




The Supergeometric Algebra

Andrew J. S. Hamilton* 

Communicated by Uwe Kaehler

Abstract. Spinors are central to physics: all matter (fermions) is made of spinors, and all forces arise from symmetries of spinors. It is common to consider the geometric (Clifford) algebra as the fundamental edifice from which spinors emerge. This paper advocates the alternative view that spinors are more fundamental than the geometric algebra. The algebra consisting of linear combinations of scalars, column spinors, row spinors, multivectors, and their various products, can be termed the supergeometric algebra. The inner product of a row spinor with a column spinor yields a scalar, while the outer product of a column spinor with a row spinor yields a multivector, in accordance with the Brauer–Weyl (Am J Math 57: 425–449, 1935, <https://doi.org/10.2307/2371218>) theorem. Prohibiting the product of a row spinor with a row spinor, or a column spinor with a column spinor, reproduces the exclusion principle. The fact that the index of a spinor is a bitcode is highlighted.

Mathematics Subject Classification. Primary 81R25, 83C60, Secondary 20C35.

Keywords. Spinors, Geometric algebra, Outer products.

1. Introduction

Spinors, objects of spin $\frac{1}{2}$, lie at the heart of physics. On the one hand, all known matter (fermions) is made of spinors. On the other, all known forces arise from symmetries of spinors.

Introduced by Cartan [4, 5] over a century ago, a spinor is the fundamental representation of the group $\text{Spin}(N)$ (the covering group of the special orthogonal group $\text{SO}(N)$) of rotations in N spacetime dimensions. An intriguing aspect of spinors is that they are indexed by a bitcode, with $[N/2]$ bits in N spacetime dimensions. The halving of dimensions is associated with the natural complex structure of spinors. A mathematical physicist would

*Corresponding author.

say that the dimension of the spinor representation of the group $\text{Spin}(N)$ is $2^{\lfloor N/2 \rfloor}$ (technically, in even N dimensions there are two isomorphic irreducible spinor representations, each of dimension $2^{\lfloor N/2 \rfloor - 1}$, of opposite chirality; chirality counts whether the number of up-bits is even or odd). Each bit can be either up (\uparrow) or down (\downarrow). A spinor in $N = 2$ or 3 dimensions is a Pauli spinor, with $\lfloor N/2 \rfloor = 1$ bit, and $2^{\lfloor N/2 \rfloor} = 2$ complex components, Appendix A. A spinor in $N = 3+1$ spacetime dimensions is a Dirac spinor, with $\lfloor N/2 \rfloor = 2$ bits, and $2^{\lfloor N/2 \rfloor} = 4$ complex components, Appendix B. The two bits of a Dirac spinor are a spin bit and a boost bit. The Dirac spinor is said to be right-handed if the spin and boost bits align, left-handed if they anti-align.

As reviewed by [1], the group $\text{Spin}(10)$ of rotations of spinors in 10 dimensions is a well-known promising candidate for a grand unified group that contains the standard model group $U_Y(1) \times SU_L(2) \times SU(3)$. As first pointed out by [25] and reviewed by [1], $\text{Spin}(10)$ characterizes a generation of fermions with $\lfloor 10/2 \rfloor = 5$ bits y, z, r, g, b , consisting of two weak bits y and z , and three colour bits r, g , and b (the labelling of bits here follows [14]). Each bit can be either up (\uparrow) or down (\downarrow), signifying a charge of $+\frac{1}{2}$ or $-\frac{1}{2}$. The charges y, z, r, g, b of each fermion are linearly related to the 5 traditional conserved charges of the standard model, which comprise hypercharge Y , weak isospin I_L , and three colours [14].

In the $\text{Spin}(10)$ model, there are $2^{\lfloor 10/2 \rfloor} = 2^5 = 32$ fermions in a generation. Each of the 32 fermions is itself a chiral (massless) fermion, also called a Weyl fermion, with 2 complex degrees of freedom, for a total of $2^6 = 64$ degrees of freedom in a generation. The components of the Weyl fermion transform under the group $\text{Spin}(3, 1)$ of Lorentz transformations. Lorentz transformations are of course part of the symmetry group of general relativity (the other symmetry of general relativity being translations, or general coordinate transformations). The usual assumption, motivated by the Coleman-Mandula no-go theorem [7, 18, 19], is that internal (grand unified) and spacetime groups combine as a direct product. However, the fact that the grand unified group $\text{Spin}(10)$ and the Lorentz group $\text{Spin}(3, 1)$ are both spin groups, and that there are a total of 2^6 degrees of freedom in a generation of fermions, suggests the alternative hypothesis that the groups might combine non-trivially in a spin group containing 2^6 spinors, which, given that there must be a time dimension, must necessarily be the group $\text{Spin}(11, 1)$ in $11 + 1$ spacetime dimensions.

Indeed, [14] has shown that the Dirac algebra (the geometric algebra associated with the Lorentz group $\text{Spin}(3, 1)$) and the standard model algebra embed non-trivially as commuting subalgebras of the $\text{Spin}(11, 1)$ geometric algebra, uniting all four forces of Nature. The proposed unification in $\text{Spin}(11, 1)$ adds one extra bit, the t -bit, or time bit, to the five bits y, z, r, g, b of $\text{Spin}(10)$, for a total of $\lfloor 12/2 \rfloor = 6$ bits. The unification obeys the Coleman-Mandula theorem because $\text{Spin}(11, 1)$ combines internal and spacetime symmetries in a single simple group. After grand symmetry breaking, the Coleman-Mandula theorem requires only that *unbroken* internal symmetries commute with spacetime symmetries; the unification proposed by

[14] achieves just that. The fact that $\text{Spin}(11,1)$ differs from the Lorentz group $\text{Spin}(3,1)$ by 8 spatial dimensions ensures, by the period-8 Cartan–Bott [2,8,21] periodicity of geometric algebras, that the discrete symmetries (of the spinor metric, Table 1, and of the conjugation operator, Table 4) of the $\text{Spin}(11,1)$ geometric algebra coincide with those of $\text{Spin}(3,1)$.

Besides their behaviour under symmetry transformations, spinors in physics have various fundamental properties, among which are:

1. Spinors possess a (complex-valued) scalar product. Among other things, the spinor scalar product allows a scalar Lagrangian for spinors to be defined, the Dirac Lagrangian, from which the dynamics of spinors flow.
2. Spinors satisfy an exclusion principle.
3. Spinors have (complex) conjugates, which define antispinors.

A major theorem, first proved by Brauer and Weyl (1935) [3], is that the algebra of outer products of spinors is isomorphic to the Clifford algebra [6,11,12] of multivectors, (see [17] for a history), also known as the geometric algebra [9,13,15,16], in any number of spacetime dimensions.

A dimension- d representation of a group is, by definition, a mapping from the group to a set of $d \times d$ matrices, multiplication among which reproduces the action of the group. The spinors of the spinor representation of the group $\text{Spin}(N)$ are column vectors of dimension $2^{\lfloor N/2 \rfloor}$. In physics there are also row spinors, a product of row and column spinors yielding a scalar product of spinors. Elements of the rotation group $\text{Spin}(N)$ acting on spinors are called rotors, represented as $2^{\lfloor N/2 \rfloor} \times 2^{\lfloor N/2 \rfloor}$ matrices. Rotors comprise the set of even multivectors obtained by exponentiating the orthonormal bivectors of the geometric algebra.

One of the core properties of spinors in physics is that they satisfy an exclusion principle. The exclusion principle underlies much of the richness of the behaviour of matter at low energy. According to the usual rules of matrix multiplication, a row matrix can multiply a column matrix, yielding a scalar, and a column matrix can multiply a row matrix, yielding a matrix, but a row matrix cannot multiply a row matrix, and a column matrix cannot multiply a column matrix:

$$\begin{pmatrix} \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \end{pmatrix} \text{ inner product} = \text{scalar} , \tag{1a}$$

$$\begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \text{ outer product} = \text{multivector} , \tag{1b}$$

$$\begin{pmatrix} \end{pmatrix} \begin{pmatrix} \end{pmatrix} = \emptyset \text{ forbidden} , \tag{1c}$$

$$\begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = \emptyset \text{ forbidden} . \tag{1d}$$

These rules resemble the rules for fermionic creation and destruction operators in quantum field theory: creation following destruction is allowed, and destruction following creation is allowed, but creation following creation is

forbidden, and destruction following destruction is forbidden. It is beyond the scope of this paper to pursue the subject, but it can be shown that the multiplication rules for row and column spinors indeed reproduce those of fermion creation (row) and destruction (column) operators in quantum field theory.

The observed properties of spinors in physics, as summarized above, motivate a principal purpose of this paper, which is to introduce the supergeometric algebra, defined to consist of (complex) linear combinations of scalars (= complex numbers), column spinors, row spinors, and multivectors, and the various multiplication rules that connect them. The name supergeometric algebra accords with the common physics practice of prepending the adjective ‘super’ to spinorial extensions of theories.

The present paper sets out the notation and groundwork for the subsequent paper [14]. From a purely mathematical perspective, most of the content of the present paper can be considered well known. However, the perspective and emphasis are different. The exposition is intended to be self-contained.

This paper adopts a trailing dot notation for row spinors, equation (30). The notation $\psi \cdot$ for a row spinor, with a trailing dot symbolizing the spinor metric, is extremely convenient. The dot immediately distinguishes a row spinor from a column spinor; and the dot makes transparent the application of the associative rule to a sequence of products of spinors, equation (36).

The Brauer–Weyl theorem [3] has practical implications. A consequence of the theorem is that manipulations commonly carried out using the geometric algebra can also be carried out using the supergeometric algebra. For example, amplitudes of processes involving spinors in quantum field theory are commonly calculated using “ γ -matrix technology,” also known as Fierz identities, e.g. [20]. Those calculations can be done more directly using the supergeometric algebra and the associative rule (36). For the same reason that it is simpler to square a quantity than to take its square root, it is simpler to go from the supergeometric algebra to the geometric algebra than vice versa.

