SOME TOPICS IN RADIATIVE TRANSFER

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I. INTRODUCTION

Photon transport theory, traditionally called “radiative transfer”, has a great deal in common with neutron transport theory in that both are essentially statistical descriptions of “particles” which migrate through a “medium” and undergo repeated random changes in their direction of propagation. Since both theories ordinarily neglect interactions between the migrating particles, their common starting point is the linear Boltzmann equation, and accordingly much of the mathematical development of the two theories is identical apart from trivial differences in notation and nomenclature. In particular, the approximation called “one-group neutron transport theory” is identical with the “grey approximation” of radiative transfer, and many of the exact results of transport theory apply only to this idealized model.

When more realistic models than the “grey” or “one-group” theory are used, some differences between neutron transport and radiative transfer begin to appear. These consist of differences in the character of parameters such as cross-sections which go into the transport equation, differences in the types of geometry and boundary conditions most often encountered in problems, and differences in the questions we ask of the theory. As a result it is no longer always trivial to translate a neutron problem into the corresponding photon problem or vice versa, and when the translation is accomplished the result may not apply to the real world. For example, despite recent advances in laser technology, it is not yet possible to start with a far-infra-red photon, with 1/40 of an electron volt energy, and trigger a reaction giving two or three gamma rays at a few MeV; i.e. there is no photon process closely analogous to fission. So the problems treated in these two areas are sufficiently different for approximations and numerical methods, which are shaped by the problems, to tend to be different also.

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In these talks I will try to survey enough selected topics in radiative transfer to illustrate these differences and the similarities that remain. I will not dwell very long on the "grey" problems unless they show some special "radiative" feature. I will, however, spend some time on a topic which is not exactly a part of transport theory itself, namely the physics of photon cross sections which are used in radiative transfer, so as to exhibit the physical context for the methods used in "non-grey" problems.

II. SCOPE OF RADIATIVE TRANSFER THEORY

Broadly defined, radiative transfer could be said to encompass all phenomena involving the propagation of electromagnetic radiation and its interaction with matter—i.e. all of optics and spectroscopy, and much of atomic, molecular, liquid, and solid structure theory. Conventionally, however, radiative transfer means the macroscopic description of a radiation field as governed by macroscopic emission, absorption, and scattering coefficients (called "optical properties" of the medium). These coefficients are regarded either as given quantities or as given functionals of the radiation field, and their detailed evaluation is a task in statistical and quantum mechanics or else experiment. Also conventionally, the rather diffuse distinction between radiative transfer and geometrical optics is that in radiative transfer the interactions of the radiation field with the matter are to some extent random events.

Types of problems to which radiative transfer can be applied vary widely, and may be classified in several ways. One practically important distinction is between "direct" problems where the optical properties and sources are prescribed and the radiation field is required, and "inverse" problems where the radiation field (or part of it) is known and the optical properties are to be deduced. The interpretation of astronomical and laboratory observations is often handled as an "inverse" problem, although an alternative is to solve several "direct" problems and see which one fits best. Typically a "direct" problem has a unique physically significant solution while an "inverse" problem does not, and so has to be supplemented by some theoretical knowledge of the optical properties. As a trivial example, the mere observation "the sky looks blue" tells us either "scattering increases toward shorter wavelengths" or "absorption decreases toward shorter wavelengths" (or both) but to decide between these possibilities requires further information, e.g. the red sunset.

Another useful classification of radiative-transfer problems is according to the assumed character of the optical properties, since it is almost always necessary to use some simplifying approximations. For example, the approximation of LTE (local thermodynamic equilibrium), which has bee
used in the vast majority of radiative transfer calculations to date, relates the emission rate to the absorption coefficient by assuming the existence of a unique local temperature for the material. In the visible layers of the sun and most stars this assumption is physically unjustified, but it simplifies the calculations greatly, and some results which are of wide generality within the LTE framework can be obtained. Similarly, the grey approximation and the isotropic scattering approximation are routinely used, even when they are really inapplicable, because of their relative simplicity. Many intermediate levels of approximation also are used and some of these will be mentioned later.

Much of what I have just said applies to both photon transport and neutron transport. If one had to pick out the most distinctive features of photon transport as contrasted with neutron transport, in my opinion there would be three:

(a) In general the optical properties of the medium in radiative transfer problems depend strongly on the radiation field, so that the full radiation problem is in fact non-linear even though photon-photon collisions are neglected. In contrast, the neutronic properties of the medium in neutron problems ordinarily depend only slightly on the neutron density field.

(b) Largely as a result, optical properties usually are continuously varying functions of position, while neutronic properties usually are at least piecewise constant in position.

(c) Line profiles in optical spectra are ordinarily so narrow and so numerous that the photon mean free path varies rapidly with wavelength and covers a range of several orders of magnitude from line centres to the “windows” between lines. Neutron resonance cross-sections show similar behaviour but are usually confined to an energy region of limited importance. In photon problems the lines often cover the whole wavelength spectrum rather densely and lead to a very “non-grey” transfer equation.

Historically, radiative transfer as a mathematical theory may be said to have begun with Rayleigh’s attempts (1871) to calculate the distribution of sunlight in the earth’s atmosphere. Schuster and Schwarzschild (1905, 1906) formulated the radiative-equilibrium stellar-atmosphere problem; the basic integral equation, which is often called the Schwarzschild–Milne equation in the literature, had, however, been derived earlier by Lommel (1889) and by Hvolson (1890) in other contexts.† The theory was developed mostly by astrophysicists for several decades (since practically all astronomical data are radiative), before the emergence of neutron transport theory accelerated its progress. Most problems in radiative transfer still are astronomical

† I am indebted to Professor Kuscher for the reference to Lommel. Hvolsen’s context was the scattering of light in translucent “milk glass”; I am indebted to Professor Zweifel for pointing out the connection between milk glass and the Galaxy (Greek γάλα, milk), i.e. the Milky Way.
or meteorological, although there are some quite practical terrestrial applications: illumination and photometry in cloudy or foggy media; design and interpretation of optical experiments such as the arc and shock-tube methods of measuring atomic parameters; energy loss from thermonuclear plasma via line radiation from highly stripped impurity ions; and, to a growing extent, problems in "radiation hydrodynamics", "hypersonic flow", or "high energy fluid dynamics", in which (because of the high temperatures generated by the flow) the radiative energy transfer rate is high enough to affect the flow pattern instead of merely serving as a probe (Bond, Watson, and Welch 1965).

III. RADIATIVE TRANSFER EQUATION; BASIC DEFINITIONS:  
THE LTE HYPOTHESIS

The fundamental quantity which describes a radiation field is the intensity (or "specific intensity") $I_\nu$, which is defined as the energy per unit area, time, frequency interval, and solid angle, crossing a small test surface which is oriented normal to the direction of propagation. It plays the same role in radiative transfer as the angular neutron flux does in neutron transport theory. The intensity $I_\nu$ has the same dimensions as the Planck radiation function $B_\nu(T)$, and indeed under the LTE assumption it will turn out that $I_\nu$ is a weighted average of the values of $B_\nu(T)$ in the radiating system. In general $I_\nu$ depends on the photon frequency $\nu$, position $r$, direction of propagation $\Omega$, and time $t$.

The intensity $I_\nu$ is insufficient to describe polarized light; four parameters are required in general. If the medium scatters light, polarization will generally be present, while if absorption and emission are the only mechanisms an isotropic medium can only emit unpolarized light. In these lectures polarization will be neglected; it is treated extensively in Chandrasekhar’s book (1950).

Other phenomena we will neglect include dispersion (dependence of refractive index on frequency) and collective effects such as would result from correlation in positions of the scattering or absorbing centres. (The phenomena are discussed in Professor Oberman’s lectures, and are very important for frequencies near the plasma frequency.) We are therefore regarding photon scatterings as independent successive isolated events and will not be able to obtain, for example, specular reflection of light by a mirror.

With the above disclaimers, we can write the radiative transfer equation:

\[
\frac{\partial}{\partial t} + \nabla \cdot \left( \frac{\mathbf{J}_\nu}{\mu} \right) = -\mathbf{E}_\nu \cdot \nabla I_\nu - \kappa_\nu I_\nu + \alpha_\nu I_\nu
\]
where $\rho = \text{local density of matter}$, $\kappa_s = \text{absorption coefficient}$, $c = \text{velocity of light}$, and $\varepsilon_s = \text{emission coefficient}$.

The absorption coefficient is the cross-section per unit mass for all processes which end flights of photons of frequency $\nu$. We shall immediately restrict ourselves to isotropic media so that $\kappa_s$ is independent of $\Omega$. Processes which contribute to $\kappa_s$ are of several kinds:

1. "Elastic" scattering
   A. Rayleigh—by atoms and molecules
   B. Thomson—by free electrons—approximation to Compton

2. "Inelastic" scattering
   A. Compton—small frequency change—energy goes into recoil of electron
   B. Raman—by atoms and molecules—energy into internal degrees of freedom

3. Capture
   A. Line absorption (excitation)
   B. Photo-ionization
   C. Inverse Bremsstrahlung
   D. Photodissociation of molecules

Other mechanisms (e.g. pair production, photonuclear reactions) could contribute at appropriately extreme photon energies. The inelastic scattering processes are seldom used in actual calculations. Resonance scattering, which has been omitted from the above list, may usually be regarded as line absorption followed by line emission.

The emission coefficient, $\varepsilon_s$, is the energy per unit mass, time, frequency interval, and solid angle, which begins flights as photons of frequency $\nu$; the energy per photon, of course, is $h\nu$. Each capture process which contributes to $\kappa_s$ has an inverse which contributes to $\varepsilon_s$; scattering processes contribute to both. The inverse of each capture process may proceed spontaneously or may be induced by the radiation field. In isotropic media the spontaneous emission will be isotropic, but induced emission propagates in the same direction as the radiation which induces it. Therefore $\varepsilon_s$ (unlike $\kappa_s$) is in general anisotropic, because of both induced emission and anisotropic scattering. It is convenient to regard the induced emission as negative absorption (i.e. regroup terms), and to work in terms of the "reduced" or "corrected" absorption and emission coefficients, $\kappa'_s$ and $\varepsilon'_s$, where $\kappa'_s$ contains capture minus induced emission, $\varepsilon'_s$ contains spontaneous emission, and both contain scattering.

We now have to express $\kappa'_s$ and $\varepsilon'_s$ in terms of atomic parameters and the formation of absorption and emission. The linear coefficients are...
Einstein coefficients, the contribution of this transition to \( \kappa'_r \) is

\[
\kappa'_r(ij) = (N_i - N_j)B_{ij}\phi_r(ij)hv_{ij}/4\pi \rho
\]  

and its contribution to \( \varepsilon'_r \) is

\[
\varepsilon'_r(ij) = N_jA_{ji}\phi_r(ij)hv_{ij}/4\pi \rho
\]

where \( N_i, N_j \) = number of atoms per unit volume in lower and upper states; 

\( hv_{ij} = E_j - E_i \); 

\( \phi_r(ij) \) = normalized line profile \( \int \phi_r \, dv = 1 \); 

\( B_{ij}, A_{ji} \) = Einstein coefficients.

