependence on propagation direction violates Lorentz reciprocity.

The promise of generating non-reciprocity in this way was first observed in the 1960s, in cascades of symmetric lattices with dual arms of time-varying inductors and capacitors [114]. In these networks, a tank circuit resonance provides the delay, and the active modulation of the circuit elements create frequency sidebands on the input signal. In more recent years, the idea has been leveraged in a general-purpose circulator proposal which is amenable for implementation in superconducting circuits [115]. The realization of that proposal is the subject of this thesis. Related proposals in the microwave [116], millimeter [?], and optical [117, 118, 119, 120] domains have utilized the same concept with non-resonant delays.

The affordances of the frequency-conversion and delay approach are a general absence of bandwidth constraints, stemming from the potential to implement the approach without any resonant physics. Related to this, resonant-based implementations are in general not subject to the constraints of the resolved sideband-limit, allowing low-frequency modulation of the active components. This is an especially attractive feature for superconducting circuits, where high-bandwidth dilution-refrigerator lines are a precious resource.

Its drawbacks are the care required in coherently erasing (or otherwise suppressing) sidebands created by the frequency converting elements, and the general circuit complexity which is sometimes needed for this task. An additional drawback, which is also common to circuits based on the parametric coupling of resonant modes, or in general any active device, is the power consumption of the control signals. This is not a problem for some applications, but in a cryogenic setting additional heat loads on the mixing chamber of a dilution refrigerator are an important consideration, especially in light of the large number of circulators which may be required in a superconducting fault-tolerant quantum computer.

3.3.4 Connection with time-reversal symmetry

As reciprocity arises in physical systems which are invariant under time-reversal, another way to classify non-reciprocal systems is via the mechanism by which they break this invariance. This perspective is utilized in recent demonstrations [107] and proposals [121, 122, 123], which break Lorentz reciprocity with judicious use of dissipation, referred to as reservoir engineering.

Generating non-reciprocity, however, does not in general require dissipation, or a complete absence of time-reversal symmetry. For example, many approaches break Lorentz reciprocity by coupling electromagnetic fields to matter with an interaction that is odd under time-reversal. Examples of this are magnetic fields, as in circulators built with gyrotropic media and quantum Hall effect circulators [75, 76, 77]; linear momentum [124, 125, 126, 127], for example through dispersion engineering [125, 126, 127] or radiation pressure in an optomechanical interaction [124]; and angular momentum [96, 70, 115], for example, by coupling electromagnetic fields to non-degenerate clockwise and counter-clockwise propagating modes in a ring resonator. The degeneracy of these modes can then be lifted with physical rotation, as in a Sagnac interferometer [128] (or in acoustic demonstrations, with the motion of the acoustic medium [70]). It may also be lifted in other ways, such as active modulation of circuit parameters, which can simulate a sense of rotation [115] or a synthetic gauge field [96].

3.4 Common non-reciprocal devices

As both Lorentz reciprocity and scattering matrices have now been discussed, we introduce in this section the scattering matrix description of several common non-reciprocal devices. Tab. 3.1 shows the microwave circuit symbols and scattering matrices for a gyrator, an isolator, and a circulator.

	gyrator	isolator	circulator
microwave circuit symbol			
ideal scattering matrix	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Table 3.1: Symbols and scattering matrices for three common non-reciprocal devices

3.4.1 Gyrators

Gyrators are impedance-matched two-port devices that transmit signals in the forward direction without changing their phase, but impart a phase shift to signals transmitted in the reverse direction [78]. This action is visible in their scattering matrix: for a gyrator, S_{21} and S_{12} are out of phase. Gyrators were first proposed in 1948 by Tellegen as a hypothetical fifth element of basic circuit theory, supplementing the resistor, inductor, capacitor, and transformer [129]. Addition of the gyrator to this set allows for the realization of passive non-reciprocal networks, and also reduces the number of basic circuit elements to three: the gyrator can replace the transformer and either the capacitor or the inductor.