Two of the most fertile ideas of modern physics are supersymmetry, and string theory, although as yet neither has received any experimental corroboration. Supersymmetry hypothesizes a local gauge theory with spinor generators [10, 23, 24]. An exciting implication of supersymmetry is that a spacetime translation is generated by a product of spinor transformations. Supersymmetry is built on the relation between spinors and vectors embodied by the part of the supergeometric algebra for which the outer products of spinors yield vectors, multivectors of grade 1. In 4 spacetime dimensions, the relation between spinors and vectors is given by equations (107).

But the supergeometric algebra contains not just vectors, but multivectors of all grades. A multivector of grade p , a p -vector, is a p -dimensional object in N -dimensional spacetime. Such things are called branes in string theory. The algebra of outer products of spinors yields branes of all dimensions. The supergeometric algebra would seem to supply a natural language for string theory.

The plan of this paper is as follows. Section 2 defines the properties of multivectors and spinors. Section 3 proves the Brauer–Weyl [3] theorem in even spacetime dimensions N . Section 4 addresses the case of odd dimensions N . Section 5 comments briefly on chiral subalgebras. Section 6 discusses the operation of conjugation, which translates spinors into their complex conjugates, antispinors, in a Lorentz-invariant fashion. The term “Lorentz-invariant” is used throughout this paper to denote invariance under rotations in an arbitrary number of spacetime dimensions, not just $3 + 1$ dimensions. Section 7 summarizes the conclusions.

2. Multivectors and Spinors

2.1. Multivectors

The orthonormal vectors γ_a , $a = 1, \dots, N$ of a geometric algebra (Clifford algebra) in N spacelike dimensions satisfy the product rule

$$\gamma_a \gamma_b = \begin{cases} \delta_{ab} & (a = b) , \\ -\gamma_b \gamma_a & (a \neq b) , \end{cases} \tag{2}$$

where the Kronecker delta δ_{ab} is the Euclidean metric. For the most part, a dimension that is timelike can be treated as the imaginary i times a spacelike vector; the only place where that algorithm is insufficient is conjugation, Sect. 6. An antisymmetric product of vectors is commonly written with a wedge sign, $\gamma_a \wedge \gamma_b$, although in the present paper the wedge sign is often omitted for brevity when there is no ambiguity. A wedge product

$$\gamma_A = \gamma_{a_1 \dots a_p} \equiv \gamma_{a_1} \wedge \dots \wedge \gamma_{a_p} \tag{3}$$

of p vectors, a multivector of grade p , or p -vector, represents a p -dimensional element in the N -dimensional space. The geometric algebra is the 2^N -dimensional algebra of linear combinations

$$\mathbf{a} = a^A \gamma_A , \tag{4}$$

of multivectors of all possible grades $p = 0$ to N .

The group $\text{Spin}(N)$ of rotations in N dimensions is generated by the $N(N-1)/2$ bivectors of the geometric algebra. Of these bivectors, $[N/2]$ are mutually commuting, and there are $[N/2]$ associated conserved charges, called central charges. Each of the mutually commuting bivectors generates a rotation in a plane defined by a pair of dimensions. The N orthonormal basis vectors γ_a thus organize into $[N/2]$ pairs, conveniently denoted γ_k^\pm . If the dimension N is odd, then one basis vector, γ_N , remains unpaired. Each of the $[N/2]$ indices k represents a conserved charge. Chiral combinations γ_k and $\gamma_{\bar{k}}$ of the orthonormal basis vectors are defined by

$$\gamma_k \equiv \frac{\gamma_k^+ + i\gamma_k^-}{\sqrt{2}} , \quad \gamma_{\bar{k}} \equiv \frac{\gamma_k^+ - i\gamma_k^-}{\sqrt{2}} , \tag{5}$$

which are normalized so that $\gamma_k \cdot \gamma_{\bar{k}} = 1$. The vectors γ_k^+ and $i\gamma_k^-$ can be thought of as, modulo a normalization, the real and imaginary parts of a complex vector γ_k whose complex conjugate is $\gamma_{\bar{k}}$. Under a right-handed

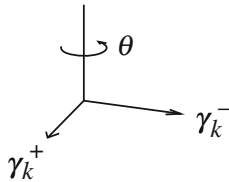


FIGURE 1. Right-handed rotation by angle θ in $\gamma_k^+ - \gamma_k^-$ plane. The $[N/2]$ conserved charges of $\text{Spin}(N)$ are eigenvalues of quantities under rotations in $[N/2]$ planes $\gamma_k^+ \gamma_k^-$, $k = 1, \dots, [N/2]$

rotation by angle θ in the $\gamma_k^+ \gamma_k^-$ plane, Fig. 1, the chiral basis vectors γ_k and $\gamma_{\bar{k}}$ transform by a phase,

$$\gamma_k \rightarrow e^{-i\theta} \gamma_k, \quad \gamma_{\bar{k}} \rightarrow e^{i\theta} \gamma_{\bar{k}}. \tag{6}$$

(If one of the two orthonormal dimensions, say $\gamma_{\bar{k}}$, is a time dimension, then the rotation in the $\gamma_k^+ \gamma_k^-$ plane becomes a Lorentz boost, and the transformation (6) becomes

$$\gamma_k \rightarrow e^{\theta} \gamma_k, \quad \gamma_{\bar{k}} \rightarrow e^{-\theta} \gamma_{\bar{k}}.) \tag{7}$$

The transformation (6) identifies the chiral basis vectors γ_k and $\gamma_{\bar{k}}$ as having k -charge equal to $+1$ and -1 . All other chiral basis vectors, γ_l and $\gamma_{\bar{l}}$ with $l \neq k$, along with the unpaired basis vector γ_N if N is odd, remain unchanged under a rotation in the $\gamma_k^+ \gamma_k^-$ plane, so have zero k -charge. The k -charge of a multivector (or tensor of multivectors) can be read off from its covariant chiral indices:

$$k\text{-charge} = \text{number of } k \text{ minus } \bar{k} \text{ covariant chiral indices.} \tag{8}$$

For “ordinary” spatial rotations, the conserved charge is the projection of the angular momentum, or spin, in the $\gamma_k^+ \gamma_k^-$ plane (in fundamental units, $\hbar = 1$). However, in other applications of the supergeometric algebra, the word charge may refer to other conserved charges, such as the conserved charges of the standard model of physics [14].

2.2. Spinors

Chiral basis spinors comprise $2^{[N/2]}$ basis spinors ϵ_a ,

$$\epsilon_a \equiv \epsilon_{a_1 \dots a_{[N/2]}} \tag{9}$$

where $a = a_1 \dots a_{[N/2]}$ denotes a bitcode of length $[N/2]$. Each bit a_k is either up \uparrow or down \downarrow . For example, one of the basis spinors is the all-bit-up basis spinor $\epsilon_{\uparrow \dots \uparrow}$. A spinor ψ ,

$$\psi = \psi^a \epsilon_a, \tag{10}$$

is a linear combination of the $2^{[N/2]}$ basis spinors ϵ_a .

Under a right-handed rotation by angle θ in the $\gamma_k^+ \gamma_k^-$ plane, basis spinors $\epsilon_{\dots k \dots}$ and $\epsilon_{\dots \bar{k} \dots}$ with k -bit respectively up and down transform as

$$\epsilon_{\dots k \dots} \rightarrow e^{-i\theta/2} \epsilon_{\dots k \dots}, \quad \epsilon_{\dots \bar{k} \dots} \rightarrow e^{i\theta/2} \epsilon_{\dots \bar{k} \dots}. \tag{11}$$

(Again, if one of the two orthonormal dimensions, say γ_k^- , is a time dimension, then the rotation in the $\gamma_k^+ \gamma_k^-$ plane becomes a Lorentz boost, and the transformation (11) becomes

$$\epsilon_{\dots k \dots} \rightarrow e^{\theta/2} \epsilon_{\dots k \dots}, \quad \epsilon_{\dots \bar{k} \dots} \rightarrow e^{-\theta/2} \epsilon_{\dots \bar{k} \dots} \tag{12}$$

The transformation (11) shows that basis spinors $\epsilon_{\dots k \dots}$ and $\epsilon_{\dots \bar{k} \dots}$ have k -charge respectively $+\frac{1}{2}$ and $-\frac{1}{2}$ in each of its $[N/2]$ bits. The k -charge of a spinor (or tensor of spinors) can be read off from its covariant chiral indices:

$$k\text{-charge} = \frac{1}{2}(\text{number of } k \text{ minus } \bar{k} \text{ covariant chiral indices}). \tag{13}$$

2.3. Rotors

A rotation in the geometric algebra is described by a rotor R , an element of the rotor group $\text{Spin}(N)$, the Lie group generated by the Lie algebra of bivectors. A multivector \mathbf{a} transforms under a rotor R as

$$R : \mathbf{a} \rightarrow R\mathbf{a}\tilde{R}, \tag{14}$$

where $\tilde{R} = R^{-1}$ is the reverse of R . A spinor ψ transforms under a rotor R as

$$R : \psi \rightarrow R\psi. \tag{15}$$

A rotor R is an even, unimodular ($R\tilde{R} = 1$) multivector, but only in dimensions $N \leq 5$ are all even, unimodular multivectors also rotors.

The action of the $N(N-1)/2$ bivectors S_A on the $2^{[N/2]}$ basis spinors ϵ_a defines a set of $2^{[N/2]} \times 2^{[N/2]}$ matrices,

$$S_A : \epsilon_a \rightarrow (S_A)_{ab} \epsilon_b. \tag{16}$$

The spinors ϵ_a and the accompanying set of matrices $(S_A)_{ab}$ define the $2^{[N/2]}$ dimensional spinor representation of the rotor group. The matrices of the spinor representation depend on the choice of basis spinors. Section 3.1 derives the spinor representation in the chiral representation, that is, with respect to the chiral basis of spinors.

