EXCITATION AND GROWTH OF RADIAL PULSATIONS*

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In the preceding paper, Baker has explained how the driving mechanism of the ionization zones works. In this paper, I shall describe the results of some calculations recently carried out in collaboration with A. N. Cox, K. H. Olsen, D. S. King, D. D. Eilers, and others in the J-15 group at the Los Alamos Scientific Laboratory. The physics of the operation of the destabilization mechanism of second helium-ionization has been studied using a hydrodynamic code for solving the complete set of nonlinear, nonadiabatic pulsation equations. A stellar model is followed in the course of time, using realistic equations of state and opacities. The basic purpose of such calculations has been twofold: (1) to check the linearized calculations that have been carried out on stellar pulsations (Baker and Kippenhahn, 1962, and Cox, 1963) to see if the mechanism of second helium-ionization will really work; and (2) to study certain nonlinear effects such as limiting amplitude, light and velocity curves, etc., of pulsating stars. Further details of these calculations will be described in a forthcoming paper.

In our calculations, we have so far considered only stellar envelopes rather than entire stellar models. The mass of the envelope is of the order of $10^{-4}$ times the mass of the entire star, and the temperature at the bottom of the envelope is of the order of $10^5$ K; the radius of the bottom of the envelope is typically 80 to 90% of the total stellar radius. The crucial region is the region which contains 50% second helium-ionization. The envelopes are chosen deep enough so that they contain the complete ionizing region plus enough material below to provide considerable radiative damping. Therefore, all the basic physics is presumably contained in the model. One of the main disadvantages of using only envelope models rather than complete stellar models is that the natural pulsation periods will be about 50% too short; the fundamental pulsations of the envelopes therefore correspond very roughly to first overtone oscillations of real stars. However, all e-folding times are relatively short because of the small mass of the envelopes, and computing time is therefore reduced.

The envelope is divided into a number of discrete mass zones, and the mass of each zone remains fixed with time. The initial conditions, i.e., the positions, velocities, and temperatures of all mass shells at time $t = 0$ must be specified. The bottom boundary of the envelope is kept fixed in time (the core is taken to be a rigid sphere) and a constant luminosity $L_0$ is fed into the envelope from below. The future time-behavior of the model can now be determined; for example it is possible to adjust the equilibrium radius $R_0$ of the model, keeping $L_0$ and $M$ (total stellar mass) fixed.

The first step in a calculation is to set the envelope into a good approximation to hydrostatic equilibrium by using some sort of implicit hydrostatic scheme. The

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hydrostatic equilibrium requirement is then removed and the time behavior of the envelope is calculated, somewhat as Christy (1962) has done. Two kinds of behavior have been observed. One type of behavior, corresponding to a stable envelope, results from using too little helium or the wrong equilibrium radius for given values of $L_0$ and $M$. Following one of the mass zones in time, a typical behavior is as follows: Initially, because the envelope has not been placed at the outset in perfect hydrostatic equilibrium, there are transients, characterized by an irregular, small-amplitude "jostling" of the mass zone; after 10 to 30 periods, however, all the transients have died out and a very-small-amplitude sine wave is left, whose amplitude in the case of a stable envelope decays in time.

The second type of behavior corresponds to an unstable envelope. To date, approximately 10 have been found. They behave as follows: For about the first 10 periods or so there are transients, as before, and the transients decay, leaving a small-amplitude sine wave; in the case of an unstable envelope, however, the amplitude of the sine wave grows in time. Eventually, a limiting amplitude is reached at which the pulsations become essentially strictly periodic. The typical growth time is of the order of a few hundred periods.

Three sets of "thin"-envelope models have been constructed and studied so far. In the first set of models (Cox, Cox, and Olsen, 1963), the composition was a pure helium–hydrogen mix, with the numerical helium–hydrogen ratio 0.15, and the envelope was divided into 20 mass zones. The equilibrium luminosity, i.e., the energy fed in from below, was chosen for computational convenience to be approximately twice that given by the Sandage mass–luminosity relation (Sandage, 1958). The total stellar mass for all models was taken to be $M = 1.073 \times 10^{34} \text{g} = 5.395M_\odot$. Figure 1 shows the location of these envelope models on the H–R diagram.