In practice, gyrators are used frequently in the design of active filters, and in the construction of other non-reciprocal circuits, such as isolators and circulators. This can be done directly, by tieing together the grounds of the gyrator's two ports and defining a third port as the differential voltage between them, resulting in a three-port circulator. Alternatively, the gyrator can be inserted between two beamsplitters (or microwave hybrids) to create a four-port circulator. This procedure is known as the Hogan construction of a circulator [130].

3.4.2 Isolators

Isolators are impedance-matched two-port devices that act as one-way valves for classical electromagnetic signals. Signals propagating in the forward direction through an isolator are transmitted unchanged, while signals propagating in the reverse direction are completely absorbed. This action is visible in the off-diagonal elements of the scattering matrix: $S_{21} = 1$, while $S_{12} = 0$.

Isolators are used in a variety of ways across the electromagnetic spectrum: in optical experiments, they are commonly employed as a way to prevent reflections in an optical network from disturbing the source laser. In telecommunication networks, where signals are frequently amplified and attenuated by many orders of magnitude, isolators serve as a way of protecting sensitive (and lower signal-amplitude) components in the network from amplifier reflections. More generally, as discussed in Ch. 1, isolators are of fundamental use in precision measurement.

3.4.3 Circulators

Circulators are impedance matched devices with three or more ports, that provide the directional signal routing needed for a variety of electromagnetic networks. As the scattering matrix in Tab. 3.1 shows, signals incident on one of the circulator's ports will be routed counter-clockwise around the device and transmitted out the subsequent port.

Circulators can be used to make isolators (by depositing the signal at all but two of the circulator ports into an entropy dump) or gyrators (by combining them with 180-degree microwave hybrids/beamsplitters). The need for a high-quality and scalable circulator which could supplant the ferrite-based commercial circulators currently used in superconducting qubit experiments is the motivation for this thesis.

Chapter 4

Theory of operation

The circulator presented in this thesis may be understood in terms of \synthetic rotation" created by the active modulation of the circuit, and analyzed rigorously with a lumped-element circuit theory or an input-output formalism [115]. Before performing a formal analysis we provide a complementary explanation for its operation based upon the frequency-domain dynamics of an analogous model system.

4.1 Model system

The model is a lumped-element network of multipliers and delays (Fig. 4.1a) which creates a gyrator (see Sec. 3.4.1). Gyration in the model system arises from the non-commutation of successive translations in frequency and time [116]: the multiplying circuits operate as frequency converters, translating an input signal up and down in frequency, and the delays translate fields forward in time. As frequency and time are Fourier duals, the time-ordering of these translations matters (the two operations do not generally commute). Transmission through the network thus depends on the propagation direction of the incident signal, breaking Lorentz reciprocity.

To see that non-reciprocity explicitly, frequency-phase diagrams are used to calculate the model's scattering parameters. The diagrams follow an incident signal at frequency ω_p as it propagates through the device, tracking its amplitude, frequency, and phase in a frame rotating at ω_p.

The insets in Fig. 4.1a depict the way that the model system's two constituent elements|

Figure 4.1: Conceptual diagram of non-reciprocity generated with frequency conversion and delay. (a) Lumped-element network that forms a gyrator. The insets show how fields are transformed by the network's two components: multiplying elements and delays. In the upper inset, an input field at frequency ω_p (top panel) is multiplied by $\cos(\Omega t)$ to create a field with spectral components at $\omega_p \pm \Omega$ (bottom panel). The real and imaginary axes of the plot shows the phase of these spectral components in a frame rotating at ω_p. In the lower inset, a delay of length $\tau = \pi/(2\Omega)$ advances (retards) the phase of spectral components at $\omega_p + (-)\,\omega_p - \Omega$ by $\pi/2$. (b) Calculation of the forward-scattering parameter for the network in (a), by following an incident field at frequency ω_p as it propagates through the device. Purple (green) arrows indicate fields propagating in the left (right) arm of the network. Fields are forward transmitted with amplitude and frequency unchanged, but phase shifted by π. (c) Backward transmission through the network in (a). Fields are transmitted with amplitude, frequency, and phase unchanged.