2.4. Spinor Metric

The spinor metric ε is as fundamental to the supergeometric algebra as the Euclidean (or Minkowski) metric is to the geometric algebra. The spinor metric ε is that spinor tensor that is invariant under rotations, suitably normalized, in the same way that the Euclidean metric is that vector tensor that is invariant under rotations, suitably normalized. Cartan [5] calls the spinor metric the fundamental polar, which he denotes C . It is worth remarking that if all dimensions are spacelike, then the spinor metric ε coincides with the conjugation operator, here denoted C , Sect. 6.1; but if one or more dimensions are timelike, then the spinor metric ε and conjugation operator C differ, equation (74).

A scalar product $\psi \cdot \chi$ of spinors ψ and χ transforms under a rotor R as

$$R : \psi \cdot \chi \equiv \psi^\top \varepsilon \chi \rightarrow \psi^\top R^\top \varepsilon R \chi = \psi^\top \varepsilon \chi, \tag{17}$$

so the spinor metric must satisfy $R^\top \varepsilon R = \varepsilon$, or equivalently

$$R^\top \varepsilon = \varepsilon \tilde{R}. \tag{18}$$

The condition (18) is determined by the commutation properties of the spinor metric ε with the orthonormal bivectors of the geometric algebra. Condition (18) holds if ε commutes with orthonormal basis bivectors whose representation is real, and anticommutes with orthonormal basis bivectors whose representation is imaginary (in this context, a timelike vector is to be construed as i times a spacelike orthonormal vector). In the chiral representation constructed in Sect. 3.1, all chiral basis vectors γ_k and $\gamma_{\bar{k}}$ are real, so in the chiral representation orthonormal basis vectors γ_k^+ (along with γ_N if N is odd) are real, while γ_k^- are imaginary, equations (5). The only matrix ε with the required commutation properties with basis bivectors is, up to a scalar or pseudoscalar normalization factor, the product of the basis vectors γ_k^+ (along with γ_N if N is odd),

$$\varepsilon = \prod_{k=1}^{[(N+1)/2]} \gamma_k^+, \tag{19}$$

where the final factor $\gamma_{[(N+1)/2]}^+$ is to be interpreted as γ_N if N is odd. An alternative version ε_{alt} of the spinor metric may be obtained by multiplying the spinor metric (19) by the chiral factor \varkappa_N , which is the pseudoscalar (product of all N vectors) normalized by a power of i so that \varkappa_N^2 equals one, equation (56),

$$\varepsilon_{\text{alt}} \equiv \varkappa_N \varepsilon = \prod_{k=1}^{[N/2]} i \gamma_k^-. \tag{20}$$

The factors of the imaginary i are introduced so that the spinor metric ε_{alt} is real. Notwithstanding the equality of ε and $\prod_k \gamma_k^+$ (or of ε_{alt} and $\prod_k i \gamma_k^-$) in the chiral representation, ε (or ε_{alt}) is defined to transform as a Lorentz-invariant spinor tensor under rotations, not as an element of the geometric algebra. The vectors γ_k^+ and γ_k^- in the expressions (19) and (20) are to be interpreted as being spacelike (timelike vectors being treated as i times a spacelike vector). The spinor metric ε or ε_{alt} is the same independent of whether orthonormal vectors are spacelike or timelike.

If N is odd, and if the odd algebra is constructed as described in Sect. 4.4, then there are potentially further options for the spinor metric, relegated to Appendix C.

The spinor metric ε , in either of the forms (19) or (20), is real and orthogonal, and its square is plus or minus the unit matrix,

$$\varepsilon^{-1} = \varepsilon^\top, \quad \varepsilon^2 = \pm 1, \tag{21}$$

where the \pm sign is as tabulated in Table 1. For odd N , the sign agrees with that of Cartan [5], eq. (13), Sect. 101 on p. 88; for even N , the sign agrees with Cartan’s for the alternative spinor metric ε_{alt} , and is minus Cartan’s for the standard spinor metric ε . Table 1 exhibits the well-known period-8 Cartan–Bott [2,8,21] periodicity of geometric algebras. For Dirac spinors in $N = 4$

TABLE 1. Symmetry of spinor metric in N spacetime dimensions

N	ε^2 $(-)^{\lfloor(N+1)/4\rfloor}$	$\varepsilon_{\text{alt}}^2$ $(-)^{\lfloor(N+2)/4\rfloor}$
1 (mod 8)	+	+
2 (mod 8)	+	-
3 (mod 8)	-	-
4 (mod 8)	-	-
5 (mod 8)	-	-
6 (mod 8)	-	+
7 (mod 8)	+	+
8 (mod 8)	+	+

spacetime dimensions, the spinor metric is the standard spinor metric (19), which is antisymmetric according to Table 1.

According to equation (47) (or (54) for ε_{alt}), the components ε_{ba} of the spinor metric ε ,

$$\varepsilon_{ba} \equiv \epsilon_b \cdot \epsilon_a \equiv \epsilon_b^\top \varepsilon \epsilon_a, \tag{22}$$

are non-vanishing only between basis spinors ϵ_b and ϵ_a that are bit flips of each other, as must be so because the scalar product must have zero charge. The sign of $\varepsilon_{\bar{a}a}$, where \bar{a} denotes the bit flip of a , follows inductively from equations (38), and is

$$\varepsilon \epsilon_a = \text{sign}(\varepsilon_{\bar{a}a}) \epsilon_{\bar{a}} \quad \text{sign}(\varepsilon_{\bar{a}a}) \equiv \text{sign}(\varepsilon_{\bar{a}_1 \dots \bar{a}_{\lfloor N/2 \rfloor} a_1 \dots a_{\lfloor N/2 \rfloor}}) = \prod_{a_k = \uparrow} (-)^{k-1}. \tag{23}$$

For the alternative spinor metric (20), the sign is

$$\varepsilon_{\text{alt}} \epsilon_a = \text{sign}(\varepsilon_{\bar{a}a}^{\text{alt}}) \epsilon_{\bar{a}} \quad \text{sign}(\varepsilon_{\bar{a}a}^{\text{alt}}) \equiv \text{sign}(\varepsilon_{\bar{a}_1 \dots \bar{a}_{\lfloor N/2 \rfloor} a_1 \dots a_{\lfloor N/2 \rfloor}}^{\text{alt}}) = \prod_{a_k = \uparrow} (-)^k. \tag{24}$$

The spinor metric is symmetric or antisymmetric as its square is positive or negative,

$$\varepsilon_{ab} = \pm \varepsilon_{ba}, \tag{25}$$

where the \pm sign is as tabulated in Table 1.

Commuting the spinor metric ε through the orthonormal basis vectors γ_a converts them to plus or minus their transposes,

$$\gamma_a^\top \varepsilon = \pm \varepsilon \gamma_a. \tag{26}$$

Table 2 tabulates the sign in equation (26) for the spinor metric ε and the alternative spinor metric ε_{alt} .

Equation (26) implies that the commutation rule of an orthonormal multivector γ_A of grade p with the spinor metric ε is

$$\gamma_A^\top \varepsilon = (\pm)^p \varepsilon \tilde{\gamma}_A = (\pm)^p (-)^{\lfloor p/2 \rfloor} \varepsilon \gamma_A, \tag{27}$$

TABLE 2. Sign of $\gamma_a^\top \varepsilon = \pm \varepsilon \gamma_a$ in N spacetime dimensions

N	ε $(-)^{\lfloor (N+3)/2 \rfloor}$	ε_{alt} $(-)^{\lfloor N/2 \rfloor}$
1 (mod 8)	+	+
2 (mod 8)	+	-
3 (mod 8)	-	-
4 (mod 8)	-	+
5 (mod 8)	+	+
6 (mod 8)	+	-
7 (mod 8)	-	-
8 (mod 8)	-	+

where $\tilde{\gamma}_A$ is the reverse of γ_A , and the \pm sign in $(\pm)^p$ is that in equation (26), which depends on dimension N as tabulated in Table 2.

Indices on a spinor are raised and lowered by pre-multiplying by the spinor metric and its inverse,

$$\epsilon^a = \varepsilon^{ab} \epsilon_b, \quad \epsilon_b = \varepsilon_{ba} \epsilon^a. \tag{28}$$

2.5. Column and Row Spinors

Corresponding to any column basis spinor ϵ_a is a row basis spinor $\epsilon_a \cdot$ defined by

$$\epsilon_a \cdot \equiv \epsilon_a^\top \varepsilon, \tag{29}$$

(or by $\epsilon_a \cdot \equiv \epsilon_a^\top \varepsilon_{\text{alt}}$ if the alternative spinor metric ε_{alt} is used). The row spinor $\psi \cdot$ corresponding to a column spinor $\psi = \psi^a \epsilon_a$ is

$$\psi \cdot \equiv \psi^\top \varepsilon = \psi^a \epsilon_a \cdot. \tag{30}$$

Whereas a column spinor ψ transforms under a rotor R as (15), a row spinor $\psi \cdot$ transforms as

$$R: \psi \cdot \equiv (R\psi)^\top \varepsilon \rightarrow \psi^\top R^\top \varepsilon = \psi^\top \varepsilon \tilde{R} = \psi \cdot \tilde{R}. \tag{31}$$

According to the usual rules of matrix multiplication, it is possible to multiply a row spinor by a column spinor, and a column spinor by a row spinor, but not a row spinor by a row spinor, nor a column spinor by a column spinor, equations (1). Rather than forbidding multiplication, it is advantageous to consider the supergeometric algebra as consisting of linear combinations of all species — scalars (= complex numbers), column spinors, row spinors, and multivectors — and setting the product of column spinors with column spinors, and row spinors with row spinors, to nothing.