We have taken the emission and absorption profiles equal, which is justifiable in nearly all circumstances; in the literature this assumption is often called "complete redistribution". It is examined, e.g. by Hummer (1962).

We note the ratio of "corrected" to uncorrected absorption coefficients;

\[
\frac{\kappa'_r(ij)}{\kappa_r(ij)} = 1 - \frac{N_j}{N_i}
\]

and recall that in local thermodynamic equilibrium at temperature \( T \),

\[
\frac{N_j}{N_i} \xrightarrow{\text{LTE}} \exp\left(-\frac{hv_{ij}}{kT}\right) < 1
\]

so that in LTE, \( \kappa'_r \geq 0 \) since \( \kappa_r \geq 0 \). Without LTE, it is possible to have \( N_j > N_i \) ("population inversion") and then \( \kappa'_r(ij) < 0 \). If the total \( \kappa'_r \) at some frequency is negative, a photon beam's intensity will grow with distance instead of attenuating as it passes through the medium, and this is the principle on which lasers operate.

The coefficients \( A_{ji} \) and \( B_{ij} \) are connected by the relation

\[
A_{ji} = 2hv^2_{ij}B_{ij}/c^2
\]

and the total contribution of line transitions to \( \kappa'_r \) and \( \varepsilon'_r \) is obtained by summing Eqs (2) and (3) respectively over all \( i \) and \( j \). Actually, the other capture processes can be handled with the same formalism as we have used for lines, if one encloses each atom, molecule, or ion in a box and takes care to count states properly. The transport equation (neglecting scattering for the moment) can then be written

\[
\left[ \frac{1}{c} \frac{\partial}{\partial t} + \Omega \cdot \mathbf{v} + \sum_{i,j \text{ \( \text{E}_i < \varepsilon_r \) \}}^{} \phi_r(ij)hv_{ij}(N_i - N_j)B_{ij}/4\pi \right] I_r =
\]

\[
\sum_{i,j}^{} \phi_r(ij)hv_{ij}N_jA_{ji}/4\pi
\]
or in the more familiar form

\[ \left[ \frac{1}{c} \frac{\partial}{\partial t} + \mathbf{\Omega} \cdot \mathbf{\nabla} \right] I_\nu = \rho \kappa'_\nu (\nu) - I_\nu + S_\nu \]  

where we have defined \( S_\nu \), the so-called source function, as

\[ S_\nu = \frac{\nu e}{\kappa'_\nu} \]  

which is in general a complicated function of \( \nu \) (it is not even certainly non-negative) and depends on knowing all the occupation numbers \( N_i \) at each point in space. The LTE assumption, however, asserts that all pairs of occupation numbers are related by Eqn (5), which results in the enormous simplification

\[ S_\nu \xrightarrow{\text{LTE}} B_\nu(T) = \frac{2h\nu^3}{e^2} \left( e^{h\nu/kT} - 1 \right)^{-1} \]  

so that in LTE a single local parameter, the temperature of the medium, specifies the source function completely for every frequency. Unlike the general (non-LTE) source function, the Planck function \( B_\nu \) is a smooth function of \( \nu \), a fact whose usefulness will become apparent later. The absorption coefficient \( \kappa'_\nu \) is still complicated, of course, but in LTE the \( N_i \) on which \( \kappa'_\nu \) depends are again known in terms of the local temperature \( T \) and density \( \rho \).

We should emphasize that the LTE hypothesis asserts nothing about the value of the local radiation field. LTE is a statement that Kirchhoff's Law, Eqn (10), holds regardless of whether \( I_\nu \) resembles \( B_\nu \) or not. In an infinite isothermal medium the solution of the LTE transport equation results in \( I_\nu = B_\nu \), so that LTE is then self-consistent, but in general this is not the case.

Finally we consider the form of the LTE transport equation when elastic isotropic scattering is present. If we let the ratio of capture cross-section to total (capture plus scattering) cross-section at frequency \( \nu \) be \( \gamma_\nu \), the source function becomes

\[ S_\nu \xrightarrow{\text{LTE with scattering}} \gamma_\nu B_\nu + (1 - \gamma_\nu)J_\nu \]  

where \( J_\nu \) is the mean intensity, analogous to the neutron "scalar flux":

\[ J_\nu = \int I_\nu \, d\Omega/4\pi \]  

The transport equation is then

\[ \left[ \frac{1}{c} \frac{\partial}{\partial t} + \mathbf{\Omega} \cdot \mathbf{\nabla} \right] I_\nu = \rho \kappa'_\nu (\nu) - I_\nu + \gamma_\nu B_\nu + (1 - \gamma_\nu)J_\nu \]
In the above, "capture" means "reduced capture", i.e. capture minus induced emission, but the "reduction" factor does not apply to the scattering cross-section (Rudkjobing 1947). If \( \gamma = 0 \) in Eqn (13), we get the so-called monochromatic radiative transfer problem in which there is no coupling between different frequencies and which can be solved as a collection of uncoupled "one-group" problems.

We close this section by mentioning several quantities which are conventionally used in the literature of radiative transfer (neutron analogues in parentheses).

\[
l_\nu = \frac{1}{\rho \kappa'} = \text{monochromatic mean free path (reciprocal macroscopic cross section)} \tag{14}
\]

\[
\pi F_\nu = \int I_\nu \Omega \, d\Omega = \text{monochromatic flux vector (energy-dependent neutron current vector)} \tag{15}
\]

\[
\pi F = \int_0^\infty \pi F_\nu \, dv = \text{integrated flux vector (total neutron current vector)} \tag{16}
\]

\[
d\tau_\nu = \rho \kappa' \, ds = \text{differential optical depth} \tag{17}
\]

In general geometry the optical depth is a function of two position vectors and \( ds \) is a differential distance along the straight line joining them. In slab geometry the optical depth is a function only of distance normal to the plane of symmetry; for a medium bounded by vacuum, \( \tau_\nu \) is conventionally taken as zero at the interface and increases (when \( \kappa' > 0 \)) into the medium, while angles are measured with respect to the outward normal to the interface. Thus for slab geometry (without azimuthal dependence) the time-independent transfer equation is written (with \( \mu = \cos \theta; \, \theta = \text{angle between outward normal and } \Omega \))

\[
\mu \frac{\partial}{\partial \tau_\nu} I_\nu(\tau_\nu, \mu) = I_\nu - S_\nu \tag{18}
\]

which differs from the usual form of the neutron transport equation in the sign of one side. The radiative convention makes \( \mu > 0 \) for the radiation leaving the surface; in the Milne problem the temperature gradient is positive but so is the flux (defined as \( 2\pi \int_0^1 I \mu \, d\mu \)). However, in some papers on terrestrial applications the neutron sign convention is employed, so the reader must keep his eyes open.

The quantities \( H_\nu = \frac{1}{4} F_\nu \) and \( H = \frac{1}{4} F \) are sometimes used to give the integral equations for mean intensity and flux a more symmetrical appearance, but can be confused with Chandrasekhar's \( H \)-function.
For problems with a constant integrated flux, a convenient measure of its magnitude is provided by the effective temperature $T_{\text{eff}}$, which is the temperature of an isothermal semi-infinite region (i.e. a “black body”) whose surface would radiate the same integrated flux. Accordingly

$$\pi F \equiv \pi B(T_{\text{eff}}) = \sigma T_{\text{eff}}^4$$

(19)

where $B(T)$ is the integrated Planck function and $\sigma$ is the Stefan–Boltzmann constant. Giving the value of $T_{\text{eff}}$, of course, in itself implies nothing about the spectral distribution of the flux. In the grey LTE Milne problem, $T_{\text{eff}}$ turns out to equal the local temperature at optical depth about 2/3.

Three averages of $\kappa'$ over frequency are in constant use for LTE problems, and will be defined here. These are the Rosseland mean, Planck mean, and transmission mean:

$$\kappa(\text{Rosseland}) = \kappa_R = \int_0^\infty \frac{\partial B_\nu(T)}{\partial T} \, dv \int_0^\infty \frac{1}{\kappa'} \frac{\partial B_\nu(T)}{\partial T} \, dv$$

(20)

$$\kappa(\text{Planck}) = \kappa_P = \int_0^\infty \kappa' B_\nu(T) \, dv \int_0^\infty B_\nu(T) \, dv$$

(21)

$$\kappa(\text{Transmission}) = \kappa_T(m) = -\frac{1}{m} \ln \left[ \int_0^\infty e^{-m\kappa_T B_\nu(T)} \, dv \int_0^\infty B_\nu(T) \, dv \right]$$

(22)

Collectively, such frequency averages are called mean opacities or simply opacities, though the latter name is sometimes applied to $\kappa'$ also. The three opacities above are relevant to the calculation of integrated intensities and fluxes in optically thick, optically thin, and isothermal regions respectively. We note that $\kappa_T$ contains a parameter $m$, and that as $m \to 0$, $\kappa_T \to \kappa_P$. One may define “local” (in frequency-space) averages also, by letting the frequency run from $\nu_1$ to $\nu_1 + \Delta \nu$ in the integrals instead of from 0 to $\infty$. If $\Delta \nu \ll kT/h$, these local averages no longer involve the Planck function and we have also the relation (for local averages only):

$$\int_0^\infty e^{-m\kappa_T(m)} \, dm = 1/\kappa_R$$

IV. REMARKS ON LTE; TREATMENT OF NON-LTE PROBLEMS

In this section we consider the time-independent version of the non-LTE transfer equation (without scattering), Eqn (7). This could be solved for $I_\nu$ if the profiles $\phi_\nu(t)$ and occupation numbers $N_\nu$ were known as functions of position (we assume the atomic constants $B_\nu$, $A_\mu$ are known already).
Under the LTE hypothesis these are all known in terms of the local thermodynamic state of the material medium—e.g. temperature and density. Two additional equations (e.g. energy conservation and hydrostatic equilibrium), plus boundary conditions for the differential transfer equation, thus are enough in LTE to make a determinate system with (at least hopefully) a unique physically meaningful solution, described by \( T \) and \( \rho \) as functions of \( r \).

But what is required without LTE? Leaving aside the line profiles \( \phi_{s \mu}(i) \), which typically depend on local velocity distribution functions (for Doppler and impact broadening) or electric-field-strength distribution functions (for quasi-static Stark broadening), we must replace the LTE assumption by a set of rate equations, one for each atomic energy level. In the steady state these take the form

\[
\frac{\partial N_i}{\partial t} = \sum_j (N_i R_{ij} - N_j R_{ji})_{\text{steady state}} \to 0
\]

where the \( R \)'s describe transition rates between levels. Each \( R \) is of the form

\[
R_{ji} = A_{ji} + B_{ji} \int J_s \phi_{s \mu}(ij) \, dv + C_{ji} N \bar{v}
\]

in which the first two terms are radiative (\( A_{ji} \) and \( B_{ji} \) are Einstein coefficients) and the last is collisional; \( C_{ji} \) is a properly averaged inelastic-collision cross section and \( N \bar{v} \) is the "flux" (in the neutron-transport sense!) of particles (chiefly free electrons, if any are present). The spontaneous inverse lifetime, \( A_{ji} \), is of course non-zero only when \( E_j > E_i \). In a Hohlraum, each term of the sum in Eqn (23) vanishes.

