Angular Momentum Transfer by Infrared Photons in Bremsstrahlung

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(Received Oct. 18, 1966)

In connection with the interpretation for infrared divergence as the statistically uncorrelated emission of many photons, it has recently been remarked that the interpretation leads to infinite angular momentum transfer to the emitted photons, and therefore might be somewhat oversimplified. Our present concern is to see the effect of virtual photons. The result of analysis shows that the angular momentum change of virtual photons during the scattering exactly cancels the angular momentum carried by radiated infrared photons. Because of this exact cancellation of angular momentum transfer, the interpretation that the photon emission processes are statistically independent is valid.

I. Introduction

The infrared divergence is one of the important problems in quantum electrodynamics and has been investigated in detail by many authors. It has been recently pointed out in this connection that the interpretation, that the photon emission processes are statistically independent, leads to infinite angular momentum transfer to the infrared photons.

According to generally accepted view, the probability of emitting a certain number of soft photons is given by the Poisson distribution. The average number of photons emitted is infinite. The average energy transferred to the photons becomes finite in the limit of vanishing photon energy. From the Poisson distribution, one may arrive at the interpretation that the photon emission processes in the infrared limit are statistically independent.

For very low energy photons, one may admit that only the electric dipole mode contributes significantly. This means that each photon carries off exactly one unit of angular momentum. Under the assumption that the emission process is random, the average angular momentum transfer becomes infinite, since the average number of radiated photons is infinite. It is, however, well-known that sequential gamma-ray emissions in an isolated system are angularly correlated.
and that the emission is certainly inhibited when the angular momentum becomes large without limit. It has been therefore remarked by Biedenharn that the interpretation of the infrared divergence as statistically uncorrelated emission of infinitely many photons might be somewhat oversimplified.\(^5\)

The present aim is to show that the infinite angular momentum transfer is eliminated by considering virtual photon effect.\(^6\) The angular momentum carried by virtual photons exactly cancels the angular momentum carried by emitted infrared photons, just as the effect of virtual photons cancels the infrared divergence coming from real photons in the scattering amplitude. The result of our analysis shows that the emission probability depends upon both the number and angular momentum of the emitted photons. The emission of any number of photons carrying a certain amount of the angular momentum is possible only when the accompanied transition of virtual photons exactly cancels the angular momentum transferred to the radiated quanta (angular momentum conservation). (It is seen in the derivation of (24) that the Poisson distribution gives the probability of emitting a number of photons independent of their carrying angular momentum.)

II. Angular Momentum Correlations

Let us consider a process in which an electron is scattered by a potential \(V(x)\) emitting a number of soft light quanta. As we are concerned about the radiative corrections, a relativistic formulation is required in order to remove the divergences associated with the self-energy and the vertex part. We shall, however, confine ourselves to the consideration of angular momentum correlations of emitted photons, and follow the non-relativistic treatment by Pauli and Fierz,\(^7\) in which the cut-off procedure is used instead of the renormalization procedure. Though its justification and the appropriate choice of the cut-off frequency should of course be given by the renormalization theory, the non-relativistic approach is sufficient for our present purpose.

The Hamiltonian with which we work is

\[
H = \frac{1}{2m} (p - eA(x))^2 + V(x) + \frac{1}{8\pi} \int (E'^2 + H^2) dx,
\]

where \(E_r\) is the transverse component of the electric field. Expanding the vector potential as

\[
\frac{e}{mc} A(x) = \sum_{s} \alpha_s \omega_s \left( P_s \cos(k_s x) + Q_s \sin(k_s x) \right),
\]

\[
\alpha_s = \frac{2e}{m} \frac{\pi \hbar}{\hbar \omega_s} \left( \alpha_i V \right)^{1/2} \varepsilon_s,
\]
\[ E_{r,l} = \hbar \omega (2q+l+\frac{3}{2}), \]  