The product of a row spinor $\psi \cdot$ and a column spinor χ is a scalar (a complex number),

$$\psi \cdot \chi = \psi^\top \varepsilon \chi. \tag{32}$$

Indeed the spinor metric ε was constructed precisely so that $\psi \cdot \chi$ transforms under a rotation as a scalar (17). In components, the scalar product of row and column spinors is

$$\psi \cdot \chi = \psi^b \varepsilon_{ba} \chi^a . \tag{33}$$

The scalar product is symmetric or antisymmetric as the spinor metric ε is symmetric or antisymmetric,

$$\psi \cdot \chi = \pm \chi \cdot \psi , \tag{34}$$

the sign being as given in Table 1.

The product of a column spinor χ and a row spinor $\psi \cdot$ defines their outer product $\chi\psi \cdot$. The outer product transforms under a rotation in the same way (14) as a multivector,

$$R : \chi\psi \cdot \equiv \chi\psi^\top \varepsilon \rightarrow (R\chi)(R\psi)^\top \varepsilon = R(\chi\psi \cdot) \tilde{R} . \tag{35}$$

Multiplication of outer products satisfies the associative rule

$$(\chi\psi \cdot)(\varphi\xi \cdot) = \chi(\psi \cdot \varphi)\xi \cdot , \tag{36}$$

which since $\psi \cdot \varphi$ is a scalar is proportional to the outer product $\chi\xi \cdot$. The associative rule (36) makes it straightforward to simplify long sequences of products of column and row spinors. The trailing-dot notation for a row spinor makes the application of the associative rule transparent.

3. Algebra of Outer Products of Spinors

This Sect. 3 proves the Brauer–Weyl [3] theorem, that the algebra of outer products of spinors is isomorphic to the geometric (Clifford) algebra of multivectors, for the case of even spacetime dimensions N . The case of odd dimensions N is addressed in Sect. 4. The claimed isomorphism is established by an explicit representation in terms of column and row vectors for spinors, and matrices for multivectors in the geometric algebra. Section 3.1 takes the spinor metric to be the standard spinor metric ε , equation (19). Then Sect. 3.2 describes the modifications that must be made if the spinor metric is taken to be the alternative spinor metric ε_{alt} , equation (20).

Examples of the isomorphism between outer products of spinors and the geometric algebra are given for the Pauli algebra ($N = 2$ or 3) in Appendix A, equations (96) and (97), and for the Dirac algebra ($N = 4$) in Appendix B, equations (103)–(108).

Since $[N/2]$, the integer part of half the number of dimensions N , appears so ubiquitously, it is convenient to define

$$n \equiv [N/2] . \tag{37}$$

3.1. Chiral Representation of the Spergeometric Algebra

The construction in this Sect. 3.1 yields the chiral representation of spinors and multivectors in even $N = 2n$ dimensions, generated inductively starting from $N = 0$. Given a representation of column and row basis spinors ϵ_A and $\epsilon_{A\cdot}$ in $N-2$ dimensions, a representation of column and row basis spinors ϵ_{Aa} and $\epsilon_{Aa\cdot}$ (with one extra index $a = \uparrow$ or \downarrow) in N dimensions are column and row matrices of length 2^n ,

$$\epsilon_{A\uparrow} = \begin{pmatrix} \epsilon_A \\ 0 \end{pmatrix}, \quad \epsilon_{A\uparrow\cdot} = (0 \ \epsilon_{A\cdot}), \tag{38a}$$

$$\epsilon_{A\downarrow} = \begin{pmatrix} 0 \\ \epsilon_A \end{pmatrix}, \quad \epsilon_{A\downarrow\cdot} = ((-)^{n-1} \epsilon_{A\cdot} \ 0), \tag{38b}$$

where 0 represents respectively a zero column or row vector of length 2^{n-1} . The induction starts at $n = 1$ where A is empty and $\epsilon_A = \epsilon_{A\cdot} = 1$. The trailing dot signifies the spinor metric tensor ϵ . The construction (38) assumes that the spinor metric ϵ is a product (19) of factors, the last factor γ_n^+ taking the form (45), so that the relation between the spinor metric in N and $N-2$ dimensions is given by equation (47).

The outer products of the column basis spinors ϵ_{Aa} and row basis spinors $\epsilon_{Bb\cdot}$ given by the inductive relations (38) are $2^n \times 2^n$ matrices

$$\epsilon_{A\uparrow} \epsilon_{B\uparrow\cdot} = \begin{pmatrix} 0 & \epsilon_A \epsilon_{B\cdot} \\ 0 & 0 \end{pmatrix}, \tag{39a}$$

$$\epsilon_{A\uparrow} \epsilon_{B\downarrow\cdot} = \begin{pmatrix} (-)^{n-1} \epsilon_A \epsilon_{B\cdot} & 0 \\ 0 & 0 \end{pmatrix}, \tag{39b}$$

$$\epsilon_{A\downarrow} \epsilon_{B\uparrow\cdot} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_A \epsilon_{B\cdot} \end{pmatrix}, \tag{39c}$$

$$\epsilon_{A\downarrow} \epsilon_{B\downarrow\cdot} = \begin{pmatrix} 0 & 0 \\ (-)^{n-1} \epsilon_A \epsilon_{B\cdot} & 0 \end{pmatrix}, \tag{39d}$$

where the 0's in equations (39) represent zero $2^{n-1} \times 2^{n-1}$ matrices. Again, the induction (39) starts at $n = 1$ where A and B are empty, and $\epsilon_A \epsilon_{B\cdot} = 1$. The outer products (39) can be rewritten

$$\epsilon_{A\uparrow} \epsilon_{B\uparrow\cdot} = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_A \epsilon_{B\cdot} & 0 \\ 0 & \pm \epsilon_A \epsilon_{B\cdot} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{40a}$$

$$\epsilon_{A\uparrow} \epsilon_{B\downarrow\cdot} = (-)^{n-1} \frac{1}{2} \begin{pmatrix} \epsilon_A \epsilon_{B\cdot} & 0 \\ 0 & \pm \epsilon_A \epsilon_{B\cdot} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{40b}$$

$$\epsilon_{A\downarrow} \epsilon_{B\uparrow\cdot} = \pm \frac{1}{2} \begin{pmatrix} \epsilon_A \epsilon_{B\cdot} & 0 \\ 0 & \pm \epsilon_A \epsilon_{B\cdot} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \tag{40c}$$

$$\epsilon_{A\downarrow} \epsilon_{B\downarrow\cdot} = \pm (-)^{n-1} \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_A \epsilon_{B\cdot} & 0 \\ 0 & \pm \epsilon_A \epsilon_{B\cdot} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \tag{40d}$$

where the upper/lower sign is for even/odd $\epsilon_A \epsilon_B \cdot$ (that is, the total charge of $\epsilon_A \epsilon_B \cdot$ is even/odd; multivectors of even and odd grade are respectively even and odd). The first matrix on the right hand sides of equations (40) is the matrix representation of the multivector $\epsilon_A \epsilon_B \cdot$ in N dimensions in terms of its representation in $N-2$ dimensions,

$$\epsilon_A \epsilon_B \cdot = (-)^{n-1} \epsilon_{A\uparrow} \epsilon_{B\downarrow} \cdot \pm \epsilon_{A\downarrow} \epsilon_{B\uparrow} \cdot = \begin{pmatrix} \epsilon_A \epsilon_B \cdot & 0 \\ 0 & \pm \epsilon_A \epsilon_B \cdot \end{pmatrix}. \tag{41}$$

The rightmost factors in equations (40) constitute the matrix representations of the chiral multivectors γ_n , $\gamma_n \gamma_{\bar{n}}$, $\gamma_{\bar{n}} \gamma_n$, and $\gamma_{\bar{n}}$ in $N = 2n$ dimensions,

$$\gamma_n = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \gamma_n \gamma_{\bar{n}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{42a}$$

$$\gamma_{\bar{n}} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \gamma_{\bar{n}} \gamma_n = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \tag{42b}$$

which have the correct normalization and commutation rules with respect to each other. The signs in equations (40) are arranged so that the correct commutation rules of the geometric algebra are respected: γ_n and $\gamma_{\bar{n}}$, which are odd, commute/anticommute with $\epsilon_A \epsilon_B \cdot$ according as the latter is even/odd; and $\gamma_n \gamma_{\bar{n}}$ and $\gamma_{\bar{n}} \gamma_n$, which are even, always commute with $\epsilon_A \epsilon_B \cdot$. In terms of scalar and wedge products, the multivectors $\gamma_n \gamma_{\bar{n}}$ and $\gamma_{\bar{n}} \gamma_n$ in equations (42) are

$$\gamma_n \gamma_{\bar{n}} = \gamma_n \cdot \gamma_{\bar{n}} + \gamma_n \wedge \gamma_{\bar{n}}, \quad \gamma_{\bar{n}} \gamma_n = \gamma_n \cdot \gamma_{\bar{n}} - \gamma_n \wedge \gamma_{\bar{n}}, \tag{43a}$$

$$\gamma_n \cdot \gamma_{\bar{n}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_n \wedge \gamma_{\bar{n}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{43b}$$

Note that chiral and orthonormal bivectors are related by

$$\gamma_n \wedge \gamma_{\bar{n}} = -i \gamma_n^+ \wedge \gamma_{\bar{n}}^-, \tag{44}$$

so that $(\gamma_n \wedge \gamma_{\bar{n}})^2 = 1$. The orthonormal basis vectors γ_n^+ and $\gamma_{\bar{n}}^-$ at the n th step are

$$\gamma_n^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{\bar{n}}^- = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{45}$$

which are traceless, unitary, and Hermitian. The orthonormal basis bivector $\gamma_n^+ \wedge \gamma_{\bar{n}}^-$ is

$$\gamma_n^+ \wedge \gamma_{\bar{n}}^- = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{46}$$

which is traceless, unitary, and skew-Hermitian. An iterative expression for the spinor metric ϵ_N follows from its expression (19) as a product of basis vectors, and is the antidiagonal matrix

$$\begin{aligned} \epsilon_N &= \epsilon_{N-2} \gamma_n^+ = \begin{pmatrix} \epsilon_{N-2} & 0 \\ 0 & (-)^{n-1} \epsilon_{N-2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \epsilon_{N-2} \\ (-)^{n-1} \epsilon_{N-2} & 0 \end{pmatrix}. \end{aligned} \tag{47}$$

The left factor in the rightmost expression in the top row of equations (47) is the matrix representation of ε_{N-2} in N dimensions in terms of its representation in $N-2$ dimensions, in accordance with equation (41). The factor of $(-)^{n-1}$ comes from the fact that the spinor metric ε_{N-2} is a product of $n-1$ basis vectors, equation (19), so is even/odd (total charge even/odd) as $n-1$ is even or odd. Equation (47), which was assumed in the initial step (38) of the construction of the chiral representation of the supergeometric algebra, proves the consistency of the construction.