The rate equations (23) must be solved self-consistently with the radiative transfer equation, since \( R_{ji} \) contains the mean intensity \( J_s \). In two situations this programme must result in LTE occupation numbers:

1. The particle velocity distribution is Maxwellian and the density is so high that every \( R \) is collision-dominated (i.e. insensitive to replacement of \( B_s \) by \( J_s \)).

2. The particle velocity distribution is Maxwellian and

\[
\int J_s \phi_{s \mu}(ij) \, dv \approx \int B_s(T) \phi_{s \mu}(ij) \, dv
\]

for every transition, regardless of the density.

The first possibility is the one usually invoked to justify LTE a priori, but it does not hold in most stellar atmospheres because of the low densities prevailing. Indeed in the solar corona and in many thermonuclear terrestrial plasmas a quite different type of equilibrium ("coronal equilibrium") holds for most important transitions: \( R_{ji} \approx A_{ji} \) for downward and \( R_{ji} \approx C_{ji} N \bar{v} \) for upward transitions.
The second LTE situation can be had in two ways: $J_v \approx B_v$ at all frequencies (this is slightly less stringent than the Hohlraum condition, $I_v = B_v$, in that $I_v$ is still permitted to be anisotropic), which is satisfied at large optical depths in stellar interiors; or a “fortuitous” radiation field which satisfies $J_v \approx B_v$ in the most important transitions but not at all frequencies. There is actually some indication (Lecar 1964) that such a fortuitous $J_v$ may occur because of the constraint of radiative equilibrium in stellar atmospheres, but the problem needs much more investigation. If this were the case it would explain the at least partial success of LTE stellar-atmosphere models in predicting observed spectra.

To treat non-LTE problems in practice, one truncates the set of rate equations and assumes a “partial LTE” of the omitted equations. A few (e.g. 2, 5, or 10) bound levels are treated explicitly. These are ordinarily the ground and low excited states, which usually have large radiative transition probabilities and relatively small collision cross-sections and so are the first to go out of LTE as the density is lowered. The very loosely bound states and the free states are thought to remain in LTE (i.e to have a Maxwellian velocity distribution) even at interstellar densities; except perhaps when strong electric or magnetic fields are present.

Without LTE the set of equations to solve is clearly so complicated that analytic results can hardly be obtained for more than, say, a two-level model of the atom or a very simple problem such as an isothermal layer (“ Isothermal” in the non-LTE context refers to an assumed kinetic temperature). Much early non-LTE work dealt with the excitation conditions in nebulae (Menzel 1962); more recent calculations have been in order to construct models of the outer solar atmosphere, that is, the so-called chromosphere and corona (Thomas and Athay 1961). These layers are optically thin except at the centres of the strongest spectral lines, so that most of the radiation field comes from the deeper-lying photosphere and is known in advance. Only very recently have self-consistent solutions of the full set of non-LTE equations for stellar photospheres been attempted, using large digital computers (Harvard-Smithsonian 1964, 1965); a comprehensive discussion of the physics of non-LTE transfer problems has been given by Thomas (1965).

We note in passing that one special case of the non-LTE equations is actually as simple as the LTE case. This is the case of resonance radiation (e.g. Holstein 1947, 1951). If essentially all the radiation travels in one spectral line, then only two atomic levels are involved and a local “excitation temperature” $T_{ex}$ can be defined by the ratio of their populations. In terms of this temperature, Eqn (7) becomes (with $u = \hbar \nu/kT_{ex}$)

$$\left[ \Omega \cdot \nabla + \frac{\phi_i B_{ji} \hbar \nu_i N_i(1 - e^{-u})}{4\pi} \right] J_v = \phi_v B_{v0} \hbar \nu_v B_v(T_{ex})(1 - e^{-u})/4\pi$$
or
\[ \Omega \cdot \nabla I_\nu = \rho \kappa'_\nu [-I_\nu + B_\nu(T_\nu)] \]  \hspace{1cm} (25)

with
\[ \rho \kappa'_\nu = N_\nu B_\nu \hbar v_\nu \phi_\nu (1 - e^{-\gamma})/4\pi \]

This resonance-line transport problem is of some practical interest in astronomy. A gaseous nebula is a low-density cloud of interstellar gas—mostly hydrogen—illuminated by a hot star which radiates strongly in the ultraviolet, beyond the Lyman limit—i.e. dilute ionizing radiation. On recombination to an excited state, spontaneous decay to the ground state occurs by a cascade process whose last step is the emission of a Lyman line \( n \to 1 \). The absorption coefficient for Lyman-line radiation is large since most hydrogen atoms will be in the ground state, so the Lyman line photons cannot escape readily and we have "trapping of resonance radiation". The strongest Lyman line is Ly\( \alpha \), \( 2 \to 1 \), and the problem can be fairly well approximated by keeping only this line in the treatment. Because of the profile \( \phi_\nu \), which is approximately of Doppler shape, the transfer problem is highly non-grey. Some early attempts to use a grey approximation (rectangular profile) failed because actually most of the photons which escape do so in the wings of the line.

V. NEUTRON COUNTERPART OF THE LTE HYPOTHESIS

The LTE radiative transfer equation together with the constraint of radiative equilibrium (energy conservation) can be written in a form showing close similarity to the energy-dependent neutron problem with an isotropic separable scattering kernel (Corngold et al., 1963). This is to be expected, since part of the essence of LTE is that the emission spectrum of a region depends only on its temperature and absorption coefficient, and not on the spectrum of the radiation it absorbs.

The (steady-state) LTE transfer equation is
\[ [\Omega \cdot \nabla + \rho \kappa_\nu] I_\nu = \rho \kappa_\nu B_\nu \]  \hspace{1cm} (26)

(we have dropped the prime on \( \kappa_\nu \)) and the condition of radiative equilibrium is
\[ \int J_{\nu'} \kappa_\nu' \, dv' = \int B_{\nu'} \kappa_\nu' \, dv' \]  \hspace{1cm} (27)

Recalling the definition of \( J_\nu \) (Eqn 12), we can combine (26) and (27) in the form
\[ [\Omega \cdot \nabla + \rho \kappa_\nu] I_\nu(r, \Omega) = \frac{\rho}{4\pi} \int d\Omega' \int dv' K(\Omega' \to \Omega, \nu' \to \nu) I_\nu'(r, \Omega') \]  \hspace{1cm} (28)
where

\[ K(\Omega' \to \Omega, \nu' \to \nu) = \frac{B_{\nu', \nu'} \kappa_{\nu'}}{\int B_{\nu', \nu'} \, d\nu'} \quad (29) \]

This is to be compared with the neutron kernel of Corngold,

\[ K(\Omega' \to \Omega, E' \to E) = \frac{M(E)\Sigma(E)\Sigma(E')}{\int M(E'')\Sigma(E'') \, dE''} \quad (30) \]

where \( M \) is the Maxwellian. The essential difference between (29) and (30) is the fact that in the photon case the temperature is determined by (27) and depends on position, so that the spectral shape of \( B_{\nu} \) (and ordinarily \( \kappa_{\nu} \), also) changes with position; in the usual neutron problems \( M(E) \) and \( \Sigma(E) \) are independent of position except possibly for discontinuities. The neutron counterpart of (29) would have a neutron-flux-dependent moderator temperature and temperature-dependent cross sections.

As will be seen later, it is possible to formulate an LTE photon problem which is mathematically equivalent to (30) at the price of an assumed restriction on the form of \( \kappa_{\nu} \). It is possible that some very special non-LTE problems can also be so formulated. Such problems can be solved as completely as the grey case.†

If the photon problem includes some scattering, the neutron analogue is obvious; as mentioned earlier, seldom is any save elastic scattering used in photon problems.

VI. SOME REPRESENTATIVE RADIATIVE-TRANSFER PROBLEMS

In this section we formulate a few problems in radiative transfer which have more or less closely analogous neutron counterparts:

A. The classical stellar atmosphere in radiative equilibrium—the conservative Milne problem.


C. Trapping of resonance radiation in a gas—a slab criticality problem.

D. Sunlight at large depths in the ocean—asymptotic neutron distribution in an infinite subcritical medium.

E. Radiative shock wave structure—two adjacent half-spaces with sources.

For simplicity, all of these will be treated in slab geometry. We will need the time-independent transfer equation, Eqn (18), and the integral expressions for specific intensity, mean intensity, and flux (= \( \pi F \)).
\[
I_\ast(\tau_\ast, \mu) = \begin{cases} 
\int_{\tau_\ast}^{\infty} S_\ast(\tau_\ast', \mu) e^{-\kappa_\ast(\tau_\ast - \tau_\ast')/\mu} \, d\tau_\ast'/\mu & (\mu > 0) \\
\int_{-\infty}^{\tau_\ast} S_\ast(\tau_\ast', \mu) e^{-\kappa_\ast(\tau_\ast - \tau_\ast')/|\mu|} \, d\tau_\ast'/|\mu| & (\mu < 0) 
\end{cases}
\] (31)

\[
J_\ast(\tau_\ast) = \begin{cases} 
\frac{1}{2} \int_{\rho}^{\infty} I_\ast(\tau_\ast, \mu) \, d\mu & \text{(in general)} \\
\frac{1}{2} \int_{-\infty}^{\infty} S_\ast(\tau_\ast') E_1(|\tau_\ast - \tau_\ast'|) \, d\tau_\ast' & \text{(isotropic } S_\ast) 
\end{cases}
\] (32)

\[
F_\ast(\tau_\ast) = \begin{cases} 
2 \int_{\rho}^{\infty} \mu I_\ast(\tau_\ast, \mu) \, d\mu & \text{(in general)} \\
2 \int_{-\infty}^{\infty} S_\ast(\tau_\ast') E_2(|\tau_\ast - \tau_\ast'|) \sgn(\tau_\ast - \tau_\ast') \, d\tau_\ast' & \text{(isotropic } S_\ast) 
\end{cases}
\] (33)

where the integrals have been extended over all space so as to include all possible radiative sources.

A. RADIATIVE-EQUILIBRIUM STELLAR ATMOSPHERES

The classical model stellar atmosphere is defined by several equilibrium equations, the transfer equation, and an equation for the absorption coefficient. In Section IV the non-LTE approach was outlined; here LTE will be used for convenience. We use \( m \) as independent variable; \( dm = \rho \, ds \) and \( d\tau_\ast = \kappa_\ast \, dm \).