and the eigenfunctions are

\[ \psi_{n,l,m}(\xi, \theta, \chi) = \left( \frac{2 \cdot q!}{I(q+l+1/2)} \right)^{1/2} \xi^{l+1/2} L_{q+l}^{(1/2)}(\xi^2)Y_{l,m}(\theta, \chi), \]  

where \( n=2q+l \) expresses the number of quanta and the normalization

\[ \int \psi_{n,l,m}(\xi, \theta, \phi) \bar{\psi}_{n,l,m}(\xi, \theta, \phi) \frac{\xi^2}{\bar{\xi}^2} \, d\xi \, d\Omega(\theta, \chi) = 1 \]

follows from the normalization of associated Laguerre polynomials,

\[ \int_0^{\infty} e^{-x} x^n L_n^{(s)}(x) L_n^{(s)}(x) \, dx = \frac{1}{s!} \frac{\Gamma(s+\alpha+1)}{\Gamma(s+1)}. \]

Degenerate states with a definite number, \( n=2q+l \), of photons are classified by the resultant spin angular momentum \( l \) and its z-component \( m \).

We are now ready to consider the Schrödinger equation,

\[ H\psi(x, \xi, \theta, \chi) = E\psi(x, \xi, \theta, \chi), \]

\[ H = \frac{p^2}{2m} + V(x - \sum_{l,\lambda} \alpha_{l,\lambda} Q_{l,\lambda}) + \frac{1}{2} \sum_{l,\lambda} \hbar \omega_{l,\lambda} \left( \frac{P_{l,\lambda}^2 + Q_{l,\lambda}^2}{\hbar^2} \right) + \sum_{l,\lambda} \alpha_{l,\lambda} \left( \frac{\pi \hbar}{\omega_{l,\lambda}} \right)^{1/2} \xi_{l,\lambda} \cdot \vec{\mathbf{E}}_{l,\lambda}. \]

The suffix \( \lambda \) stands for the three directions of polarization, and we assign \( \xi_{l,\lambda} \) to the direction of propagation. Introducing the wave function \( \phi(p; n, l, m) \) in terms of

\[ \Psi(x, \xi, \theta, \chi) = (2\pi \hbar)^{-3/2} \sum_{(n, l, m, \mu)} \int d\mathbf{p} \phi(p; n, l, m, \mu) e^{i\mathbf{p} \cdot \mathbf{x}/\hbar} \]

we have

\[ E\phi(p; n, l, m) = \left( \frac{p^2}{2m} + \hbar \sum_{\lambda} \omega_{l,\lambda} \right) \phi(p; n, l, m) \]

\[ + \sum_{(n', l', m')} \int d\mathbf{p}'(p' V p') \Pi_{l,l'}(n, l, m, \mu; p, K, n', l', m', p') \phi(p'; n', l', m'), \]

where

\[ (p' V p') = (2\pi \hbar)^{-1} \int dx V(x) \exp \left( i (p' - p) x/\hbar \right), \]
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\( (n, l, \mu, ; p, K, n', l', \mu' ; p') = \int d\xi \exp \{ i (p' - p) \alpha \xi \} \)

\( \times \phi^*_{n, l, \mu} (\xi, \theta, \varphi) \phi_{n', l', \mu'} (\xi, \theta, \varphi) \) \hspace{1cm} (12)

\( \xi = \sum_{\lambda=1}^{l} \mathcal{E}_{n, l, \lambda} Q_{n, l, \lambda} \hspace{1cm} \alpha = \mathcal{A}_{n, \lambda} \)