multipliers and delays—transform input fields to output fields. In the multiplying elements, that transformation occurs via multiplication by a bias signal—in this case, $\cos(\Omega t)$. The trigonometric product-to-sum identity states

$$\cos(\omega_p t)\cos(\Omega t + \phi) = \frac{1}{2}\left[\cos([\omega_p + \Omega]\,t + \phi) + \cos([\omega_p - \Omega]\,t - \phi)\right]; \qquad (4.1)$$

which has a simple interpretation in the frequency domain: multiplication creates two sidebands, each detuned from ω_p by the bias frequency Ω. Importantly, the phases of these sidebands depend on the phase ϕ of the multiplier's bias signal. We choose a phase reference such that multiplication by $\cos(\Omega t)$ creates two sidebands with the same phase.

In the delay elements, inputs are transformed to outputs by way of a phase shift. In the rotating frame, a delay of length $= = (2\ )$ leaves the phase of spectral components at $!_p$ unchanged, while advancing the phase of spectral components in the upper sideband $!_p +$ by $= 2$, and retarding the phase of components in the lower sideband $!_p$ by $= 2$.

With the action of the multiplying and delay elements defined, calculation of the scattering parameters is straightforward. Forward transmission through the model system is shown in Fig. 4.1b. A signal incident on port 1 with frequency $!_p$ (Fig. 4.1b, i.) is first divided equally into the network's two arms. Fields in both arms encounter a first multiplying element, a delay, a second multiplying element, and are then recombined.

Critically, the modulation sidebands at $!_p\ \ 2$ created in the network's two arms are out of phase and interfere destructively at the device's output (Fig. 4.1b, iv). Conversely, the components at the frequency $!_p$ interfere constructively. Comparison of Fig. 4.1b, iv. with Fig. 4.1b, i. shows that the incident signal has been transmitted through the device with its frequency and amplitude unchanged, but its phase shifted by . The scattering parameter S_{21} for the network is therefore 1.

The reverse path is traced out in Fig. 4.1c, for a signal incident on the network's second port. As with forward transmission, destructive interference occurs at $!_p\ \ 2$ (Fig. 4.1c, iv.). Likewise, this is accompanied by constructive interference at the frequency $!_p$. Now, however, comparison of Fig. 4.1c, iv. with Fig. 4.1c, i. shows that the frequency, amplitude, and phase of the incident signal were unchanged by the network. Therefore, in contrast to the forward transmission, the backwards transmission is characterized by a scattering parameter $S_{12} = 1$. The network in Fig. 4.1a is thus described by the gyrator scattering matrix in Tab. 3.1, and forms an ideal gyrator.

In the model system, the convert-delay-convert process happens simultaneously in both arms of the network. Consequently, each arm is individually non-reciprocal. Alone, though, a single arm creates unwanted modulation sidebands. To create an ideal gyrator, two arms, with the bias signals of their multiplying elements separated in phase by $= 2$, are connected in parallel. This balanced architecture engineers destructive interference of the spectral components at $!_p\ \ 2$.

Such a strategy for suppressing the creation of spurious sidebands, which we refer to as \coherent cancellation," may be contrasted with that used in non-reciprocal devices that operate with the parametric coupling of resonant modes in the resolved-sideband limit, discussed in Sec. 3.3.3.1. In that scheme, parametric modulation of a resonant system creates sidebands at the parametric drive frequency, and a second resonant mode is used to enhance the density of states at the desired frequency, while simultaneously diminishing it at the undesired frequency. To work in the resolved sideband limit, however, the parametric modulation must be many times the resonant system's linewidth. In microwave frequency implementations, this typically requires GHz modulation tones. In contrast, the coherent cancellation approach lifts the resolved-sideband constraint, and can therefore be used with lower-frequency control tones.