**Radiative transfer:** \( J_\ast(m) = \frac{1}{2} \int_{0}^{\infty} B_\ast(T(m')) E_1(|\tau_\ast - \tau_\ast'|) \kappa_\ast(m') \, dm' \) (34)

**Radiative equilibrium:** \( \int_{0}^{\infty} \kappa_\ast(m) [J_\ast(m) - B_\ast(T(m))] \, dv = 0 \) (35)

**Hydrostatic equilibrium:** \( P(m) = gm \) (\( g = \) gravitational acceleration) (36)

**Absorption coefficient:** \( \kappa_\ast = \kappa_\ast(P, T) \) (37)

Two parameters, which may be taken as \( g \) and \( T_{\text{eff}} \), specify a model if the chemical composition is known. Equations (34) and (35) may be replaced by Eqn (33) for the flux and the statement

\[
F(m) = \int_{0}^{\infty} dvF_\ast(m) = \text{constant} = B(T_{\text{eff}})
\]

To obtain the model, i.e. \( T(m) \), the equations have to be solved self-consistently by iteration. Afterwards the emergent intensity can be found from Eqn (31) with \( \tau_\ast = 0 \).
To make the solution easier, one sometimes makes one of a number of possible assumptions about the dependence of \( \kappa \) on \( v, P \), and \( T \). For example:

1. If \( \kappa(P, T) = K(P, T) \cdot X_v(T) \), let \( K \, \text{d}m = \text{d}r / X_{\text{rel}}(T) \)

Then the hydrostatic equation splits off and the radiative equations can be solved (numerically) for \( T(\tau) \). This case is actually correct for pure hydrogen (except for the pressure broadening of the line profiles), and has been treated by Mustel (1940; see also Ambarzumian 1958) with further simplifying approximations.

2. If in addition \( X_v \) is independent of \( T \), we have the “Milne–Eddington model” absorption coefficient. The radiative equations can then be written in terms of \( X \) rather than \( v \) (Stewart 1964) and lead to the integral equation

\[
B(\tau) = \int_0^\infty \, \text{d}r' B(\tau') \frac{1}{2} \kappa_v(\tau, \tau')
\]

where

\[
\kappa_v(\tau, \tau') = \int_0^{l_{\text{max}}} \, \text{d}l \frac{P(\tau', l)}{l^2} \cdot E(l) \left( \frac{\tau - \tau'}{\tau} \right)
\]

(note that this is not a displacement kernel, or even symmetric) and the “distribution function for mean free paths”, \( P(\tau, l) \), is defined by the condition

\[
\int_0^{l_{\text{max}}} P(\tau, l) f(l) \, \text{d}l = \int_0^\infty B_v(T(\tau)) f(l_v) \, \text{d}v / B(T(\tau))
\]

for any function \( f \). (Here \( l_v = X_{\text{rel}} / X_v \), and \( B = \int_0^\infty B_v \, \text{d}v \).) The purpose of making the transformation is that \( P(\tau, l) \) is likely to be more manageable than \( J_v(\tau) \) if there are many lines in the spectrum. The function \( P(\tau, l) \) depends on \( \tau \) only through the temperature, \( T(\tau) \).

Even this case, it seems, cannot be solved analytically for the Milne problem. Its purpose (in the original \( v \)-dependent form) was to calculate absorption-line intensities in a known temperature field (Eddington 1926).

3. If in addition \( P(\tau, l) \) is independent of \( \tau \), a completely soluble but still non-grey Milne problem results. The kernel (Eqn 39) is now of displacement type. The function \( P(l) \) is determined by the model chosen for the frequency dependence of the absorption coefficient; e.g. the grey model corresponds to a single delta-function, the “picket-fence” model (Chandrasekhar 1935) to two delta-functions, etc. The approximation that \( P \) is independent of \( \tau \) means physically that even though the Planck spectrum shifts to higher frequencies as the temperature rises, the fraction of the spectrum allocated to
each value of the (properly normalized) mean free path does not change. The effect on the radiative-equilibrium condition is then the same as if no spectral shift had occurred. This would indeed be the case if absorption lines were distributed uniformly across the spectrum, so following Unno (1962) we may call this model the "uniform-line-blanking" (ULB) model. It is mathematically identical with the neutron problem whose kernel is given by Eqn (30). The solution of the Milne problem for the ULB model is considered in Section VII.

(4) Finally, the familiar grey model is recovered for \( \kappa \), independent of \( \nu \). The transfer and radiative-equilibrium equations can be immediately integrated over frequency and give the Schwarzschild–Milne integral equation

\[
B(\tau) = \frac{1}{2} \int_{0}^{\infty} B(\tau')E_{1}(|\tau - \tau'|) \, d\tau'
\]

whose solution by numerous approximate and exact methods is the subject of most of Kourganoff’s (1952) treatise.

The Milne problem, in its various levels of approximation, has received preponderant attention among radiative-transfer problems in the astrophysical literature, since it provides a connection between the basic physical parameters of a star—mass \( M \), radius \( R \), luminosity \( L \) (total radiated power), chemical composition—and the emergent starlight which is observed spectroscopically in such impressive detail. The atmospheric parameters \( g \) and \( T_{\text{eff}} \) are related to \( M \), \( R \), and \( L \) as follows:

\[
g = GM/R^2 \quad (G = \text{gravitational constant}) \quad (42)
\]

\[
\pi B(T_{\text{eff}}) = L/4\pi R^2 \quad (43)
\]

and the theory of stellar structure and evolution (e.g. Schwarzschild 1958) provides, in principle at least, \( M \), \( L \), and \( R \) as functions of the star’s age, initial chemical composition, and initial mass. For chemically homogeneous (thoroughly-mixed) stars in the steady state, with \( L \) equal to the thermonuclear power generated in the star, \( L \) and \( R \) depend only on the present mass and composition; if all stars were chemically alike and homogeneous, or if the composition depended only on the mass, a single parameter (e.g. mass) would then fix \( g \) and \( T_{\text{eff}} \), and all stellar spectra would lie on a single curve in the \((g, T_{\text{eff}})\) plane. The observational fact that they do not establishes the existence of differing chemical composition even among stars of the same mass. However, the spectra of stars belonging to a single cluster do lie approximately on a single curve and are consistent with the view that all the stars in the cluster were formed at about the same time from a chemically homogeneous cloud.
Of course, many parameters besides $g$ and $T_{\text{eff}}$ can be inferred by matching
an observed stellar spectrum with a calculated one; the relative abundances
of the elements in a stellar atmosphere reveal themselves through the
strengths of the spectral lines. The observed line strengths (and profiles)
are related to the abundances in a rather complicated way, which is sensitive
to the structure of the model atmosphere because the degree of excitation of
each atomic level depends on depth. The observed spectrum of a star
ordinarily consists of the intensity averaged over the visible hemisphere,
i.e. $F_\nu(0)$, which according to Eqn (33) involves the source function over a
range of depths. For the Milne problem $S_\nu(\tau_\nu') = 0$ for $\tau_\nu' < 0$, and the
important depths for $F_\nu(0)$ are those for which $\tau_\nu'$ is of the order of unity.
In the case of the sun, the distribution of intensity across the visible disk
(the "limb-darkening curve"), $I_\nu(0, \mu)$, can also be observed, and since (for
isotropic $S_\nu$) this is essentially the Laplace transform of $S_\nu(\tau_\nu')$ some empirical
information about the depth dependence of $S_\nu$ can be obtained by "inverting"
the transform. The "inversion" remains partly ambiguous since the observa-
tions are of finite precision and scan only a finite range of the transform
variable $1/\mu$ (Böhm 1961).

The basic assumptions underlying the "classical" model stellar atmo-
sphere—steady-state radiative equilibrium, hydrostatic equilibrium, slab
geometry (i.e. thickness of observable atmosphere $\ll$ stellar radius), spherical
symmetry, and usually LTE—are all violated to some extent by actual
stars. Thus, for example, convective instability anywhere in the star leads to
non-radiative energy transfer. Even if the convectively unstable region is
situated at unobservably large optical depths, it can generate acoustic noise
which propagates outward into the visible atmosphere and dissipates energy
(through shock-wave formation) in the tenuous outer layers, invalidating
radiative equilibrium. Some stars have atmospheres so distended as to
require curvature to be considered, and some rotate so rapidly that $g$ varies
with latitude. The well-known solar granulation is a small-scale deviation
from spherical symmetry, and the chromosphere, corona, and solar wind
do not fit into the "classical" model atmosphere at all. As remarked earlier,
LTE via collisions is untenable in most stellar atmospheres. These various
complications are still a long way from satisfactory theoretical treat-
ment, and their effect on the derived stellar abundances is an unsettled
question.

Detailed treatises on the analysis of stellar atmospheres, mostly by
"classical" methods, have been published by Unsöld (1955) and Aller (1963);
for a review and bibliography of recent work in model atmospheres, see
Pecker (1965). Other sources are Barbier's (1958) Handbuch article and
Greenstein's (1960) compendium.
B. PLANETARY ATMOSPHERES

The radiative-transfer problem in the atmosphere of a planet illuminated by the sun is analogous to the neutron "slab-albedo" problem except for the usual complication that the optical properties of the medium depend on position. Actually two nearly-uncoupled albedo problems arise, since the radiation field in the atmosphere can be split into the visible and near-visible radiation (whose source is direct and reflected sunlight) and the far-infra-red radiation (whose source is nearly all thermal emission by the planetary surface). Qualitatively, the earth's atmosphere scatters visible light but absorbs and emits infra-red. The scattering is partly Rayleigh scattering by molecules and partly Mie scattering by larger aggregates such as water droplets and dust particles, and so is markedly anisotropic. The infra-red absorption is due to excitation of vibration and rotation of molecules (especially water vapour and carbon dioxide), and the band structure of the absorption coefficient is highly non-grey.