The matrix representation of the column and row basis spinors (38) and of their outer products (40) is entirely real (with respect to i). The expansion of the 2^N outer products $\epsilon_a \epsilon_b \cdot$ of spinors in terms of the 2^N basis multivectors γ_A of the spacetime algebra, and vice versa, define the matrix of coefficients γ_A^a and its inverse γ_A^{ab} ,

$$\epsilon_a \epsilon_b \cdot = \gamma_{ab}^A \gamma_A, \quad \gamma_A = \gamma_A^{ab} \epsilon_a \epsilon_b \cdot. \tag{48}$$

The coefficients γ_{ab}^A and γ_A^{ab} in the chiral representation are

$$\gamma_{ab}^A = \frac{1}{2^n} \epsilon_b \cdot \gamma^A \epsilon_a, \quad \gamma_A^{ab} = \text{sign}(\varepsilon^2) \epsilon^a \cdot \gamma_A \epsilon^b, \tag{49}$$

where $\text{sign}(\varepsilon^2)$ is the symmetry of the spinor metric, Table 1.

The trace of the outer product of a pair of basis spinors gives their scalar product,

$$\text{Tr } \epsilon_a \epsilon_b \cdot = \epsilon_b \cdot \epsilon_a = \varepsilon_{ba}. \tag{50}$$

More generally, the trace of an outer product of spinors equals their scalar product,

$$\text{Tr } \chi \psi \cdot = \psi \cdot \chi. \tag{51}$$

The above construction establishes the claimed isomorphism between outer products of spinors and the geometric algebra in even dimensions N ,

$$\text{outer products of spinors} \cong \text{geometric algebra} \quad (N \text{ even}). \tag{52}$$

Both spaces are complex 2^N -dimensional vector spaces. Their representation as $2^n \times 2^n$ dimensional matrices is minimal: there is no representation of the geometric algebra with matrices of smaller dimension.

3.2. Chiral Representation of the Supergeometric Algebra Using the Alternative Spinor Metric

The chiral representation of the supergeometric algebra with the alternative spinor metric (20) is the same as the construction in Sect. 3.1, but with the replacement

$$(-)^{n-1} \rightarrow (-)^n \tag{53}$$

in equations (38) to (41). Analogously to equation (47), an iterative equation for the alternative spinor metric follows from its expression (20) as a product

of basis vectors, and is the antidiagonal matrix

$$\begin{aligned} \varepsilon_N^{\text{alt}} &= \varepsilon_{N-2}^{\text{alt}} i\gamma_n^- = \begin{pmatrix} \varepsilon_{N-2}^{\text{alt}} & 0 \\ 0 & (-)^{n-1} \varepsilon_{N-2}^{\text{alt}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varepsilon_{N-2}^{\text{alt}} \\ (-)^n \varepsilon_{N-2}^{\text{alt}} & 0 \end{pmatrix}. \end{aligned} \tag{54}$$

3.3. Properties of Orthonormal Basis Multivectors in the Chiral Representation

In the chiral representation constructed in sections 3.1 or 3.2, all orthonormal basis vectors γ_a , and all orthonormal basis p -vectors $\gamma_{a_1 \dots a_p} \equiv \gamma_{a_1} \wedge \dots \wedge \gamma_{a_p}$, are traceless (except for the unit basis element 1), unitary, and either Hermitian (if $[p/2]$ is even, i.e. $p = 0, 1, 4, 5, \dots$) or skew-Hermitian (if $[p/2]$ is odd, i.e. $p = 2, 3, 6, 7, \dots$) $2^n \times 2^n$ matrices. All matrices have determinant 1, except that for $N = 2$ the vectors (grade $p = 1$) have determinant -1 . The unit element is represented by the unit matrix. Most of these assertions can be proved by induction using the expression (41), which gives the representation of a multivector in N dimensions in terms of its representation in $N-2$ dimensions.

4. Supergeometric Algebra in Odd Dimensions

The natural complex structure of spinors means that supergeometric algebras are naturally even-dimensional. Section 3 proved the Brauer–Weyl theorem for even dimensions. What about odd dimensions N ?

There are essentially two options. One option, considered in Sect. 4.1, is to project the odd N -dimensional geometric algebra into one lower dimension. The other option, considered in Sect. 4.4, is to embed the odd N -dimensional geometric algebra in one higher dimension.

4.1. Supergeometric Algebra in Odd Dimensions, Version 1

The first approach, projecting the odd-dimensional algebra into one lower dimension, requires identifying the chiral operator \varkappa_N with 1, equation (59). This is the standard convention for geometric algebras in odd dimensions, including the Pauli algebra, $N = 3$.

The pseudoscalar I_N of the geometric algebra, in either even or odd dimensions N , is the product of all N orthonormal vectors (the orthonormal vectors γ_k^\pm and γ_N here are being taken to be spacelike, as in Sect. 3; if some of the vectors are timelike, then the usual physics convention for the pseudoscalar would have extra factors of i , one for each timelike dimension, but for simplicity that adjustment is not made here),

$$I_N \equiv \gamma_1^+ \wedge \gamma_1^- \wedge \dots \wedge \gamma_n^+ \wedge \gamma_n^- \{ \wedge \gamma_N \text{ if } N \text{ is odd} \} = i^n \varkappa_N \tag{55}$$

(with $n \equiv [N/2]$ as before). The chiral operator \varkappa_N is the pseudoscalar normalized by a phase so that its square is the unit matrix,

$$\varkappa_N \equiv \gamma_1 \wedge \gamma_1 \wedge \dots \wedge \gamma_n \wedge \gamma_n \{ \wedge \gamma_N \text{ if } N \text{ is odd} \}. \tag{56}$$

In the chiral representation (40), the representation of the chiral operator \varkappa_N in N even dimensions in terms of its representation \varkappa_{N-2} in $N-2$ dimensions is the diagonal matrix

$$\varkappa_N = \begin{pmatrix} \varkappa_{N-2} & 0 \\ 0 & -\varkappa_{N-2} \end{pmatrix} \quad (N \text{ even}) . \tag{57}$$

The chiral operator is diagonal in the chiral representation by construction. The square of the pseudoscalar is $I_N^2 = (-)^n$, so the square of the chiral operator is the unit matrix 1,

$$\varkappa_N^2 = 1 . \tag{58}$$

Like the pseudoscalar I_N , the chiral operator \varkappa_N is invariant under rotations.

For even N , the chiral operator \varkappa_N is defined through equation (56) as a prescribed member of both algebras, the algebra of spinor outer products and the geometric algebra. But for odd N , since the definition (56) involves γ_N which has (as yet) no expression in the algebra of outer products of spinors, there is the possibility that \varkappa_N could be a distinct element not belonging to the algebra of spinor outer products. The element \varkappa_N is a rotationally invariant scalar that squares to 1, and that, for odd N , commutes with all basis vectors γ_k^\pm , $k = 1, \dots, n$ other than γ_N . The other element of the odd- N algebra of spinor outer products that possesses those properties is (up to a possible sign) the unit element. Thus if the chiral operator \varkappa_N is identified with 1,

$$\varkappa_N = 1 \quad (N \text{ odd}) , \tag{59}$$

then there is an isomorphism between the algebra of outer products of spinors in $N-1$ dimensions and the geometric algebra in N dimensions modulo the chiral operator \varkappa_N ,

$$\text{outer products of spinors} \cong \text{geometric algebra (mod } \varkappa_N) \quad (N \text{ odd}) . \tag{60}$$

Given the identification (59) of the chiral operator with 1, it then follows from the definition equation (56) of \varkappa_N that the final element γ_N of the geometric algebra is

$$\gamma_N = \varkappa_{N-1} = \gamma_1 \wedge \gamma_{\bar{1}} \wedge \dots \wedge \gamma_n \wedge \gamma_{\bar{n}} \quad (N \text{ odd}) . \tag{61}$$

In the case $N = 3$, this gives

$$\gamma_3 = \gamma_1 \wedge \gamma_{\bar{1}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{62}$$

in agreement with the usual expression for the third Pauli matrix. With the identification (59), the pseudoscalar I_N itself is, equation (55),

$$I_N = i^n \quad (N \text{ odd}) . \tag{63}$$

The final basis vector γ_N , equation (61), of the odd algebra has the same properties as the other orthonormal basis vectors γ_k^\pm , $k = 1, \dots, n$: its square is 1, it anticommutes with the other orthonormal basis vectors, it is represented by a traceless, unitary, Hermitian matrix, and its reverse is (by

definition) itself, $\tilde{\gamma}_N = \gamma_N$. And, like the other orthonormal basis vectors γ_k^+ of odd index, the representation of γ_N is real.