4.2 Superconducting implementation

We make use of the unique properties of superconducting circuitry to realize compact on-chip multiplier and delay elements. Specifically, Josephson junctions form widely tunable inductors, while vanishing conductor loss permits high-quality on-chip microwave resonators. Fig. 4.2 shows how a single arm of the model system (Fig. 4.2a) is made with a network of capacitors and dynamically tunable inductors (Fig. 4.2b).

4.2.1 Multiplying elements

The multiplying elements in the circuit representation are created with reactive bridge circuits (also known as symmetrical lattices [114]), built with two tunable pairs of nominally identical inductors l_+ and l arranged opposite one-another (gray box in Fig. 4.2b). Two differential ports are defined by the left-and-right and top-and-bottom bridge nodes. Importantly, the inductors tune in a coordinated fashion: when one pair of inductors increases, the other pair decreases. We parametrize this tuning with a base inductance l_0 and an imbalance variable , by writing

$$l \quad = l_0 = (1 \quad) : \tag{4.2}$$

Figure 4.2: Multiplying elements and delays realized in a superconducting lumped-element circuit. (a) The model system (Fig. 4.1a) is constructed from two parallel instances of this network. (b) A lumped-element version of the network in (a), created with capacitors and tunable inductors arranged in a bridge geometry. (c) To create an inductive bridge circuit in a superconducting microwave environment, four series-arrays of SQUIDs are arranged in a figure-eight geometry, and tuned with an off-chip magnetic coil producing a uniform flux Φ_u and an on-chip bias line creating a gradiometric flux Φ_g. (d) Simulated group delay for the circuit in (b) when its ports are connected to 50 Ohm transmission lines. The bridge inductors are parametrized according to Eq. (4.2), with $c = 1$ pF, $l_0 = 1$ nH, and $\epsilon = 0.2$.

When a bridge is coupled to transmission lines of characteristic impedance Z_0, its forward scattering parameter at angular frequency ω is

$$\begin{aligned} S_{21} &= \frac{2i\epsilon\omega l_0 Z_0}{Z_0^2(1-\epsilon^2) - 2i\omega l_0 Z_0 + \omega^2 l_0^2} \\ &= \frac{2i\epsilon\omega l_0 Z_0}{(Z_0 + i\omega l_0)^2} + O(\epsilon)^2. \end{aligned} \tag{4.3}$$

As the imbalance in the bridge determines its transmission, changing ϵ allows the circuit to act as a switch or a multiplying element [131, 132].

The bridge circuit's tunable inductors are realized with series-arrays of superconducting quantum interference devices (SQUIDs), formed by the parallel arrangement of two Josephson junctions. Arrays are used in place of individual SQUIDs to increase the linearity of the inductors [115]. The inductance l of an N SQUID array depends on the magnetic flux that threads through each SQUID [58]:

$$l = N\frac{'_0}{2I_0}\sec\frac{}{2'_0} + O(I{=}I_0)^2: \qquad (4.4)$$

Here $'_0 = ~{=}2e$ is the reduced flux quantum, and the Josephson junctions in each SQUID are assumed to be identical with critical current I_0 and negligible geometric inductance. The SQUIDs are also assumed to be identical.

To realize the coordinated tuning of inductors described in Eq. (4.2), we arrange the SQUID arrays in a figure-eight geometry (Fig. 4.2c). Two flux controls determine the imbalance in the bridge. First, an off-chip coil threads a uniform magnetic flux $_u$ through all the SQUIDs. Second, an on-chip bias line, which bisects the figure-eight, threads a gradiometric flux $_g$ through the SQUIDs. SQUIDs on the left side of the bias line therefore experience an overall magnetic flux which is the sum of the uniform and gradiometric contributions, whereas SQUIDs to the right of the line are threaded by the difference of the uniform and gradiometric fluxes.