The visible transfer problem can be treated monochromatically since the scattering is, to good approximation, elastic. To the extent that the temperature profile of the atmosphere is controlled by the infra-red, the optical properties for visible light are independent of the visible radiation field and we get a linear transfer problem. Letting

\[ \sigma(\tau, \mu_1) \, d\mu_1 = \text{cross-section, at depth } \tau \text{, for scattering through angle } \arccos \mu_1 \text{ into } d\mu_1 \]

\[ f(\tau, \mu, \mu') = \text{normalized scattering kernel} \]

\[ \int_0^\infty d\phi \sigma(\tau, \mu_1 \mu + \sqrt{1 - \mu^2} \sqrt{1 - \mu'^2} \cos \phi) \]

\[ \pi \int_{-1}^1 \sigma(\tau, \mu_1) \, d\mu_1 \]

\[ \mu = \text{direction cosine with respect to outward normal} \]

\[ \tau = \text{optical depth measured downward into atmosphere} \]

\[ I(\tau, \mu) = \text{azimuth-averaged intensity excluding direct (unscattered) solar beam} \]

we have as the transfer equation

\[ \mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \int_{-1}^1 d\mu' I(\tau, \mu') f(\tau, \mu', \mu) - I_0 e^{-\mu_0 \sigma I(\tau, -\mu_0, \mu)} \]  

where the last term represents the contribution of solar rays scattered for the first time at depth \( \tau \). The direct solar beam has been taken as collimated; i.e. outside the atmosphere \( I(\text{direct}) = I_0 \delta(\mu + \mu_0) \), where \( \mu_0 \) is the cosine of the sun's zenith angle.
The boundary condition at the top of the atmosphere is \( I(0, \mu) = 0 \) for \( \mu < 0 \) (i.e. no incident radiation except the direct solar beam); at the planetary surface (\( \tau = \tau_1 \)) the boundary condition depends on the reflecting properties of the ground. Here an adequate approximation (except for water surfaces) is Lambert’s law for diffuse reflection, i.e. the reflected intensity is isotropic in the upward hemisphere and its flux is some definite fraction \( a \) (the ground “albedo”) of the flux incident on the ground:

\[
I(\tau_1, \mu) = 2a \left[ \int_0^\tau I(\tau, -\mu') \mu' \, d\mu' + I_0 \mu_0 e^{-\tau_1/\mu_0} \right] \quad (\mu > 0)
\]  

(45)

With this boundary condition a further separation can be made: the intensity \( I(\tau, \mu) \) consists of a part \( I_1(\tau, \mu) \) which has never been reflected from the ground, plus a part \( I_2(\tau, \mu) \) which has been reflected at least once. \( I_1 \) satisfies the above transfer equation (44) and a ground condition of zero albedo. \( I_2 \) satisfies the transfer equation with \( I_0 = 0 \) and a ground condition \( I_2(\tau_1, \mu) = A = \text{constant} (\mu > 0) \). Both \( I_1 \) and \( I_2 \) satisfy the previous boundary condition at \( \tau = 0 \). Thus we have two independent problems—evaluation of \( i_1 = I_1(\tau, \mu)/I_0 \) and evaluation of \( i_2 = I_2(\tau, \mu)/A \)—and neither one involves the ground albedo. After both are solved, the original ground condition is invoked to evaluate \( A \):

\[
\frac{A}{I_0} = \frac{2a \left[ \int_0^\tau i_1(\tau, -\mu') \mu' \, d\mu' + \mu_0 e^{-\tau_1/\mu_0} \right]}{1 - 2a \int_0^\tau i_2(\tau, -\mu') \mu' \, d\mu'}
\]  

(46)

The evaluation of \( i_1 \) or \( i_2 \), it will be recognized, is a “standard” one-group transfer problem—the finite-thickness slab, bounded by vacuum, with anisotropic scattering, exposed to (respectively) a collimated beam or an isotropic incident distribution on one side. This reduction of the original problem to “standard” problems, of course, rests on the linearity of the transfer equation, and would no longer be possible if we allowed, for example, heating of the atmosphere by visible light and a resulting effect on the visible optical properties.

Turning now to the far-infra-red transfer problem, we can obtain an approximate picture of the temperature structure and radiation field in the atmosphere by assuming radiative and hydrostatic equilibrium. LTE, no direct solar radiation, and a ground which emits black body radiation. (A realistic atmospheric model demands consideration of convective heat transfer near the ground and direct solar heating at high altitudes, among other corrections.) The relevant equations are then Eqns (34–37) for \( 0 < m < m_1 \) and the statement \( T(m') = T_0 \) for \( m' > m_1 \) (this expresses the
ground condition. Here $m_1$ is the total mass per unit area of the atmosphere and $T_g$ is the ground temperature. The main source of difficulty is the absorption coefficient $\kappa_1$, which consists almost entirely of a multitude of molecular absorption lines which occur in clusters known as "bands". The absorption coefficient is very much smaller between bands than at line centres, so the atmosphere is optically thin at some frequencies and thick at others. The remarks of Section VI-A on assumed forms for $\kappa_1$ apply to the present problem; the separation of variables employed there is only approximate, since the line profiles are pressure-broadened.

Numerous "band models" (Goody 1964) have been proposed as representations for the complicated molecular absorption coefficient. The basic idea is to replace the actual array of absorption lines in a frequency interval by an idealized array which has similar statistical properties; within the interval the absorption coefficient is evaluated as if all other intervals had these properties also. To specify a model one starts with distribution functions for line shapes, strengths, and spacings, and calculates the mean transmission function $\theta(m)$ for the frequency interval in question:

$$\theta(m) = \frac{1}{\Delta \nu} \int \Delta \nu e^{-m \nu} = e^{-m \kappa_1 (m)}$$

(47)

where an average of $\theta$ with respect to the distribution functions is implied. The transmission mean opacity $\kappa_1(m)$ here is a "local" average. The transmission function can also be regarded as the Laplace transform of a probability distribution $P(\kappa)$ whose first moment is the local Planck mean and whose minus-first moment is the local reciprocal Rosseland mean. In the ULB model, the same $\theta(m)$ applies to all frequency intervals.

The two band models which have found widest use both employ Lorentz line profiles of identical half-width. In Elsasser's regular model, the lines are uniformly spaced and have identical strengths; in the Mayer–Jacobsohn–Goody random model, both spacings and strengths are exponentially distributed. In either model the ratio of line half-width to mean line spacing governs the shape of $\theta(m)$. A particularly simple result emerges if this ratio becomes vanishingly small in the random model: letting $m \kappa_R = t = \text{Rosseland optical depth}$, we get

$$\theta = e^{-\sqrt{t}}$$

(48)

in contrast to the grey result $\theta = e^{-t}$. The Elsasser model, in the same narrow-line limit, yields a transmission function intermediate in shape between the grey and random models.

Any of the band models may be used to construct a ULB model; the connection is established by expressing the transmission in terms of optical
depth (Rosseland units are usually convenient), using the same $\theta(t)$ throughout, and constructing the kernel of (e.g.) the flux equation (33) from $\theta(t)$. In this way we find after a little manipulation:

$$ F(t) = 2 \int_{-\infty}^{\infty} B(\tau') e_2(t') \, d\tau' $$

where

$$ e_2(t) = -t \int_{t'}^{\infty} \frac{d\theta(x)}{x^2} $$

with all optical depths in Rosseland units. The kernel $e_2(t)$ reduces to the second exponential integral $E_2(t)$ in the grey case; for the narrow-line limit of the random model it becomes

$$ e_2(t) = 2E_4(\sqrt{2t})/\sqrt{2t} $$

The planetary infra-red problem has been considered by King (1955, 1956) for a model equivalent to the ULB assumption: he obtained solutions for the source function and emergent intensity in a discrete-ordinates approximation applied to the Elsasser band model. Some identities in the second of these papers are based on a postulated exact solution to the grey finite-atmosphere problem (Yamamoto 1955), which was subsequently shown by Sobolev (1962) to be incorrect.

C. IMPRISONMENT OF RESONANCE-LINE RADIATION

Since the mean number of photons emerging from a photon-atom collision is less than unity unless the laser condition exists, and since induced emission propagates in the same direction as the inducing radiation, there is at first sight no photon process closely analogous to the multiplication of neutrons with isotropic emission of secondaries. Accordingly in most of the radiative-transfer literature one finds only the cases $\omega_0 \ll 1$ treated ($\omega_0 =$ mean number of secondaries per collision); then multiple-scattering expansions (Neumann-series solutions of the integral transfer equation) always converge and critical-size problems do not occur.

However, time-dependent radiative problems do occur, and sometimes lead to "equivalent" time-independent problems with $\omega_0 > 1$, hence to criticality problems. We illustrate with a simple example: the decay of excitation in a gas which is optically thick at the centre of the resonance line (Holstein 1947, 1951). We consider only the ground state 1 and first excited state 2 to have appreciable populations, and further take $N_2 \ll N_1$ so that induced emission is negligible. We also neglect collisional excitation and deexcitation, and photon flight times. Then for slab geometry the
time-dependent transfer equation (7) and the rate equation (23) become
\[
\left[ \mu \frac{\partial}{\partial x} + \phi_v h v N_1 B_{12} / 4\pi \right] I_v(x, \mu, t) = \phi_v h v N_2 A_{21} / 4\pi
\]  
(52)
and
\[
\frac{\partial N_2}{\partial t} = -N_2 A_{21} + N_1 B_{12} \int J_v \phi_v \, dv
\]  
(53)
from which we get, on inserting time dependence \( e^{-\alpha t} \), letting \( \phi_v / \phi_{\text{max}} = g_v \), \( \phi_{\text{max}} h v N_1 (B_{12} / 4\pi) \, dx = \alpha \), \( N_2 A_{21} / N_1 B_{12} = S \), and using (32),
\[
\left( 1 - \frac{\alpha}{A_{21}} \right) S(\tau) = \frac{1}{\alpha} \int_0^{\tau_1} S(\tau') \left[ \frac{dv g_v^2 E_1(|\tau - \tau'| g_v)}{\int dv g_v} \right] \, d\tau'
\]  
(54)
This is of a form closely similar to the integral equation of radiative equilibrium in the ULB model; the differences are the presence of the factor \( [1 - (\alpha / A_{21})] \) and the finite value of \( \tau_1 \). Apart from the non-greyness of the kernel, this is precisely the one-velocity criticality condition for a bare slab reactor: the smallest value of \( \tau_1 \) for which a solution exists is the critical optical thickness of a slab of material which yields \( [1 - (\alpha / A_{21})]^{-1} \) secondaries per collision. In the resonance-imprisonment context, it relates the decay rate \( \alpha \) to the optical thickness at the line centre \( \tau_1 \). Clearly as \( \tau_1 \to 0 \), \( \alpha \to A_{21} \), which is the unhindered spontaneous decay rate. For large \( \tau_1 \) it may be anticipated that \( \alpha \ll A_{21} \).

If \( g_v \) were a rectangular profile, diffusion theory would yield for \( \tau_1 \gg 1 \):
\[
\frac{\alpha}{A_{21}} \approx \frac{1}{3} \left( \frac{\pi}{\tau_1} \right)^2 \quad \text{(rectangular)}
\]  
(55)
On the other hand, for any realistic line profile, \( g_v \tau_1 \ll 1 \) at some frequency in the line wing; photons emitted near this frequency escape readily, producing major changes in the resulting decay rate. Diffusion theory is inapplicable; Holstein used the Rayleigh–Ritz variational principle and obtained for \( \tau_1 \gg 1 \):
\[
\frac{\alpha}{A_{21}} \approx \frac{1.875}{\tau_1 \sqrt{\pi \ln \tau_1}} \quad \text{(Doppler)}
\]  
(56)
\[
\frac{\alpha}{A_{21}} \approx \frac{1.150}{\sqrt{\pi \tau_1}} \quad \text{(Lorentz)}
\]  
(57)
Physically, this means a photon does not random-walk out of the slab, but gets absorbed and re-emitted until it happens to be emitted at a frequency for which the slab is not optically thick. The ratio \( \tau_1 \ll 1 \) may be regarded as the mean number of emission processes per escaping photon.
D. Solar Illumination of the Deep Sea

The distribution of solar radiation under the surface of the ocean poses a rather novel problem in radiative transfer, owing to the extreme anisotropy of the scattering function and the fact that it is possible to make quite detailed measurements of the intensity as a function of direction and depth. The scattering function for small scattering angles is difficult to measure accurately, however, simply because it is so large and changes so rapidly with angle near the forward direction.