We are interested in the problem in which the incident electron in a plane wave state is scattered by an external potential \( V(x) \) emitting a number of soft light quanta. We consider the solution with the form

\[ \phi(p; n, l, \mu) = \delta(p - p_0) \prod_{s} \delta(n_s) \delta_{l_s, 0} \delta_{\mu_s, 0} + \frac{f(p; n, l, \mu)}{(p_0^2 - p^2)/2m - \hbar \sum n_s \omega_s} \]

where

\[ f(p; n, l, \mu) = \prod_{s} \Pi(n_s, l_s, \mu_s ; p, K, 0, 0, 0 ; p_0) \]

\[ + \sum_{(n', l', \mu')} \int dp' \frac{\prod_{s} \Pi(n_s, l_s, \mu_s ; p, K, n_s, l_s, \mu_s ; p, K, n_s, l_s, \mu_s ; p') f(p'; n_s, l_s, \mu_s)}{(p_0^2 - p^2)/2m - \hbar \sum n_s \omega_s} \]

\hspace{1cm} (13)

The scattering cross section for the given angle of deflection and the arbitrary values of \( n, l, \mu_s \) is

\[ d\sigma = \sigma = 16 \pi^2 \hbar^2 m^2 \frac{p_0^2}{p_0^2} f(p_0, n, l, \mu) \]

\hspace{1cm} (14)

and the energy relation is

\[ p_0^2 = 2m - \hbar \sum n_s \omega_s \]

\hspace{1cm} (15)

In a quick collision the first Born approximation is sufficiently good. Under this approximation, we find from (13)

\[ f(p; n, l, \mu) = \prod_{s} \Pi(n_s, l_s, \mu_s ; p, K, 0, 0, 0 ; p_0) \]

The product form which results from the dipole approximation proves to be convenient for the calculation to come. The recoil momentum can be ignored for the case in which all emitted photons are very soft.

The straightforward evaluation of the integral,

\[ (n, l, \mu ; p, K, 0, 0, 0 ; p_0) = \int d\xi \phi^*_{n, l, \mu} (\xi) e^{i \xi \cdot \xi} \phi_{0, 0, 0} (\xi) \]

\hspace{1cm} (16)

gives the result
\[ \delta_{\mu, \nu} i^{\nu} 2l+1 \left\{ \frac{\pi}{2} \left( \frac{\pi}{2} \right) \right\} \frac{1}{\Gamma(l+1+\frac{\nu}{2})} \int_0^{\infty} d^2 \xi \xi^{l+\frac{\nu}{2}-1} e^{-t} L_{\mu}^{l+1/2} (\xi) \]

\[ \times J_{l+1/2} (t, \xi) = \delta_{\nu, \mu} i^{\nu} 2l+1 \left\{ \frac{1}{2} \frac{\sqrt{\pi}}{3} \frac{W}{2} \right\} \frac{1}{\Gamma(l+1+\frac{\nu}{2})} \left( \frac{\xi}{2} \right)^{\nu} e^{-\xi}, \]

(17)

where \( t = \alpha (p_0 - p) \), and \( W = t^2 / 2 \). If we assume

\[ V(x) = V_0 \exp (-\kappa r) / r, \]

(18)

where \( V_0 \) and \( \kappa \) are constant, we find:

\[ (p, V, p_0) = \sum_{l=0}^{\infty} (2l+1) P_l (\cos \theta) \eta_l, \]

\[ \eta_l = \frac{V_0}{2\pi^2 \hbar^2} \int_0^{\infty} j_l (k_r r) j_{l+1/2} (kr) e^{-it} r dr, \]

\[ = \frac{V_0}{2\pi^2 \hbar^2} \left\{ \frac{Q_l \left( \frac{x + h^2 + k^2}{2h_0 k} \right)}{2h_0 k} \right\}. \]

(19)

Here \( \hbar k_0 = p_0 \), \( \hbar k = p \) and \( Q_l (x) \) is the Legendre function of the second kind:

\[ Q_l (x) = 2^{-l-1} \int_{-1}^{1} (1-t^2)^l (x-t)^{l-1} dt. \]

Let us denote a set of suffixes \((n,l,\mu)\) by \( \rho \) or \( \sigma \). As the entire system of the eigenfunctions, \( \psi_{\rho} (t) ' s \), forms a complete orthonormal set, we can expand as

\[ \exp (it \cdot \xi) \psi_{\rho} (\xi) = \sum_{\sigma} c_{\rho} (\sigma ; W) \psi_{\sigma} (\xi), \]