When the gradiometric bias line is driven with a sinusoidal current source at frequency , the flux through the SQUIDs varies in time as $= \ _u \ \ _g \cos(\ t + \ )$. This process creates a bridge of inductors which tune according to Eq. (4.2), with a simple sinusoidal variation in the imbalance $= \ _0 \cos(\ t + \ )$ and a rescaling of the base inductance l_0. Such a periodic variation in makes SQUID bridges biased in this way into dissipationless multiplying elements. The precise mapping between the flux controls $_u$ & $_g$ and the circuit parameters l_0 & $_0$ is [133]

$$\begin{aligned} _0 &= \ 2\tan(\ )\frac{J_1(\ )}{J_0(\ )} + O(\ ^2); \\ l_0 &= \ N\frac{'_0}{2I_0}\frac{1}{\cos(\ )J_0(\ )} + O(\ ^2); \end{aligned} \qquad (4.5)$$

with

$$\frac{{}_u}{{}_0};\qquad \frac{{}_g}{{}_0}; \tag{4.6}$$

and J_n the n^{th} Bessel function of the first kind.

4.2.2 Delays

The second primitive needed for the model system is a delay, realized in our circuit with a resonant mode. Conveniently, the SQUIDs in the bridge circuits are inductive, so the addition of a single capacitor is enough to create a resonance. This resonance delays fields near its center frequency by a timescale characterized by the inverse of its linewidth. More quantitatively, when a harmonic field incident on port of a resonant network is scattered to port , it acquires a group delay = d\ S =d! [78]. Here ! is the frequency of the harmonic field, and \ S is the phase ofS . Fig. 4.2d shows delay as a function of frequency, simulated for the resonant circuit in Fig. 4.2b. Fields near the circuit's resonant frequency experience a delay of several nanoseconds. As a confirmation of these simulations, an analytical study of the same circuit was also made, using a graph theoretical approach [134, 135] to solve for the circuit's eigenfrequencies with Kirchhoff's current and voltage laws. This is described in App. A.

Delays realized with resonant modes allow for a deeply sub-wavelength implementation, which is critical for the \coherent cancellation" approach. While these lumped-element delays are necessarily narrower in bandwidth than those created with, for example, a length of transmission line, their finite bandwidth is mitigated by the tunable inductance of the bridge circuits, which allows the frequency ! $_0$ of the resonant delay to be tuned (over several GHz) with the uniform magnetic flux $_u$. As the multiplying elements are broadband [132], this tunability of the delay is inherited by the full circulator. Likewise, the duration of the delay depends on the imbalance in the bridges, and may be tuned with the gradiometric flux $_g$, facilitating satisfaction of the requirement that = = (2).

Tuning of the resonant delay takes a simple form when expressed in terms of the circuit parameters l_0 and ${}_0$. When two of the arms in Fig. 4.2b are combined in parallel to create the fully assembled circuit, the resonant delay occurs at the frequency [115]

$$\omega_0 = \sqrt{\frac{4 \quad {}_0^2}{2l_0c}}; \tag{4.7}$$

and its duration is approximately the inverse of the resonant mode's linewidth,

$$\frac{8Z_0c}{{}_0^2}: \tag{4.8}$$

Here Z_0 is the characteristic impedance of the surrounding transmission lines.

4.2.3 Assembly of the full circulator

Construction of a superconducting version of the model system requires the parallel combination of two of the circuits shown in Fig. 4.2. This process is depicted Fig. 4.3, which shows the model system (Fig. 4.3a) alongside a lumped-element schematic of an equivalent network made with variable inductors and capacitors (Fig. 4.3b).

In principle, a circulator could be created from the gyrator shown in Fig. 4.3b using any of the methods detailed in Sec. 3.4.1. Alternatively, the same circuit could be designed without differential ports, where instead each of the four nodes define their own port via comparison to a common ground, as shown in Fig. 4.3c. Such a design allows direct realization of a four-port circulator, without any additional embedding network. The ideal scattering matrix for this four-port clockwise circulator is

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \tag{4.9}$$

and reversing the direction of gyration also allows the same network to act as a counterclockwise circulator with scattering matrix S^T. This device is the four-port version of the circulator described in Tab. 3.1.