For odd N , the chiral operator \varkappa_N defined by equation (56) is (before \varkappa_N is identified with 1) an odd element of the geometric algebra. Thus for odd N , the odd part of the geometric algebra is isomorphic to \varkappa_N times the even geometric algebra. Only the odd geometric algebra is affected by the identification (59) of the chiral operator with unity; the even geometric algebra is unaffected. The square of the chiral operator is always 1, equation (58), so the product of two odd multivectors yields the correct even multivector regardless of the identification (59).

The imaginary i must be introduced already in the earliest steps of the construction of the supergeometric algebra, where vectors are grouped into complex conjugate pairs, equation (5). One might ask where that imaginary came from? An intriguing observation is that if N is odd and $[N/2]$ is odd (thus $N = 3, 7, 11, \dots$), then the pseudoscalar I_N squares to -1 and commutes with all elements of the geometric algebra, just like the imaginary i . One might take the view that maybe that's where i comes from. Taking the view that I_N is indeed the imaginary is equivalent to indentifying the chiral operator \varkappa_N with unity, equation (59), in which case i is, up to a sign, the pseudoscalar I_N , equation (63).

In summary, the algebra of spinor outer products in $2[N/2]$ dimensions is isomorphic to the geometric algebra for both even and odd N , modulo \varkappa_N in the case of odd N . The algebra is a complex (with respect to i) vector space of dimension 2^n , represented in the chiral construction (40) by $2^n \times 2^n$ matrices. For example, the $N = 2$ geometric algebra is the algebra generated by $1, \gamma_1^+, \gamma_1^-, \gamma_1^+ \wedge \gamma_1^-$, while the $N = 3$ geometric algebra (the Pauli algebra) is the algebra generated by $1, \gamma_1^+, \gamma_1^-, \gamma_3$, the pseudoscalar I_3 being identified with the imaginary i .

4.2. Extra Symmetry of the Supergeometric Algebra in Odd Dimensions

Given that, if \varkappa_N is identified with 1, the geometric algebra for odd N is isomorphic to the geometric algebra for even $N-1$, what is the difference between the two algebras? Since the algebras are isomorphic, there is of course no difference. However, bivectors are special in that they are the only generators that generate transformations that preserve grade, and therefore correspond to what one usually thinks of as rotations. If one restricts to rotations generated by bivectors, then the odd algebra has a higher degree of symmetry. The equivalence (61) means that the pseudoscalar \varkappa_{N-1} in the even algebra is promoted to a vector γ_N in the odd algebra, and pseudovectors $\gamma_k^\pm \varkappa_{N-1}$ in the even algebra become bivectors $\gamma_k^\pm \wedge \gamma_N$ in the odd algebra. Thus the odd geometric algebra has $N-1$ more rotations than the even algebra.

The Pauli algebra in $N = 3$ dimensions offers a familiar example. In both 2 and 3 dimensions there are just 2 basis spinors, ϵ_\uparrow and ϵ_\downarrow , which one commonly conceptualizes as being up and down along a “3-axis”. But whereas in 2 dimensions there is just one rotation, generated by the bivector $\gamma_1^+ \wedge \gamma_1^-$ (rotation about the “3-axis”), in 3 dimensions there are 2 more

TABLE 3. Parity P and time reversal T operators in $K + M$ spacetime dimensions

K	M	$X = \gamma_a$ (spacelike)	$X = \gamma_m$ (timelike)
Even	Even	P	T
Odd	Odd	T	P
Even	Odd	PT	–
Odd	Even	–	PT

rotations, generated by the bivectors $\gamma_1^+ \wedge \gamma_3$ and $\gamma_1^- \wedge \gamma_3$ (rotations about the “2-axis” and “1-axis”).

4.3. Parity and Time Reversal

Parity reversal P is the operation of reflecting an odd number of spatial axes. Time reversal T is the operation of reflecting an odd number of time axes. The combination PT reflects an odd number of spatial axes and an odd number of time axes. Parity and time reversal are discrete transformations that cannot be achieved by any continuous rotation. A reflection of an even number of spatial axes can be accomplished by a continuous rotation in spatial dimensions, while a reflection of an even number of time axes can be accomplished by a continuous rotation in time dimensions.

Consider a transformation X that transforms spinors ψ and multivectors \mathbf{a} according to

$$X : \quad \psi \rightarrow X\psi, \quad \mathbf{a} \rightarrow X\mathbf{a}X^{-1}. \tag{64}$$

The form (64) of the X transformation, which is similar to the form of transformations (14) and (15) under rotations, ensures consistency between transformations of spinors and multivectors. If X is set equal to an orthonormal dimension γ_a , then the transformation (64) flips all orthonormal axes *except* γ_a . This assertion remains true in the situation considered in Sect. 4.1, where the number of N of spacetime dimensions is odd, and the final vector γ_N is identified with the $(N-1)$ -dimensional chiral operator, equation (61).

Table 3 summarizes how setting the operator X in the transformation (64) either to a spacelike orthonormal basis vector γ_a , or to a timelike orthonormal basis vector γ_m , yields a parity transformation P , a time reversal transformation T , a product PT of both, or neither. For example, if X is a spacelike orthonormal basis vector γ_a , then the transformation (64) flips $K-1$ of the K spacelike axes, and M of the M timelike axes, so flips P if K is even, and T if M is odd; and similarly if X is a timelike orthonormal basis vector γ_m , then the transformation (64) flips K of the K spacelike axes, and $M-1$ of the M timelike axes, so flips P if K is odd, and T if M is even. Setting X equal to products of spacelike/timelike basis vectors yields corresponding products of P , T , PT , or $-$. As Table 3 shows, individual P and T operators are available only if the total number $N = K+M$ of spacetime dimensions is even. If on the other hand the total number $N = K+M$ of

spacetime dimensions is odd, then only a combination PT operator is available (provided that there is at least one spatial dimension and at least one time dimension).

A solution to the problem of constructing an odd- N supergeometric algebra that incorporates individual parity and time reversal operators is presented in the next Sect. 4.4.

4.4. Supergeometric Algebra in Odd Dimensions, Version 2

The previous Sect. 4.3 brought up the fact that the geometric algebra in odd N dimensions does not contain a parity operator P or time reversal operator T .

The problem is not that the operations of parity or time reversal do not exist, but rather, how to construct such operators as elements of the supergeometric algebra.

A solution is to embed the odd N -dimensional algebra in the even $(N+1)$ -dimensional algebra, and to treat either the final (odd) dimension γ_N or the extra (even) orthonormal dimension γ_{N+1} as a scalar dimension. The dimension is scalar in the sense that, although it transforms under rotations in the enlarged group $\text{Spin}(N+1)$, it is fixed under rotations in the $\text{Spin}(N)$ subgroup of the enlarged group. Setting X equal to the scalar dimension,

$$X = \gamma_N \text{ or } \gamma_{N+1} , \quad (65)$$

yields an operator that flips the N other orthonormal vectors. Since N is odd, X flips an odd number of orthonormal vectors. X is a parity operator P if the number K of spatial dimensions is odd, or a time reversal operator T if K is even.

As usual, there is a spin bit (the $[(N+1)/2]$ th bit) associated with the pair γ_N and γ_{N+1} of axes. Normally a rotation in the $\gamma_N \wedge \gamma_{N+1}$ plane would rotate spinors by a phase $e^{\mp i\theta/2}$ with sign \mp depending on whether the spin bit is up \uparrow or down \downarrow . But since one of γ_N or γ_{N+1} is a scalar, there is no such rotation. Notwithstanding the absence of a rotation by a phase, the spin bit is still there, part of the bitcode index $a = a_1 \dots a_{[(N+1)/2]}$ of a basis spinor ϵ_a .

5. Chiral Subalgebras

Chirality, the eigenvalue of the chiral operator \varkappa_N , counts whether the number of up-bits of a spinor is even or odd, with right-handed defined as the all-bit-up spinor. In even spacetime dimensions N , rotations preserve chirality. [5] refers to spinors of definite (right- or left-handed) chirality as semi-spinors. In 3+1 spacetime dimensions, Dirac spinors of definite chirality are called Weyl spinors.

In odd N dimensions, if the path proposed in Sect. 4.1 is followed, where the algebra is projected into one lower dimension, which requires identifying the chiral operator \varkappa_N with unity, equation (59), then rotations mix right- and left-handed spinors, and chirality is not a rotationally invariant property of spinors.

If on the other hand N is odd and the path proposed in Sect. 4.4 is followed, where the algebra is embedded in one higher dimension, then a basis spinor ϵ_a has $[(N+1)/2]$ bits, and chirality is that \varkappa_{N+1} of the algebra in one higher dimension. In the rest of this Sect. 5, replace N by $N+1$ if N is odd and Sect. 4.4 is followed.

Right- and left-handed chiral multivectors are eigenvectors of the chiral operator \varkappa_N (or \varkappa_{N+1} if N is odd and Sect. 4.4 is followed), with eigenvalues ± 1 ,

$$\varkappa_N \mathbf{a}_{R/L} = \pm \mathbf{a}_{R/L} . \tag{66}$$

Right- and left-handed chirality projection operators $P_{R/L}$ may be defined by

$$P_{R/L} \equiv \frac{1}{2}(1 \pm \varkappa_N) = \frac{1}{2}(1 \pm i^{-n} I_N) , \tag{67}$$

which are projection operators because they are idempotent, $(P_R)^2 = P_R$ and $(P_L)^2 = P_L$, and their product is zero, $P_R P_L = 0$. A multivector \mathbf{a} splits into right- and left-handed chiral parts,

$$\mathbf{a} = \mathbf{a}_R + \mathbf{a}_L, \quad \mathbf{a}_{R/L} \equiv P_{R/L} \mathbf{a} . \tag{68}$$

Since the chiral operator \varkappa_N is proportional to the pseudoscalar I_N , a purely right- or left-handed multivector is necessarily a linear combination of a multivector \mathbf{a} and its Hodge dual $I_N \mathbf{a}$.