(20)

where the coefficient \( C_{\rho} (\sigma ; W) \) is

\[ C_{\rho} (\sigma ; W) = \int d^2 \xi \psi_{\rho}^* (\xi) \exp (it \cdot \xi) \psi_{\sigma} (\xi). \]

(21)

We can derive the orthogonality and the completeness relation, too:

\[ \sum_{\sigma} C_{\rho}^{*} (\sigma ; W) C_{\sigma} (\sigma ; W) = \delta_{\rho, \sigma}, \]

\[ \sum_{\rho} C_{\rho} (\sigma ; W) C_{\rho}^{*} (\sigma' ; W) = \delta_{\sigma, \sigma'}. \]

(22)

Consequently, one can expand any function \( f(\sigma) \), which is defined for the integral values of the independent variable \( \sigma \), in terms of linear combination of \( C_{\rho} (\sigma ; W) \):

\[ f(\sigma) = \sum_{\rho} a_{\rho} (W) C_{\rho} (\sigma ; W), \quad a_{\rho} (W) = \sum_{\sigma} f(\sigma') C_{\rho}^{*} (\sigma' ; W). \]

This is a generalization of Charlier's expansion* associated with Charlier's polynomials and Poisson distribution. It is noticed from (22) that the transition to any state from all the states is equally probable and the total transition probability from an arbitrary state to all the states is equal to unity. The latter is connected
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as well with the unitary character of the operator K in (16).

For the special case ρ=(0,0,0), the average number and the average angular momentum of radiated photons in a certain mode can be calculated as

\[ \sum_{(n, l, \mu)} (2q+l) \, C_{0,0,0} (n, l, \mu ; W)^2 = W, \]

\[ \hbar^2 \sum_{(n, l, \mu)} l (l+1) \, C_{0,0,0} (n, l, \mu ; W)^2 = 2 \hbar^2 W. \]  

(23)

The average angular momentum of radiated photons becomes infinite in the infrared limit as is expected.

We will now pursue physical analogy. Suppose that a hydrogen atom is in the ground state and an event occurs in which a proton abruptly begins to move with momentum \(-\hbar \ell\) by, e.g., neutron bombardment. It is then conceivable that there will occur various excitations (including ionization) of the bound electron in the hydrogen atom due to the recoil of the proton. We look at the event from the coordinate system fixed at the moving proton after the collision. If the collision time is very short compared with the time \(\hbar/\ell_0\) (\(\ell_0\)=ground state energy of a hydrogen atom), we can apply the "sudden-approximation" method to the calculation of excitation probability of the electron. Under this approximation, the electron wavefunction before and after the impact of neutron is connected with each other at the instant of the impact;

\[ \sum_{(n, l, \mu)} A_{n, l, \mu} \, \psi_{n, l, \mu} (x) = \exp (i t \cdot x) \, \psi_{0,0,0} (x), \]

where \(A_{n, l, \mu}\) gives the transition matrix element to the excited state characterized by \(n, l, \mu\). To describe the ionization process, we understand that the quantum number \(n\) and the summation over \(n\) are adequately modified. We actually obtain the transition matrix element of the same form with (16). One can realize in this example that the transition to the states with infinitely large angular momentum does not occur for finite magnitude of \(t\), since the available angular momentum is of the order, \(\hbar \ell \propto\) (Bohr radius), and, therefore, finite.