For radii that were too small or too large, the envelopes were stable. The unstable envelopes lie in the vicinity of the observed instability strip for classical cepheids. This agreement is encouraging, but the closeness of the agreement is not

![Figure 1. H-R diagram for envelope models with helium/hydrogen ratio = 0.15 (by numbers). Points labeled by slanted numbers represent unstable envelopes.](image-url)
entirely significant, because our envelopes have the wrong periods. The total kinetic energy of the envelope as a function of time, as the limiting amplitude is attained, is shown in Figure 2. The limiting amplitude was reached, in this case, after about 1000 periods of growth.

The kinetic energy leveled off at about $10^{41}$ ergs for all envelope models calculated. The limiting pulsation properties of the two unstable envelopes in this series are summarized in Table I.

In order to insure that the limiting amplitude was independent of starting conditions, a large perturbation was applied to one of the envelopes after it had attained limiting amplitude. The perturbation quickly decayed, and, after some 10 periods or so, the original limiting amplitude had been recovered to within a

Figure 2. Total kinetic energy of the envelope as a function of time. Ordinates are in units of $10^{16}$ ergs. The leveling off of the upper envelope of the curve at about $10^{41}$ ergs represents the attainment of the limiting amplitude. The curious appearance of the lower envelope of the curve is a spurious effect introduced by the method of plotting the data. Approximately 750 periods are shown in the figure.
Table I. Limiting Pulsation Properties of Unstable Envelopes
(Helium–Hydrogen Mix, $B = 0.15$)

<table>
<thead>
<tr>
<th>PM</th>
<th>Radius semiamplitude ($\delta R/R_0$)</th>
<th>Total bolometric light range</th>
<th>Total velocity amplitude (km/sec)</th>
<th>Maximum kinetic energy (ergs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$\approx 0.04$</td>
<td>$\approx 0''32$</td>
<td>64</td>
<td>$1.0 \times 10^{41}$</td>
</tr>
<tr>
<td>19</td>
<td>$\approx 0.045$</td>
<td></td>
<td>61</td>
<td>$8 \times 10^{39}$</td>
</tr>
</tbody>
</table>

Few percent; the detailed properties of the pulsations were also practically the same as before the perturbation.

To test the effect of zoning, one of the unstable envelopes was divided into 50 zones and the calculation repeated. There was no important change in behavior.

The second set of models was constructed in an attempt to investigate the effects of a chemical composition which included some metals, i.e., a population I type composition. The Aller mix, $Z = 0.02$, was used, with the helium–hydrogen ratio $= 0.161$ (by numbers), and the models were divided into 50 mass zones. The models were constructed for the same luminosity and mass as for the pure helium–hydrogen models. The envelopes were found to be stable for all reasonable radii. This stability was interpreted as being a consequence of the higher average opacity above the ionizing region, caused by the higher metal abundance, which leads to a lower density in the ionizing region. Very roughly, the density in the ionizing region varies as

$$\rho \propto \frac{M}{\alpha L}$$

where $\alpha$ is an average value of the opacity in the regions above the ionizing region. The lower the density, the less sensitive is the opacity to the temperature. Thus, the kappa-mechanism, as described by Baker, is less effective in driving pulsations.

In order to raise the densities, a lower luminosity was chosen and a third set of models (with the same composition and zoning) was calculated with a luminosity

![Figure 3. Location of the two sets of envelope models on the H-R diagram. Points labeled by slanted numbers represent unstable envelopes. The arrows indicate the estimated edges of the computed instability strips.](image-url)
Figure 4. Total kinetic energy of a stable Aller-mix envelope lying just to the left of the instability strip as a function of time. Ordinates are in units of $10^{44}$ ergs and abscissae are in units of the pulsation period of the envelope. This envelope is barely stable, as is indicated by the very slow rate of decrease of the upper envelope of the curve with time.

about half the previous value, that is, the luminosity was now taken as equal to that given by the Sandage mass–luminosity law (Sandage, 1958). $L_0$ and $M$ were kept constant, and the equilibrium radius was varied. Again, a group of unstable models was found; those having too small or too large a radius were stable, as in the first set of models. The results are shown in Table II.