An outer product of a right-handed column spinor with any row spinor (right- or left-handed) is a right-handed multivector. An outer product of a left-handed column spinor with any row spinor is a left-handed multivector.

6. Conjugation

Supergeometric algebras have a natural complex structure. Complex conjugation is a discrete operation that transforms a spinor into its antispinor partner. However, complex conjugation of a spinor is by itself not a Lorentz-invariant operation. The conjugation operator C is introduced to provide a Lorentz-invariant version of complex conjugation.

6.1. Conjugation Operator

In the chiral representation, chiral basis spinors ϵ_a are real, and the complex conjugate of a spinor $\psi \equiv \psi^a \epsilon_a$ is

$$\psi^* \equiv \psi^{a*} \epsilon_a . \tag{69}$$

The complex conjugate spinor ψ^* transforms under a rotor R not like the spinor ψ , but rather as

$$R : \psi^* \rightarrow R^* \psi^* . \tag{70}$$

The rotationally-invariant conjugation operator C is defined such that commutation with it converts rotors R to their complex conjugates,

$$C R^* = R C . \tag{71}$$

Note that since a rotor R is a real linear combination of even orthonormal basis multivectors, the complex conjugate R^* of a rotor R is a rotor. The conjugate $\bar{\psi}$ of a spinor is defined by

$$\bar{\psi} \equiv C\psi^* . \tag{72}$$

Thanks to the condition (71) on the conjugation operator C , the conjugate spinor $\bar{\psi}$ transforms under a rotor R in the same way as the spinor ψ ,

$$R : \bar{\psi} \equiv C\psi^* \rightarrow C(R\psi)^* = CR^*\psi^* = RC\psi^* = R\bar{\psi} . \tag{73}$$

If all dimensions are spacelike, then rotors are unitary, $R^{-1} = R^\dagger$, in which case the condition (71) on the conjugation operator C is the same as the condition (18) on the spinor metric ε , and the conjugation operator is just equal to the spinor metric, $C = \varepsilon$ (or to the alternative spinor metric, $C_{\text{alt}} = \varepsilon_{\text{alt}}$). But if there are M timelike dimensions, then each timelike dimension can be treated as i times a spacelike dimension, introducing an extra minus sign in the condition (71) compared to the spacelike case, for each timelike factor. The result is that the conjugation operator C equals, modulo a normalization factor, the product of the spinor metric tensor ε (or the alternative spinor metric tensor ε_{alt}) with the product of all timelike orthonormal basis vectors,

$$C = \varepsilon \mathbf{\Gamma}^\top , \quad \mathbf{\Gamma} \equiv c \prod_m \gamma_m(\text{timelike}) , \tag{74}$$

where the phase factor c is chosen such that the square of $\mathbf{\Gamma}$ is the unit matrix,

$$\mathbf{\Gamma}^2 = 1 . \tag{75}$$

The conjugation operator C is unitary, as is $\mathbf{\Gamma}$. The condition (75) implies that the eigenvalues of $\mathbf{\Gamma}$ are ± 1 , which ensures that $\bar{\psi} \cdot \psi$ is real, equation (79). If there is at least one time dimension (and, if N is odd and Sect. 4.1 is followed, at least one spatial dimension), then the trace of $\mathbf{\Gamma}$ is zero, implying that there are equal numbers of $+1$ and -1 eigenvalues. If there is just one time dimension γ_0 , then $\mathbf{\Gamma}$ is (the minus sign is optional, but conventional in the Dirac algebra)

$$\mathbf{\Gamma} = -i\gamma_0 . \tag{76}$$

Notwithstanding equation (74), the conjugation operator C is defined by equation (71), and does not transform as an element of the geometric algebra, but rather is a spinor tensor constructed such that conjugation, equation (72), is a Lorentz-invariant operation.

The double conjugate of a spinor is

$$\bar{\bar{\psi}} = C(C\psi^*)^* = CC^*\psi = \pm\psi , \tag{77}$$

where the sign is $+$ or $-$ depending on whether the conjugation operator is symmetric, $C = C^\top$, or antisymmetric (the symmetry condition $C = C^\top$ is equivalent to $CC^* = 1$ in view of the unitarity of C). Table 4 shows the symmetry of the conjugation operator C for the standard and alternative spinor metrics. Table 4 is essentially identical to the earlier Table 1, except that the number $N = K+M$ of spacetime dimensions in Table 1 is changed

TABLE 4. Symmetry of the conjugation operator C in $K+M$ spacetime dimensions

$K - M$	C	C_{alt}
1 (mod 8)	+	+
2 (mod 8)	+	-
3 (mod 8)	-	-
4 (mod 8)	-	-
5 (mod 8)	-	-
6 (mod 8)	-	+
7 (mod 8)	+	+
8 (mod 8)	+	+

to the difference $K-M$ of numbers K and M of space and time dimensions in Table 4. For Dirac spinors in 3+1 dimensions, for which $K-M = 3-1 = 2$, the conventional choice is the standard spinor metric (19), which ensures that the conjugation operator is symmetric, hence that the double conjugate of a Dirac spinor is itself, $\bar{\bar{\psi}} = \psi$.

The scalar product of a conjugate spinor $\bar{\psi}$ with a spinor χ is

$$\bar{\psi} \cdot \chi = \psi^\dagger C^T \varepsilon \chi = \psi^\dagger \mathbf{\Gamma} \chi, \tag{78}$$

which is a complex (with respect to i) number. The scalar product of a spinor ψ with its conjugate is

$$\bar{\psi} \cdot \psi = \psi^\dagger \mathbf{\Gamma} \psi, \tag{79}$$

which is real given that the eigenvalues of $\mathbf{\Gamma}$ are real. In zero time dimensions the scalar product of a spinor with its conjugate is always positive, but with one or more time dimensions the scalar product of a spinor with its conjugate can be either positive or negative (or zero).

The scalar product $\bar{\psi} \cdot \mathbf{\Gamma} \chi$ is

$$\bar{\psi} \cdot \mathbf{\Gamma} \chi = \psi^\dagger \chi. \tag{80}$$

In particular, $\bar{\psi} \cdot \mathbf{\Gamma} \psi$ is real and positive,

$$\bar{\psi} \cdot \mathbf{\Gamma} \psi = \psi^\dagger \psi. \tag{81}$$

6.2. Conjugate Multivectors

The complex conjugate (with respect to i) \mathbf{a}^* of a multivector $\mathbf{a} = a^A \gamma_A$ is defined to be its complex conjugate in the chiral representation (40) of multivectors,

$$\mathbf{a}^* = a^{A*} \gamma_A^*. \tag{82}$$

In the representation (40), the chiral basis vectors γ_k and $\gamma_{\bar{k}}$ (and the final vector γ_N if N is odd) are real, so the orthonormal basis vectors γ_k^+ and γ_k^- are respectively real and imaginary. The conjugate $\bar{\mathbf{a}}$ of a multivector

$\mathbf{a} = a^A \gamma_A$ is defined to be, consistent with the definition (72) of the conjugate of a spinor,

$$\bar{\mathbf{a}} \equiv C \mathbf{a}^* C^{-1} . \tag{83}$$

The conjugate multivector $\bar{\mathbf{a}}$ rotates under a rotor R in the same way as the multivector \mathbf{a} ,

$$\begin{aligned} R : \bar{\mathbf{a}} \equiv C \mathbf{a}^* C^{-1} &\rightarrow C(R \mathbf{a} \tilde{R})^* C^{-1} = C R^* \mathbf{a}^* \tilde{R}^* C^{-1} \\ &= R C \mathbf{a}^* C^{-1} \tilde{R} = R \bar{\mathbf{a}} \tilde{R} . \end{aligned} \tag{84}$$

Conjugation is multiplicative over multivectors, and over multivectors with spinors,

$$\overline{\mathbf{a}\mathbf{b}} = \bar{\mathbf{a}}\bar{\mathbf{b}} , \quad \overline{\mathbf{a}\psi} = \bar{\mathbf{a}}\bar{\psi} . \tag{85}$$

The conjugate $\bar{\mathbf{a}}$ of an outer product $\mathbf{a} \equiv \psi \bar{\chi} \cdot$ of spinors is related to the outer product $\bar{\psi} \chi \cdot$ by

$$\mathbf{a} \equiv \psi \bar{\chi} \cdot , \quad \bar{\mathbf{a}} \equiv \overline{\psi \bar{\chi} \cdot} = \pm \bar{\psi} \chi \cdot , \tag{86}$$

where the \pm sign is the symmetry of the spinor metric, Table 1.

The conjugate of a basis multivector γ_A is defined to be

$$\bar{\gamma}_A \equiv C \gamma_A^* C^{-1} , \tag{87}$$

so that a conjugate multivector $\bar{\mathbf{a}}$ is

$$\bar{\mathbf{a}} = a^A \bar{\gamma}_A . \tag{88}$$

The conjugate of a spacelike or timelike orthonormal basis vector γ_m is (if γ_m is a timelike basis vector, γ_m here means the timelike vector itself, whose square is -1)

$$\bar{\gamma}_m = \pm (-)^M \gamma_m , \tag{89}$$

where the \pm factor is as given in Table 2.

A real subalgebra of the complex geometric algebra may be obtained by restricting to multivectors satisfying the reality condition that they are their own conjugates,

$$\bar{\mathbf{a}} = \mathbf{a} . \tag{90}$$

Conjugates of orthonormal basis vectors are plus or minus themselves per equation (89). If the overall sign $\pm (-)^M$ in equation (89) is $+$, as it is for example in the (3+1)-dimensional Dirac algebra, then the real subalgebra consists of real linear combinations of orthonormal basis multivectors. If the sign $\pm (-)^M$ in equation (89) is $-$, then the real subalgebra consists of linear combinations of odd-grade orthonormal multivectors with pure imaginary coefficients and even-grade orthonormal multivectors with pure real coefficients.