Figure 4.3: (a) The model system of Sec. 4.1 forms a gyrator, which breaks Lorentz reciprocity with a combination of frequency conversion and delay. (b) A realization of the network shown in (a), made with tunable, reactive circuit elements. (c) When four ports are formed by comparing the nodes of the circuit in (b) to a common ground, the network forms a four port circulator. Circulation is accomplished through the interference of the common and differential modes of the \left" (ports 1 & 3) and \right" (ports 2 & 4) circuit ports. This is analogous to the circulation generated by a Hogan circulator (d), which comes from the interference between two arms of an interferometer, one of which contains a gyrator, and the other of which is a direct short between ports.

The transformation between gyrator and circulator can be understood in the following way: driving any one of the four ports in Fig. 4.3c involves simultaneously exciting the common and differential modes of the circuit. The gyrating differential mode is non-reciprocal, whereas the prompt scattering of the non-resonant common mode is reciprocal. The interference of these two scattering processes results in circulation [115].

In a sense, this method for realizing a four-port circulator is a virtual form of the Hogan construction [130] depicted in Fig. 4.3d. In (a physical implementation of) the Hogan construction, an incident signal is routed through a 180° hybrid and directed out its differential (Δ) and common (Σ) ports. The differential port leads to a gyrator and then to the differential port of a second hybrid, whereas the common port is directly connected to the second hybrid's common port. The result is an interferometer with two arms: the Δ arm contains a gyrator (and is therefore non-reciprocal), and the Σ arm is a direct connection between the two microwave beamsplitters, which is reciprocal. The circulator in Fig. 4.3c relies on the same interference effect, but in this case the two “arms” of the interferometer are not spatially separated. Instead, they are encoded in the even and odd excitations of ports 1 & 3 and 2 & 4. In both cases, however, the circulation is accomplished through the interference of a non-reciprocal (gyrating) signal pathway and a reciprocal signal pathway.

4.3 Circuit analysis

We have attempted in the preceding sections to establish intuition for how the circuit in Fig. 4.3c realizes a four-port circulator. To support that argument we now directly calculate the elements of its scattering matrix using circuit theory, following Ref. [115].

Consider the bridge circuit shown in Fig. 4.4a. The constitutive relations for this circuit are

$$\frac{1}{l}\begin{pmatrix}1 & \epsilon(t)\\ \epsilon(t) & 1\end{pmatrix}\begin{pmatrix}\Phi_1(t)\\ \Phi_q(t)\end{pmatrix} = \begin{pmatrix}I_1(t)\\ I_q(t)\end{pmatrix}. \tag{4.10}$$

The matrix on the left-hand side of Eq. (4.10) is the reluctance of the circuit. The reluctance matrix gives the current response I_i to an applied flux Φ_i, defined as the time integral of the voltage at port i: $\Phi_i \equiv \int_{-\infty}^{t} V_i(\tau)d\tau$. It is a sort of inverse inductance, what was historically called “magnetic resistance” before the term “reluctance” was coined by Heaviside [136].

When four of these circuits are connected together, in the manner shown in Fig. 4.4b, the

Figure 4.4:

constitutive equation for the network is

$$\frac{{}_0}{l}\begin{pmatrix} 2={}_0 & 0 & \cos(\ t) & \sin(\ t) & \cos(\ t) & \sin(\ t) \\ 0 & 2={}_0 & \sin(\ t) & \cos(\ t) & \sin(\ t) & \cos(\ t) \\ \cos(\ t) & \sin(\ t) & 3={}_0 & 1={}_0 & 1={}_0 & 1={}_0 \\ \sin(\ t) & \cos(\ t) & 1={}_0 & 3={}_0 & 1={}_0 & 1={}_0 \\ \cos(\ t) & \sin(\ t) & 1={}_0 & 1={}_0 & 3={}_0 & 1={}_0 \\ \sin(\ t) & \cos(\ t) & 1={}_0 & 1={}_0 & 1={}_0 & 3={}_0 \end{pmatrix}\begin{pmatrix} {}_p(t) \\ {}_q(t) \\ {}_1(t) \\ {}_2(t) \\ {}_3(t) \\ {}_4(t) \end{pmatrix} = \begin{pmatrix} I_p(t) \\ I_q(t) \\ I_1(t) \\ I_2(t) \\ I_3(t) \\ I_4(t) \end{pmatrix}: \tag{4.11}$$