The location of the two sets of unstable envelopes on the H–R diagram is shown in Figure 3. The unstable envelopes with the higher value of $Z$ are shifted to the right because of the greater mean opacity in the regions above the ionizing region. This mean opacity is increased by 20 to 30% over that for the pure helium–hydrogen models. This calculated instability strip (for the Aller-mix composition) is close to that observed for the classical cepheids. However, if the fact were taken
Figure 5. Total kinetic energy of an unstable Aller-mix envelope located near the middle of the instability strip as a function of time. Units are the same as in Figure 4. The steep initial rise in the upper envelope of the curve to about $10^{40}$ ergs was artificially induced in the calculations to save computer time.

Table II. Results for Envelopes with Aller-Mix Composition (50 Zones)

<table>
<thead>
<tr>
<th>PM</th>
<th>$M_{bol,0}$</th>
<th>$\Pi_{NH}$</th>
<th>$R_o$</th>
<th>log $T_{eff}$</th>
<th>$\tau_d/\Pi_{NH}$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>44</td>
<td>$-3.720$</td>
<td>2.359</td>
<td>36.8</td>
<td>3.827</td>
<td>$-200$</td>
<td>Unstable, He-H mix</td>
</tr>
<tr>
<td>47</td>
<td>$-3.055$</td>
<td>1.444</td>
<td>27.1</td>
<td>3.828</td>
<td>+260</td>
<td>Stable</td>
</tr>
<tr>
<td>45</td>
<td>$-3.055$</td>
<td>1.510</td>
<td>28.5</td>
<td>3.817</td>
<td>+2500</td>
<td>Barely stable</td>
</tr>
<tr>
<td>46</td>
<td>$-3.055$</td>
<td>1.678</td>
<td>30.0</td>
<td>3.806</td>
<td>-150</td>
<td>Unstable</td>
</tr>
<tr>
<td>48</td>
<td>$-3.055$</td>
<td>1.806</td>
<td>31.4</td>
<td>3.796</td>
<td>-100</td>
<td>Unstable</td>
</tr>
<tr>
<td>49</td>
<td>$-3.055$</td>
<td>1.942</td>
<td>33.0</td>
<td>3.787</td>
<td>-125</td>
<td>Unstable</td>
</tr>
<tr>
<td>50</td>
<td>$-3.055$</td>
<td>2.089</td>
<td>34.7</td>
<td>3.774</td>
<td>-</td>
<td>Unstable</td>
</tr>
<tr>
<td>52</td>
<td>$-3.055$</td>
<td>2.411</td>
<td>38.1</td>
<td>3.754</td>
<td>-135</td>
<td>Unstable</td>
</tr>
<tr>
<td>53</td>
<td>$-3.055$</td>
<td>2.782</td>
<td>41.9</td>
<td>3.733</td>
<td>+340</td>
<td>Stable</td>
</tr>
</tbody>
</table>
into account that these envelopes have periods about 50% shorter than the fundamental periods of the corresponding whole stars, the instability strip would be shifted further to the right (by about $\Delta \log T_e \approx -0.036$) away from the observed strip. This shift would then bring these results into approximate agreement with those obtained by Baker and Kippenhahn (1962) on the basis of their linearized calculations. Thus, there is still not very good agreement between theory and observations.

The kinetic energies of Aller-mix envelopes lying immediately to the left of, in, and just to the right of the instability strip are shown in Figures 4 to 6. If the radius is too small or too large, the energy decays, while for the unstable envelopes it grows and again levels off at about $10^{41}$ ergs.

Figure 7 shows radius versus time for some of the mass zones for one of the unstable Aller-mix envelopes after limiting amplitude has been attained. Notice
that the nonlinearity becomes appreciable only for the outermost few zones. In other words, the pulsations are still quite small, even at the limiting amplitude.

The luminosities and radial velocities of some of the zones as a function of time for one of the unstable Aller-mix models after attainment of the limiting amplitude are shown in Figures 8 and 9. The details of the luminosity curves, especially for the outermost zones, are probably not significant because of some rather crude approximations used in treating the radiation flow in these regions (for example, the entire region of hydrogen ionization was contained in only one or two mass zones).