A survey of the relevant experimental data on solar and artificial underwater illumination has been given by Duntley (1963). The optical properties of natural water are characterized by an elastic scattering cross-section, sensibly independent of frequency, and an absorption cross-section which has a broad minimum in the blue or green part of the spectrum. Coupling between different visible frequencies (e.g. fluorescence) is ordinarily too small to measure; the absorption cross-section represents conversion of visible light to thermal and chemical energy. Accordingly the transfer problem for each frequency is a one-group problem. We will consider here only the asymptotic behaviour of the radiation field at large depths in optically uniform water; this problem formally resembles the determination of the neutron distribution produced by a distant source in an infinite scattering and absorbing medium. Thus at large depths $I(z, \mu) \rightarrow I(\mu) e^{-zL}$, with $z = \text{depth below surface}$ and $L = \text{diffusion length}$; the angular distribution $I(\mu)$ must satisfy the integral equation

$$
\left( \sigma_s + \frac{\mu}{L} \sigma_a \right) I(\mu) = \int_{-1}^{1} \sigma(\mu, \mu') I(\mu') d\mu'
$$

(58)

where $\mu = \text{direction cosine with respect to the downward vertical direction}$;

$\sigma(\mu_0) d\mu_0 = \text{differential cross-section for scattering through angle } \arccos \mu_0$;

$$
\sigma(\mu, \mu') = \frac{1}{\pi} \int_0^\pi \; d\phi \; \sigma(\mu \mu' + \sqrt{1 - \mu^2} \sqrt{1 - \mu'^2} \cos \phi) = \sigma(\mu', \mu);
$$

$$
\sigma_s = \int_{-1}^{1} \sigma(\mu_0) d\mu_0 = \int_{-1}^{1} \sigma(\mu, \mu') d\mu' = \text{total scattering cross-section};
$$

$\sigma_a = \text{absorption cross section};$

and all cross-sections are "macroscopic", i.e. have dimensions of reciprocal length.

In Duntley's paper, experimental measurements of $\sigma(\mu_0)$ for natural waters are compiled from numerous sources. All show very pronounced peaking in the forward direction, i.e. $\sigma(\mu_0)$ changes by several orders of
magnitude between, say, \( \mu_0 = 0 \) and \( \mu \) near 1. It is not clear from the data that \( \sigma_\mu \) is even finite. The usual methods (e.g. Davison 1957) for solving Eqn (58) suppose that \( \sigma(\mu_0) \) can be adequately represented by a few terms of its Legendre-polynomial expansion, a condition which is manifestly violated in the present case. Measurements of \( I(\mu) \) quoted by Duntley, though highly anisotropic, are much better behaved, and suggest that one might try to treat Eqn (58) as an "inverse" problem and obtain \( \sigma(\mu_0) \). It should be noted that a numerical solution of the "direct" problem for all depths has been carried out by Preisendorfer (1965), using a discrete grid in both depth and direction; since the intensity then depends nonseparably on depth and azimuth as well as on \( \mu \), a rather coarse grid had to be used and the asymptotic \( I(\mu) \) did not agree very well with the measured values.) Here we outline a possible approach (Stewart, unpublished) to this inverse problem.

We start by defining \( 1/k = L\sigma_\mu \) = diffusion length/absorption mean free path:

\[
\frac{\sigma(\mu_0)}{\sigma_*} = F(\mu_0)
\]

\[
\frac{\sigma(\mu, \mu')}{\sigma_*} = F(\mu, \mu')
\]

and transferring the term \( \sigma_\mu I(\mu) \) to the right-hand side of Eqn (58), to obtain

\[
(1 - k\mu)I(\mu) = \int_{-1}^{1} F(\mu, \mu')[I(\mu') - I(\mu)] d\mu'
\]

which is to be solved for \( F(\mu_0) = F(1, \mu_0) \) and \( k \), given \( I(\mu) \). We must have physically \( I(\mu) > 0 \), \( F(\mu_0) > 0 \), and we assume that the Legendre expansion of \( I(\mu) \) exists:

\[
I(\mu) = \sum_{n=0}^{\infty} \frac{2n + 1}{2} A_n P_n(\mu); \quad A_n = \int_{-1}^{1} I(\mu)P_n(\mu) d\mu
\]

Writing formally a corresponding expansion for \( F(\mu_0) \), we have

\[
F(\mu_0) = \sum_{n=0}^{\infty} \frac{2n + 1}{2} F_n P_n(\mu_0); \quad F_n = \int_{-1}^{1} F(\mu)P_n(\mu) d\mu
\]

\[
F(\mu, \mu') = \sum_{n=0}^{\infty} \frac{2n + 1}{2} F_n P_n(\mu)P_n(\mu')
\]

and if we insert (60) and (62) into (59), we get

\[
1 - kg_n = F_n - F_0
\]

where

\[
g_n = \frac{\int_{-1}^{1} I(\mu)P_n(\mu) d\mu}{\int_{-1}^{1} I(\mu)P_n(\mu) d\mu} = \frac{(n + 1)A_{n+1} + nA_{n-1}}{(2n + 1)A_n}
\]
To get \( F(\mu_0) \), then, we evaluate \( A_n \) from the given \( I(\mu) \), obtain \( F_n - F_0 \) from (63), and sum the series (61) (This leaves \( F(\mu_0) \) indeterminate to within an additive \( \delta \)-function at \( \mu_0 = 1 \).) As a workable procedure this leaves much to be desired, since we already know \( F(\mu_0) \) will require a very long series, and the numerical evaluation of the integrals \( A_n \) for large \( n \) will present computational troubles due to the rapidly oscillating integrand. We employ two tricks:

(1) Represent

\[
I(\mu) = \sum B_m e^{\mu m}
\]

Then

\[
A_n = \sum B_m f_n(\beta_m)
\]

where

\[
f_n(\beta) = \sqrt{2\pi/\beta} I_{n+1/2}(\beta);
\]

\( I_{n+1/2}(\beta) \) is the modified Bessel function of the first kind. It is actually an elementary function, but for large \( n \) it is again awkward to compute. The Bessel-function recurrence relation

\[
f_{n-1}(\beta) = f_{n+1}(\beta) + \frac{2n+1}{\beta} f_n(\beta)
\]  

(65)

or its nonlinear two-term version

\[
\frac{1}{R_{n-1}} = R_n + \frac{2n+1}{\beta} \quad R_n = \frac{f_{n+1}(\beta)}{f_n(\beta)}
\]  

(66)

provides a very convenient way to obtain the \( f_n \); it is numerically stable if one starts at large \( n \) and works down. The asymptotic (large \( n \)) behaviour

\[
R_n \approx \frac{\beta}{2n+3} - O\left(\frac{\beta^3}{n^3}\right)
\]

or merely the bounds \( 0 < R_n < \infty \), can be used to get started. The expansion for \( I(\mu) \) as a sum of exponential functions is well suited to the experimental data presented by Duntley; two terms (with four adjustable parameters \( B_1, B_2, \beta_1, \beta_2 \)) provide a fit to within the graphical precision of the data.

(2) At large \( n \), it turns out that \( F_n \sim n \) and so the formal series for \( F(\mu_0) \) cannot be summed numerically. However, the observation (Yennie, Ravenhall and Wilson, 1954)

\[
(1 - \mu_n)F(\mu_n) = \sum \frac{2n+1}{\alpha} G_n P_n(\mu_n)
\]  

(67)
where
\[ G_n = F_n - \frac{(n + 1)F_{n+1} + nF_{n-1}}{2n + 1} \] (68)
yields a series in which \( G_n \sim 1/n \). This converges except at \( \mu_0 = 1 \), but it is convenient to subtract from it an appropriate multiple of the series
\[ \sum_{n=0}^{\infty} P_n(\mu_0) = \frac{1}{\sqrt{2(1 - \mu_0)}} \] (69)
and sum the remaining series (which converges like \( 1/n^2 \) at \( \mu_0 = 1 \) and faster elsewhere) numerically. Again, the transformation from \( F_n \) to \( G_n \) leaves open the possibility of a forward \( \delta \)-function in \( F(\mu_0) \).

The above programme has been carried out for Duntley’s data, with only partial success. Specifically, the deduced \( F(\mu_0) \) behaves like \((1 - \mu_0)^{-3/2}\) as \( \mu_0 \to 1 \) and agrees rather well with the shape of the measured differential scattering cross-section in the forward hemisphere, but is negative in part of the backward hemisphere near \( \mu_0 = -1 \). This unphysical feature may be only a result of overanalyzing the data; there is no guarantee that a slightly different but equally close analytic fit to \( I(\mu) \) might not produce a markedly different \( F(\mu_0) \).

We note that if the result \( F(\mu_0) \sim (1 - \mu_0)^{-3/2} \) as \( \mu_0 \to 1 \) is correct, the ratio \( \sigma_\text{e}/\sigma_\text{s} \) is infinite although the transport cross section, \( \int_{-1}^{1} (1 - \mu_0)\sigma(\mu_0) d\mu_0 \), is finite. This points up the fact that in macroscopic transfer theory the concept of “total cross-section” is actually meaningless unless \( \sigma(\mu_0) \) is representable by a finite Legendre series; we have noted that a forward \( \delta \)-function in \( F(\mu_0) \) is undetectable in the present inverse problem, and the physical meaning of this is clear: “elastic scattering in the forward direction” is operationally identical with “no scattering at all”. The usual formulations of transfer theory do not manifest this identity, since the “total cross section” is often used to define the length scale and the ratio \( \sigma_\text{e}/\sigma_\text{s} \) is used as a basic parameter—a harmless practice ordinarily, but one which could conceal physical relationships in highly anisotropic problems. Consider the “gedanken calculation” of the radiation field for two problems in which the physical geometry, boundary conditions, and absorption cross sections are identical, and \( \sigma(\mu_0)/\sigma_\text{s} \) is equal to \((1 - \mu_0 + 10^{-10})^{-3/2}\) in one problem and \((1 - \mu_0 + 10^{-14})^{-3/2}\) in the other. Intuitively one expects the two radiation fields to be practically identical; but since \( \sigma_\text{e}/\sigma_\text{s} \) differs by a factor of 100, the formulation in terms of total cross section would superficially lead to two very different problems. Therefore, one could argue for a different formulation, e.g. in terms of the transport cross section \( \sigma_\text{t} \).
such that any forward-scattering singularities which do not affect the physics also do not affect the formalism.