Analysis is simplified by changing to a left-right, even-odd basis for ports 1-4, and a rotating, circular basis for ports p and q:

$$\begin{pmatrix} I_{l,e}(t) \\ I_{r,e}(t) \\ I_{l,o}(t) \\ I_{r,o}(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} I_1(t) \\ I_2(t) \\ I_3(t) \\ I_4(t) \end{pmatrix}; \tag{4.12}$$

$$\begin{pmatrix} I_+(t) \\ I\ \ (t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\ t} & ie^{i\ t} \\ e^{\ i\ t} & ie^{\ i\ t} \end{pmatrix}\begin{pmatrix} I_q(t) \\ I_p(t) \end{pmatrix}; \tag{4.13}$$

and similarly for the branch fluxes. In these new bases, Eq. (4.11) separates into two uncoupled systems of equations:

$$\frac{2}{l}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}{}_{l,e}(t)\\ {}_{r,e}(t)\end{pmatrix}=\begin{pmatrix}I_{l,e}(t)\\ I_{r,e}(t)\end{pmatrix}; \tag{4.14}$$

$$\frac{1}{l}\begin{pmatrix}2 & 0 & {}_0 & i\,{}_0\\ 0 & 2 & {}_0 & i\,{}_0\\ {}_0 & {}_0 & 4 & 0\\ i\,{}_0 & i\,{}_0 & 0 & 4\end{pmatrix}\begin{pmatrix}{}_+(t)\\ (t)\\ {}_{l,o}(t)\\ {}_{r,o}(t)\end{pmatrix}=\begin{pmatrix}I_+(t)\\ I\ (t)\\ I_{l,o}(t)\\ I_{r,o}(t)\end{pmatrix}: \tag{4.15}$$

To complete the circulator, we shunt ports p and q with capacitances c. This fixes the relation between the currents and the branch fluxes at these ports:

$$c\frac{d^2}{dt^2}\ {}_{q;p} = I_{q;p}; \tag{4.16}$$

or equivalently in the circular, rotating basis,

$$c\left(\frac{d}{dt}\ \ i\ \right)^2\ \ = I\ \ : \tag{4.17}$$

With this constraint, Eq. (4.15) has a simple representation in the frequency domain, which we indicate with square brackets []:

$$\frac{1}{l}\begin{pmatrix}2\ \ lc(!\ \ )^2 & 0 & {}_0 & i\,{}_0\\ 0 & 2\ \ lc(!+\ )^2 & {}_0 & i\,{}_0\\ {}_0 & {}_0 & 4 & 0\\ i\,{}_0 & i\,{}_0 & 0 & 4\end{pmatrix}\begin{pmatrix}{}_+[!\ ]\\ [!\ ]\\ {}_{l,o}[!\ ]\\ {}_{r,o}[!\ ]\end{pmatrix}=\begin{pmatrix}0\\ 0\\ I_{l,o}[!\ ]\\ I_{r,o}[!\ ]\end{pmatrix}: \tag{4.18}$$

At this point we pause and comment on the constitutive relations for the even and odd excitations of the left and right ports of the circuit, given in Eq. (4.14) and Eq. (4.18), respectively. Even excitations of the left and right ports are uncoupled from nodes p and q, and are therefore non-resonant. Contrastingly, odd excitations of the left and right ports depend on the branch fluxes across nodes p and q, and on the value of the capacitors c. These dynamics are resonant. To show that they are also non-reciprocal, we write Eq. (4.18) as a 2 2 system of block matrices

$$\frac{1}{l}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\\ {}_{l,r}\end{pmatrix}=\begin{pmatrix}0\\ I_{l,r}\end{pmatrix}; \tag{4.19}$$