The limiting pulsation properties of the unstable envelopes for the Aller mixture are summarized in Table III. Note that the limiting values of $\delta R/R_0$, velocity amplitude, and total kinetic energy are at least reasonable compared to
Figure 8. Luminosity (in bolometric magnitudes) versus time for several mass zones for the envelope of Figure 7. Luminosity increases toward the top of the figure. The outermost zone is represented by the curve marked with the asterisk (*) symbols; the curve marked with the plus (+) symbols lies the next farthest in; the curve marked with the minus (−) symbols lies still farther in; and the curve not marked with any symbols represents the innermost zone.

Table III. Limiting Pulsation Properties of Unstable Envelopes (Aller Mix)

<table>
<thead>
<tr>
<th>PM</th>
<th>Radius semiamplitude (δR/R₀)</th>
<th>Total bolometric light range</th>
<th>Total velocity amplitude (km/sec)</th>
<th>Maximum kinetic energy (× 10⁶⁰ ergs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>44*</td>
<td>0.047</td>
<td>0º⁴⁰</td>
<td>69.5</td>
<td>9</td>
</tr>
<tr>
<td>46</td>
<td>0.036</td>
<td>0º³⁶</td>
<td>63</td>
<td>7</td>
</tr>
<tr>
<td>48</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>8</td>
</tr>
<tr>
<td>49</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>9</td>
</tr>
<tr>
<td>50</td>
<td>0.042</td>
<td>0º⁴¹</td>
<td>71</td>
<td>9</td>
</tr>
<tr>
<td>52</td>
<td>0.041</td>
<td>0º⁴⁰</td>
<td>67</td>
<td>9</td>
</tr>
</tbody>
</table>

* Helium–hydrogen mix.
observed properties of classical cepheids. In addition, the relatively small values of $\delta R/R_0$ at the limiting amplitude favor the suggestion by Ledoux and Whitney (1961) that the limiting amplitude is determined by a "saturation effect" of the second-helium ionization-driving mechanism.

Perhaps the most important conclusion to be drawn from this set of calculations is that "soft" self-excited radial oscillations (see Ledoux and Walraven, 1958) can arise in a stellar envelope (and therefore presumably also in a whole star) from the action of second-helium ionization in the envelope, provided that the parameters of the star (luminosity, mass, radius, and composition) form a favorable combination. Moreover, the attainment of a stable limiting amplitude shows that a "limit cycle" kind of behavior is possible. Because of the good overall agreement with the results of the linearized theory (Baker and Kippenhahn, 1962, and Cox, 1963)
(reasonable growth rates, well-defined instability strip in approximately the right location, stability when insufficient helium is present, etc.), the instabilities found here are very likely of an actual physical origin and not merely a reflection of spurious mathematical instabilities.

The stability of the second set of models (Aller-mix composition, $L_0$ about twice the Sandage value) leads to the following conclusions (these might also have been deduced from a linearized theory): (1) If the density in the critical region becomes too small, instability is likely to disappear. This means, with population I cepheids, that the ratio $L/M$ should not be too large. For a given luminosity, a fairly large mass is required in order that these stars be unstable. (2) Going up along the cepheid-instability strip, the luminosity increases as about $M$ to the third or fourth power; therefore, the density in the critical region should decrease in going up the strip, and it may well be that this decrease is what cuts off the instability strip at the top. (3) Since the ratio $L/M$ is probably larger for population II stars than for population I stars, it may be that low metal abundance is a necessary condition for instability in population II stars. These last three conclusions are based on studying envelope models alone and are therefore rather tentative when applied to whole stars. Models with much deeper envelopes are needed to test these conclusions; work with such models is now in progress.

REFERENCES


DISCUSSION

G. Wallerstein: From your remarks, it seems that a helium-ionization zone is needed to get a population II cepheid to pulsate. If one needs a low metal content combined with small mass, which is reasonable for population II, perhaps it should be concluded that there are stars, namely, these stars, with low metal content but high helium content. This is of considerable interest to the question of the primordial helium content of the galaxy, which might be considerable.

N. Baker: Convection is important in determining the stratification of the equilibrium model. The H$^+$ absorption can change the outer boundary conditions of the convection zone. There can be a fairly large change in the stratification in the case where there is a convection zone. For the stars you are considering, I don’t know how important this is.

J. P. Cox: Some of the Los Alamos calculations were carried out using your code for convection. We found that convection may very well become important in some of these stars in the second helium-ionization region, but we have not treated convection adequately yet.