It is amusing, but probably fruitless at this point, to speculate on physical mechanisms which could produce the observed differential scattering cross-section of natural water. It is clear that the extreme anisotropy can only be produced by scattering centres which are very much larger than the wavelength; if one imagines large transparent spheres, with refractive indices slightly different from that of water, to be embedded in the water, one can obtain \( \sigma(\mu_0) \sim (1 - \mu_0)^{-3/2} \) near \( \mu_0 = 1 \) by averaging the geometric-optics scattering pattern for such a sphere (van de Hulst 1957) over an appropriate distribution of sphere sizes and refractive indices. Duntley attributes the anisotropic scattering of natural water to the presence of plankton; however, one must seek another mechanism for distilled water, which shows a rather similar scattering pattern.

E. Influence of Radiative Transfer on Shock-Wave Structure

The classical model of a plane steady shock wave is as follows. Take a coordinate system in which the plane \( x = 0 \) is the location of the shock front. At every point the material may be fully described by its temperature \( T(x) \), density \( \rho(x) \), and bulk velocity (with respect to the shock front) \( v(x) \). At \( x = -\infty \) the values of \( T, \rho, \) and \( v \) describe the unshocked or "upstream" conditions. From these the conditions at all other points (including the "downstream" conditions at \( x = +\infty \)) are determined by the three hydrodynamic conservation equations for mass, momentum, and energy. At the shock front \( T, \rho, \) and \( v \) undergo (in general) discontinuous changes, which are the mathematical representation of physical non-equilibrium (irreversible) processes confined to distances of the order of few gas-kinetic mean free paths from the shock front. The conservation equations may be written (e.g. Heaslet and Baldwin 1963)

\[
\rho v = \text{const.} = \text{mass flux} \tag{70}
\]

\[
\rho v^2 + P = \text{const.} = \text{momentum flux} \tag{71}
\]

\[
\frac{1}{2} \rho v^3 + \rho vh - \pi F = \text{const.} = \text{total energy flux} \tag{72}
\]

where \( P = \) pressure, \( h = \) enthalpy per unit mass, \( \pi F = \) net radiative flux directed upstream. (Note \( P \) and \( h \) are functions of \( T \) and \( \rho \) through the thermodynamic equation of state of the material.) For simplicity, we have neglected effects such as radiation pressure, material viscosity, and non-radiative heat conduction in the above.

Shocks which are cool enough or opaque enough for \( F \) to be negligible enter the simple solution in which each thermodynamic quantity equals
its upstream value for all $x < 0$ and its downstream value for all $x > 0$. Denoting upstream and downstream values by subscripts 1 and 2 respectively, one finds $T_1 < T_2$, $\rho_1 < \rho_2$, $v_1 > v_2$; the entropy of the material must increase on passing through the shock front. The qualitative effect of radiation will evidently be to deliver energy upstream, from the hot shocked material to the cold unshocked material, resulting in an altered temperature profile near the shock front. The first treatments of this effect employed diffusion theory and led to the incorrect conclusion that radiation would always eradicate the temperature discontinuity, since (in diffusion theory) finite radiative fluxes require finite temperature gradients. The correct situation was elucidated by Zel'dovich (1957), and later by Heaslet and Baldwin (1963): when integral transport theory (even highly simplified) is used, a temperature jump occurs for sufficiently strong shocks.

The transfer problem arises as follows: Use the mass and momentum conservation equations to express, say, $\rho$ and $P$ as functions of $v$. Then through the equation of state, $T$ and $h$ are also known functions of $v$. The energy conservation equation then reads $F = F(v)$. On the other hand, Eqn (33) (assuming LTE so that $S = B_v$) expresses the radiative flux in terms of the temperature profile in optical-depth space. Equating these two expressions for the flux gives a non-linear integral equation to solve for the function $v(\tau)$, or equivalently for $T(\tau)$. Once $v(\tau)$ is obtained, all the thermodynamic functions and the flux are known in terms of $\tau$. It is remarkable that the actual magnitude of the opacity nowhere enters the problem; this is because the conservation equations do not contain $x$ explicitly. (The opacity is required only to convert to geometrical depth afterward.)

Again for simplicity, consider the case of a grey absorption coefficient and an ideal-gas equation of state. Then $P = \rho RT$, $h = \gamma RT/(\gamma - 1)$; letting $C_1$, $C_2$, $C_3$ denote mass, momentum, and (total) energy fluxes, we obtain

$$RT(v) = vC_2/C_1 - v^2$$  \hspace{1cm} (73)

$$\pi F(v) = -\frac{\gamma + 1}{2(\gamma - 1)} C_1 v^2 + \frac{\gamma}{\gamma - 1} C_2 v - C_3$$  \hspace{1cm} (74)

$$\pi F(\tau) = 2 \int_{-\infty}^{\infty} d\tau' \sigma T^4(\tau')E_2(\tau - \tau') \text{sgn}(\tau' - \tau)$$  \hspace{1cm} (75)

where the positive $\tau$-direction is the positive $x$-direction. If $F(\tau)$ is regarded as known (as, for example, in an iterative solution) the last equation is a two-sided Milne problem with sources. The equations can be put in dimensionless form by introducing the reduced velocity $u = v/v_1$; the reduced temperature $\tau = R T / v_1^2$ the upstream Mach number $M = v_1 / \sqrt{\gamma R T_1}$, and
the radiation strength parameter $A \equiv (\gamma - 1)v_{r1}^2/\rho_{r1}R^2$. Then
\[
\gamma + \frac{1}{2}u^2 + \left(\gamma + \frac{1}{M^2}\right)u - \left(\gamma - \frac{1}{2} + \frac{1}{M^2}\right)
= 2A \int_{-\infty}^{\infty} d\tau' \psi^2(\tau') E_2(\tau - \tau') \text{sgn}(\tau' - \tau)
\] (76)
where
\[
t = \left(1 + \frac{1}{\gamma M^2} - u\right) \tau.
\] (77)

The solution, $\psi(\tau)$, thus depends on the two dimensionless parameters $M$ and $A$. Heaslet and Baldwin (1983) approximated the $E_2$ function by an exponential and solved the resulting non-linear differential equation numerically. This approximation allows temperature discontinuities and is not equivalent to diffusion theory. The solutions for strong shocks (large $M$) with weak radiation (small $A$) resemble the radiationless shock, as expected. For strong radiation, a rise in temperature (called a "radiative precursor") extends many mean free paths upstream from the shock front; for a strong shock the temperature attains its downstream value just ahead of the shock, jumps to a large value immediately behind the shock, and then rapidly (within a fraction of a mean free path) relaxes to its final downstream value; for a weak shock the temperature rise is continuous and monotonic everywhere. In every case the radiative flux has a maximum at the shock front.

It is worth noting that the final results of the shock, i.e. the Rankine-Hugoniot relationships giving downstream quantities in terms of upstream, are entirely unaffected by the presence of the radiation; this is because the radiative flux vanishes far from the shock, as required in the grey approximation. The presence of a spectral region which is transparent upstream, for example, would affect the Rankine-Hugoniot relations by allowing some of the thermal energy generated in the shock to escape altogether instead of pre-heating the unshocked material.

VI. TRACTABLE TRANSFER PROBLEMS

A wide class of transfer problems lead to a transfer equation of the form
\[
\mu \frac{\partial}{\partial \tau} I(\tau, \mu) = I(\tau, \mu) - \int_{0}^{\infty} [I(\tau, \mu') + I(\tau - \mu')] \psi(\mu') d\mu'
\] (78)
or to its inhomogeneous form, which has a prescribed source term added to the right-hand-side. These problems include:

1. The radiative equilibrium $I(\mu)$ problem with $\mu$-independent absorption coefficient (Stewart 1965).

II. The radiative equilibrium $I(\mu)$ problem with $\mu$-independent absorption coefficient (Stewart 1965).

III. The radiative equilibrium $I(\mu)$ problem with $\mu$-independent absorption coefficient (Stewart 1965).
(2) The non-LTE problem for a two-level atom with complete redistribution and depth-independent line profile (e.g. Ivanov 1963).

(3) The energy-dependent neutron problem with isotropic separable scattering kernel (Bednarz and Mika 1963; Williams 1964).

(4) The “pseudo-problems” which arise in the treatment of one-group anisotropic scattering (Chandrasekhar 1950).

(5) The “generalized Milne problems” considered by Busbridge (1960). The physical meaning of the quantities in Eqn (78), of course, differs from problem to problem. In problems (1), (2), and (3), the quantity \( \mu \) has the significance of a direction cosine times a mean free path in units of a reference mean free path, and \( d\tau \) is a differential distance measured in terms of the same reference mean free path; the reference mean free path, thanks to the plane-parallel geometry assumed, may itself depend on \( \tau \). The limits \((-A, A)\) on the relevant range of \( \mu \) can be transformed to \((-1, 1)\) by choosing as reference the longest mean free path, if it is finite. Meaningful transfer problems exist, however, for which \( A = \infty \). Examples are problem (2) above, and the use of the random band model in problem (1).

The characteristic function \( \psi(\mu) \) in all of the above problems has the property \( \psi(\mu) \geq 0 \) for \( 0 < \mu < A \). In some of the above, other properties also hold; e.g. in (1), (2), and (3) \( d\psi/d\mu \leq 0 \) for \( \mu > 0 \); and in (4) and (5) \( A = 1 \), \( \psi(\mu) \) is regular for \( 0 < \mu < 1 \) (in (4), \( \psi(\mu) \) is a polynomial), and \( \int_0^A \psi(\mu) \, d\mu \leq \frac{1}{2} \). In general, the “single-scattering albedo” or “mean number of secondaries per collision” is given by \( \omega_0 = 2 \int_0^A \psi(\mu) \, d\mu \) and characterizes problems as conservative (\( \omega_0 = 1 \)) or nonconservative (\( \omega_0 \neq 1 \)).

The integral form of Eqn (78) may be written as

\[
S(\tau) = \int_{-\infty}^{\infty} K(\tau - \tau') S(\tau') \, d\tau' + g(\tau) \tag{79}
\]

where \( S(\tau) \) is the integral term on the right-hand side of (78), and

\[
K(\tau) = \int_0^A \psi(\mu) e^{-\mu \tau} \frac{d\mu}{\mu} \tag{80}
\]

Any boundary conditions and sources for (78) have been incorporated into \( g(\tau) \) and into the definition of \( S(\tau') \) outside the region where (78) holds. For example, the Milne problem corresponds to \( g(\tau) = 0 \) at \( \tau = 0 \), \( S(\tau') = 0 \) for \( \tau' < 0 \); the half-space albedo problem has the same condition on \( S \), but \( g(\tau) = e^{-\mu \tau} \) for \( \tau > 0 \), etc.

Because of the diverse physical interpretations of Eqn (78), solutions of it for particular source distributions, boundary conditions, and characteristic functions have been derived and re-derived by numerous authors. The half-space Milne problem, in particular, has been solved by the Wiener-Hopf...
method (Busbridge 1960, Williams 1964), the Case method (Bednarz and Mika 1963), the Wick-Chandrasekhar discrete-ordinates method for infinite order (Stewart 1962), and by Nagirner (1965), under fairly mild restrictions on the form of $\psi(\mu)$.