Let us observe how the exponential factor, \(\exp (i t \cdot \xi)\), is derived in (12). By the canonical transformation (4), a bare electron is clothed with the virtual photon cloud. As the self-field depends upon the momentum of a bare electron, the states of the self-field are different between before and after the scattering. This difference appears as the exponential factor, \(\exp (i t \cdot \xi)\). For the photon with vanishing energy, the magnitude \(t\) and, therefore \(W\), too, become large without limit. The result follows from the fact that since the interaction between an electron and a light quantum becomes strong as the energy of the photon decreases, the virtual photon field for infrared virtual photons becomes infinitely intense. (In the example of a hydrogen atom, this corresponds to the case that the wave packet acquires infinitely large momentum, which does not occur.) The
total angular momentum of the electron and the electromagnetic field is a constant of motion since we assume that the potential function is spherical symmetric. We see from (12) that the bare electron does not undergo any change of the angular momentum. The resultant angular momentum of the virtual photons and the real photons is therefore conserved before and after the scattering. The exponential factor, exp \((i \mathbf{t} \cdot \mathbf{\xi})\), corresponds to the difference in the self-field before and after the scattering. Let us decompose it into partial waves by the usual expansion of a plane wave and denote by \(\partial L\) the angular momentum carried by each wave. We designate also the angular momenta of initial and final states by \(L_i\) and \(L_f\), respectively. It is observed in (16) that the transition to the state with angular momentum \(L_f\) from the state \(L_i\) (=0) occurs only in such a state that \(L_f-L_i=\delta L\). In other words, each angular momentum transfer to the real photon is necessarily compensated by the angular momentum change of the self-field.

The Poisson distribution (1) can be derived for the probability of emitting \(n\)-photons. When one uses the value (17) for \(C_{\omega \omega} (\sigma; W)\) and takes the sum over the quantity \(q\), from 0 to \((n/2)\) (which stands for \(n/2\) or \((n-1)/2\) according as \(n\) is even or odd), one finds

\[
\frac{\sqrt{\pi}}{2} \left( \frac{W}{2} \right)^{n\omega} e^{-W} \sum_{q=0}^{(n/2)} \frac{2(n-2q+1/2)}{\Gamma(n-q+3/2)q!} = \frac{W^n}{n!} e^{-W}.
\]

(24)

The well-known effect of virtual photons, which cancels the infrared divergence coming from real photons in the scattering amplitude, also appears in the present calculation. The insertion of (15) into (14) leads to

\[
d\sigma = d\mathbf{q}^2 \frac{P_f^\mathbf{V}}{P_0^\mathbf{V}} \sum_{\sigma_s} C_{\mathbf{p}_s} (\sigma_s; W_s) \cdot \mathbf{V} \cdot \mathbf{p}_s \cdot \mathbf{V} \cdot \mathbf{p}_s \cdot \mathbf{V} \cdot \mathbf{p}_s \cdot \mathbf{V}
\]

where \(\mathbf{p}_s = (0, 0, 0)\) and \(\sigma_s = (n_s, L_s, \mu_s)\). The summation over all \(\sigma_s\) for \(\sum \sigma_s C_{\mathbf{p}_s} (\sigma_s; W_s)\) yields the value unity from the orthogonality (22).

III. Concluding Remark

The interaction between the electron and the electromagnetic field establishes the virtual photon cloud around the electron. One must take into account the virtual photons together with the radiated photons, since the angular momentum carried by only the radiated quanta is not a constant of motion. We obtain from the orthogonality of \(C_\omega (\sigma; W)\) the angular correlations among the emitted photons and the emission probability which depends upon the number of emissions. They are completely governed by the variation of virtual photon clouds during the scattering. We see that the infinite angular momentum transfer between the self-fields, before and after the scattering, exactly cancels the infinite angular
momentum transferred to the radiated photons. This is nothing but the damping effect of virtual photons which cancels the infrared divergence coming from emitted photons in the scattering amplitude. We obtain the Poisson distribution again for the emission of \( n \) soft quanta of any mode. The angular momentum carried by those photons in (24) has been already cancelled by the accompanying virtual photons. Because of this exact cancellation of angular momentum transfer, the interpretation of photon emission processes as being statistically independent is valid.

It is a pleasure to acknowledge the hospitality extended to the author at Joint Institute for Laboratory Astrophysics (Colorado, U.S.A.). He wishes to thank Professor L. C. Biedenharn for suggesting this problem.

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