As is well known (e.g. Sobolev 1963), from the Milne-problem solution of (78) one can construct explicitly the Green's function ("resolvent kernel") for the half-space, which in turn gives the solution for any distribution of plane isotropic sources. Anisotropic sources can be trivially reduced to isotropic ones by treating the uncollided intensity separately; the particles emerge isotropically from their first collision. Thus all half-space problems for the transfer Eqn (78) have in effect been solved exactly. Together with certain problems for two half-spaces (see Prof. Zweifel's lectures) this class of problems apparently exhausts the presently known ability of transport theory to yield explicit exact solutions.

The Milne-problem solution of (78) is completely specified by giving $S(\tau)$, since the intensity $I(\tau, \mu)$ can be reconstructed using Eqn (31). The normalization of $S(\tau)$ is arbitrary; taking $S(0) = 1$, we have in the conservative case:

$$
S(\tau) = 1 + \frac{\tau}{\sqrt{\int_0^\infty \psi(\mu) \mu^2 d\mu}} + \int_0^\infty d\mu [1 - e^{-\tau}\mu] \frac{\sin^2 \pi G(\mu)}{\pi^2 \mu^2 H(\mu) \psi(\mu)}
$$

(81)

where $H(\mu)$ is the (generalized) Ambarzumian-Chandrasekhar $H$-function; its physical meaning for the ULB or two-level-atom context is the specific intensity emerging at a particular angle $\cos^{-1}/\mu_0$, averaged over the frequencies for which $I_r = \mu/\mu_0$, and divided by the grazing intensity ($\mu_0 = 0^+$) averaged over the same frequencies. Here $I_r$ is the mean free path in units of the reference mean free path. The $H$-function is thus related to the $S$-function by

$$
H(\mu) = \int_0^\infty S(\tau) e^{-\tau} d\tau / \mu
$$

(82)

and is given explicitly by

$$
H(\mu) = \exp \int_0^\infty \left( \frac{1}{y} - \frac{1}{y + \mu} \right) [1 - G(y)] dy
$$

(83)

where

$$
G(y) = \frac{1}{\pi} \arccot \left[ \int_0^\infty dx \frac{2x^2 \psi(x)}{y^2 - x^2} \right]
$$

(84)

with

$$
0 \leq G(y) \leq 1
$$

(85)
and the integral in (84) is a Cauchy principal value. Written in this form the
integrals over \(x\) and \(\mu\) terminate at \(A\) if \(\psi(\mu) = 0\) for \(\mu > A\), and still cover
the case \(A = \infty\). We note that while the usual asymptotic behaviour of
\(S(\tau)\) is linear in \(\tau\), the two-level atom problem has \(\int_0^\infty \psi(\mu) \mu^2 \, d\mu\) infinite and
the term linear in \(\tau\) vanishes; then the asymptotic form of \(S(\tau)\) is approxi-
mately \(\sim \tau^{1/2}\) for a Doppler line profile and \(\sim \tau^{1/4}\) for a Lorentz profile.

The "standard" problems for slabs of finite optical thickness \(\tau_1\) (diffuse
reflection and transmission, uniform internal sources, isotropic incident
intensity, etc.) bring in the \(X\)- and \(Y\)-functions instead of the \(H\)-function.
These are proportional, respectively, to the angular intensity distributions
(multiplied by \(\mu\)) emerging from the top and bottom of an assembly consisting
of a slab with an isotropic plane source on its top surface. The function
\(S(\tau)\) in such a system satisfies the integral equation
\[
S(\tau) = \int_0^\tau K(\tau - \tau') S(\tau') \, d\tau' + K(\tau)
\]  
(86)
where the source is at \(\tau = 0\); then
\[
X(\mu) = 1 + \int_0^\mu S(\tau) e^{-r/\mu} \, d\tau
\]  
(87)
\[
Y(\mu) = e^{-r/\mu} + \int_0^\mu S(\tau) e^{-(r - \tau)/\mu} \, d\tau
\]  
(88)
The emergent intensities for the rest of the standard problems turn out to be
also simple in terms of these functions; for example, the intensity emerging
from the top of the atmosphere in the planetary infra-red problem (King 1956)
is proportional to \(X(\mu) + Y(\mu)\). Recently Ivanov (1964), using a method due
to Sobolev (1963) based on a probabilistic picture, has obtained the Green's
function for the finite slab in terms of the infinite-medium Green's function
and the \(X\)- and \(Y\)-functions. His derivation assumed isotropic scattering
(\(\psi(\mu) = \text{const.}\)) but presumably works for general \(\psi(\mu)\), at least if \(\omega_0 < 1\); for
the conservative case the infinite-medium Green's function diverges. How-
ever a similar derivation using the half-space Green's function appears
possible.†

Thus all finite-slab problems for the transfer equation (78) appear to be
reducible to the evaluation of the \(X\)- and \(Y\)-functions, or equivalently to the
solution of (86) with the kernel (80). Numerous integral equations for the
\(X\)- and \(Y\)-functions have been derived (Chandrasekhar 1950; Busbridge
1955, 1960; Sobolev 1957, 1963) but none has been solved explicitly. Numeri-
cally, the uncoupled non-singular inhomogeneous Fredholm equations of
Busbridge (1955) look promising for large \(\tau_1\), while direct iteration of (86)
may suffice for small $\tau_1$. Explicit expressions in the discrete-ordinates approximation (finite order) have been obtained by King (1956) following Chandrasekhar's (1950) treatment for isotropic scattering; the passage to infinite order does not appear possible. Asymptotic properties of $X$ and $Y$ for large $\tau_1$ are discussed by Ivanov (1965).

In summary, the various transfer problems which can be written in the form of (78) appear to be amenable to the same techniques as the grey or one-group case.

VIII. NUMERICAL METHODS

We have seen repeatedly in the preceding sections that in order to get a tractable transfer problem—in effect, to get Eqn (78)—we have had to sacrifice physical realism, sometimes to an uncomfortable degree. Once we venture outside of Eqn (78), however, all of the methods capable of giving exact solutions appear to become impotent. The innocent-looking transfer equation

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \frac{w_0(\tau)}{2} \int_{-1}^{1} I(\tau, \mu') d\mu'$$

(89)

for example, seems so far to be insoluble. The condition for tractability, it would seem, is translational invariance.

Certain exact statements can be made for limiting situations, which then become constraints on approximate or numerical solutions of the complete problem. For example, the Milne problem in a stellar atmosphere with unrestricted $\kappa_\tau$, Eqns (34–37), must yield at sufficiently large depths the diffusion-theory limit

$$\frac{1}{\kappa_R} \frac{dB}{dm} = \frac{dB}{dT_R} = \frac{3}{4} B(T_{\text{eff}}) = \text{constant}$$

(90)

i.e. $T^4$ is linear in $\tau_R$, the Rosseland optical depth. Likewise the non-LTE rate equations must produce LTE occupation numbers at large enough depths. There is thus a well-defined asymptotic solution, but unlike the corresponding neutron case (general energy-dependent scattering kernel) the photon spectrum does not separate asymptotically into a function of energy times a function of depth.

Many iterative numerical procedures have been devised for the LTE Milne problem; the most successful employ some combination of the conditions $F = \text{constant}$ and total emission = total absorption, which operate effectively at large and small depths respectively. The review by Pecker (1965) describes many of these methods.
For finite slab problems, with arbitrary variation of optical properties with depth, the differential transfer equation encounters numerical instability if solved straightforwardly, because half the incident boundary conditions are given at each face. A recently proposed device (Rybczinski and Usher 1966) not only yields a numerically stable system of differential equations, but provides a manipulative (rather than heuristic) derivation of an "invariant-imbedding" result.

The transformation of Rybczinski and Usher is as follows: consider a plane-parallel slab with arbitrary but known optical properties, internal sources, and incident intensities. Let \( f_1 \) denote a vector whose components are the intensities at various frequencies, polarizations, and angles with \( \mu > 0 \), and \( f_2 \) a similar vector with \( \mu < 0 \). The transfer equation and its boundary conditions can then be written in the form

\[
\frac{df_1}{d\tau} = \Gamma_{11} f_1 + \Gamma_{12} f_2 + h_1; \quad f_1(0) = E_1, \quad (91)
\]

\[-\frac{df_2}{d\tau} = \Gamma_{21} f_1 + \Gamma_{22} f_2 + h_2; \quad f_2(0) = E_2 \quad (92)\]

where \( \tau \) is optical depth in any convenient units, \( \tau_0 \) is optical thickness of the slab, the \( \Gamma \)'s are matrices containing all the local optical properties, the \( h \)'s are internal source terms, and the \( E \)'s are incident intensities. The \( \Gamma \)'s and \( h \)'s, as well as the \( f \)'s, are in general functions of \( \tau \). The basic idea of the transformation is to define a vector \( \psi \) to replace \( f_1 \):

\[
\psi_1 = f_1 - Rf_2 \quad (93)
\]

where \( R \) is a matrix (unspecified so far). Inserting (93) into (92) gives

\[-\frac{df_2}{d\tau} = \Gamma_{21} \psi_1 + (\Gamma_{21} R + \Gamma_{22}) f_2 + h_2; \quad f_2(0) = E_2 \quad (94)\]

and differentiation of (93) with respect to \( \tau \) gives

\[
\frac{d\psi_1}{d\tau} = \frac{df_1}{d\tau} + dR \frac{df_2}{d\tau} = \frac{d\psi_1}{d\tau} + R \frac{df_2}{d\tau} \quad (95)
\]

Elimination of \( f_1, df_1/d\tau, \) and \( df_2/d\tau \) between (91, 92, 93, 95) gives

\[
\frac{d\psi_1}{d\tau} = (\Gamma_{11} + R \Gamma_{21}) \psi_1 + h_1 + Rh_2
\]

\[
+ \left( \Gamma_{12} + \Gamma_{11} R + R \Gamma_{22} + R \Gamma_{31} R \frac{dR}{d\tau} \right) f_2 \quad (96)
\]
and (96) uncouples from (94) if we choose $R$ to be the solution of

$$\frac{dR}{dt} = \Gamma_{12} + \Gamma_{11}R + R\Gamma_{22} + R\Gamma_{21}R; \quad R(\tau_0) = 0 \quad (97)$$

Then (96) reduces to

$$\frac{d\psi}{dt} = (\Gamma_{11} + R\Gamma_{21})\psi + h_1 + Rh_2; \quad \psi(\tau_0) = E_1 \quad (98)$$

and one solves successively (97), then (98), then (94) as initial-value problems. Finally $f_1$ is given by (93). From the properties of the $\Gamma$ matrices (at least without the laser condition) one can show that each initial-value problem is numerically stable.

A physical interpretation can now be attached to $R$: it is the reflection matrix of the part of the slab between $\tau$ and $\tau_0$. Thus for spatially uniform one-group anisotropic scattering (97) reduces to Eqn (29) of Chapter VII of Chandrasekhar (1950), where, however, it resulted from the physical statement of the principles of invariance. A similar but more general deduction of (97) from invariance principles is given by Preisendorfer (1965, p. 94).

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