7. Conclusions

All matter (fermions) is made from spinors. All forces arise from symmetries of spinors. It would not be surprising to find that spinors are at the heart of a theory that unifies all forces of nature.

The index of a spinor in N dimensions is a bitcode with $[N/2]$ bits. It is delightful to contemplate that the laws of physics might be written in a language whose letters are a spinor bitcode.

Whereas a vector sticks out in one dimension at a time, a spinor sticks out in all dimensions at once. A basis spinor is either up (\uparrow) or down (\downarrow), not zero, in each pair of dimensions.

While it is common to consider the geometric (Clifford) algebra as the foundation upon which spinors are built, this paper advances the alternative view that spinors are more fundamental than the geometric algebra. To this end, the paper introduces the supergeometric algebra, which consists of complex linear combinations of scalars (complex numbers), column spinors, row spinors, and multivectors, and the various multiplication rules that connect them, equations (1). The exclusion principle, which is a central property of spinors in physics, is not apparent in the geometric algebra, but becomes apparent in the language of the supergeometric algebra.

Among other things, this paper gives a proof, Sect. 3, of the Brauer–Weyl (1935) [3] theorem, that the algebra of outer products of spinors is isomorphic to the geometric algebra of multivectors. The theorem formalizes the notion that the geometric algebra is the ‘square’ of the supergeometric algebra, or equivalently that the supergeometric algebra is the ‘square root’ of geometry [22].

The algebra of outer products of spinors yields not only vectors but the entire geometric algebra of multivectors. Since multivectors of grade p are associated with geometric objects of dimension p , it is intriguing to infer that geometric objects of arbitrary dimension, which are called branes in string theory (M theory), are an intrinsic part of a theory built from spinors.

This paper introduces the convenient notation $\psi\cdot$ with a trailing dot to signify a row spinor, equation (30). The trailing dot represents the spinor metric tensor ε . A row spinor $\psi\cdot$ times a column spinor χ defines their inner product $\psi\cdot\chi$, a scalar. A column spinor χ times a row spinor $\psi\cdot$ defines their outer product $\chi\psi\cdot$, which transforms like a multivector. The trailing-dot notation makes transparent the application of the associative rule to a sequence of products of row and column spinors, equation (36).

Acknowledgements

This research was supported in part by FQXI mini-grant FQXI-MGB-1626.

Data statement Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix A. The Pauli supergeometric algebra

The supergeometric algebra in 2 or 3 spatial dimensions consists of Pauli spinors and the Pauli algebra, and their various products.

In $N = 2$ or 3 dimensions, there is just $[N/2] = 1$ bit, and $2^{[N/2]} = 2$ basis column spinors with respectively spin up and spin down,

$$\epsilon_a = \{ \epsilon_{\uparrow}, \epsilon_{\downarrow} \}. \tag{91}$$

As column vectors in the chiral representation, the two basis spinors are

$$\epsilon_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \epsilon_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{92}$$

In Dirac bra-ket notation, the spinors would be written $\epsilon_{\uparrow} = |\uparrow\rangle$ and $\epsilon_{\downarrow} = |\downarrow\rangle$. The outer products of the 2 column basis spinors ϵ_a with the 2 row basis spinors $\epsilon_b \cdot$ form 4 outer products, yielding the 2D geometric algebra with $2^2 = 4$ basis elements. The 3D geometric algebra, the Pauli algebra, has $2^3 = 8$ basis elements, but is projected into 2D by identifying the pseudoscalar I_3 with i times the unit matrix, equation (63).

The orthonormal vectors γ_a of the Pauli algebra are commonly denoted $\sigma_a = \{ \sigma_1, \sigma_2, \sigma_3 \}$. In the notation of paired orthonormal vectors γ_k^{\pm} of the present paper, the orthonormal Pauli vectors are, equations (45),

$$\gamma_1^+ = \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1^- = \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{93}$$

The 2D and 3D pseudoscalars I_2 and I_3 are

$$I_2 \equiv \gamma_1^+ \gamma_1^- = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_3 \equiv \gamma_1^+ \gamma_1^- \gamma_3 = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{94}$$

The third vector γ_3 is, modulo a phase, equal to the 2-dimensional pseudoscalar, equation (61), $\gamma_3 = -iI_2$. The chiral basis vectors (5) of the Pauli algebra are

$$\gamma_1 = \frac{\sigma_1 + i\sigma_2}{\sqrt{2}} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \gamma_{\bar{1}} = \frac{\sigma_1 - i\sigma_2}{\sqrt{2}} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}. \tag{95}$$

The bivectors of the 3D Pauli algebra are the pseudovectors $I_3\sigma_a = i\sigma_a$.

The algebra of outer products of spinors in 2 or 3 dimensions is as follows. The antisymmetric outer products of spinors form a scalar singlet,

$$[\epsilon_{\downarrow}, \epsilon_{\uparrow}] \cdot = 1, \tag{96}$$

where the 1 on the right hand side denotes the unit element, the 2×2 identity matrix. The trailing dot on the commutator indicates that the right partner of each product is a row spinor, $[\epsilon_{\uparrow}, \epsilon_{\downarrow}] \cdot = \epsilon_{\uparrow} \epsilon_{\downarrow} \cdot - \epsilon_{\downarrow} \epsilon_{\uparrow} \cdot$. The spin charge of the singlet (96) is zero according to the rule (13). The spin charge, zero, of the left and right hand sides match, as they should.

The symmetric outer products of spinors yield the chiral vectors of the Pauli algebra,

$$\{\epsilon_{\uparrow}, \epsilon_{\uparrow}\} \cdot = \sqrt{2} \gamma_1, \quad \{\epsilon_{\uparrow}, \epsilon_{\downarrow}\} \cdot = -\gamma_3, \quad \{\epsilon_{\downarrow}, \epsilon_{\downarrow}\} \cdot = -\sqrt{2} \gamma_{\bar{1}}. \tag{97}$$

The spin charges of the vectors (97) are respectively +1, 0, -1 according to the rules (8) and (13). The spin charges of the left and right hand sides match, as they should.

The spinor metric ε in 3 dimensions is, equations (19) or (20),

$$\varepsilon \equiv i \gamma_{\bar{1}}^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{98}$$

The spinor metric ε is antisymmetric, in agreement with Table 1. Despite the equality of ε and $i \gamma_{\bar{1}}^-$ in the Pauli representation, ε is defined to transform as a spinor tensor under spatial rotations, not as an element of the geometric algebra. As always in a geometric algebra with only spatial (no time) dimensions, the conjugation operator C coincides with the spinor metric ε , equation (74),

$$C = \varepsilon. \tag{99}$$

Appendix B. The Dirac supergeometric algebra

The supergeometric algebra in $N = 3+1$ spacetime dimensions consists of Dirac spinors and the Dirac algebra, and their various products. The Dirac algebra is summarized in a notation consistent with the present paper in Appendix A of [14], and that summary will not be repeated here. The present Appendix gives the relation between outer products of basis Dirac spinors and the basis multivectors of the Dirac algebra, equations (103)–(108).

The Dirac algebra has $[N/2] = 2$ bits, a boost bit \uparrow or \downarrow , and a spin bit \uparrow or \downarrow . There are $2^{[N/2]} = 4$ basis spinors, which group into two massless Weyl spinors of opposite chirality, right- and left-handed. A Dirac (Weyl) spinor is right-handed if its boost and spin bits align, left-handed if its bits antialign. The 4 Dirac basis spinors, ordered with the right-handed pair first, then the left-handed pair, are

$$\epsilon_a = \{\epsilon_{\uparrow\uparrow}, \epsilon_{\downarrow\downarrow}, \epsilon_{\downarrow\uparrow}, \epsilon_{\uparrow\downarrow}\}. \tag{100}$$

Orthonormal basis vectors in the Dirac algebra are conventionally denoted γ_m , $m = 0, 1, 2, 3$. In the notation of paired orthonormal vectors γ_k^\pm

of the present paper, the traditional basis vectors are

$$\{\gamma_1, \gamma_2, \gamma_3, \gamma_0\} = \{\gamma_1^+, \gamma_1^-, \gamma_2^+, i\gamma_2^-\} . \tag{101}$$

A common notation for chiral vectors in the Newman-Penrose community is, in terms of orthonormal vectors, $\gamma_v \equiv (\gamma_0 + \gamma_3)/\sqrt{2}$, $\gamma_u \equiv (\gamma_0 - \gamma_3)/\sqrt{2}$, $\gamma_+ \equiv (\gamma_1 + i\gamma_2)/\sqrt{2}$, $\gamma_- \equiv (\gamma_1 - i\gamma_2)/\sqrt{2}$. In the notation γ_k and $\gamma_{\bar{k}}$ of paired chiral vectors used elsewhere in this paper, equations (5), the chiral (Newman-Penrose) basis vectors are

$$\{\gamma_+, \gamma_-, \gamma_v, \gamma_u\} = \{\gamma_1, \gamma_{\bar{1}}, \gamma_2, -\gamma_2\} . \tag{102}$$

The 8 outer products of basis spinors of like chirality map to even multivectors of the spacetime algebra as follows. The boost and spin charges of the left and right hand sides of each of equations (103)–(106) below match, as they should. The antisymmetric outer products of right-handed spinors form a right-handed scalar singlet,

$$[\epsilon_{\downarrow\downarrow}, \epsilon_{\uparrow\uparrow}] \cdot = \frac{1}{2}(1 + \varkappa_4) , \tag{103}$$

where \varkappa_4 is the chiral operator in 4 dimensions, equation (56), commonly denoted γ_5 in the traditional Dirac community. The trailing dot on the commutator indicates that the right partner of each product is a row spinor, $[\epsilon_{\downarrow\downarrow}, \epsilon_{\uparrow\uparrow}] \cdot = \epsilon_{\downarrow\downarrow}\epsilon_{\uparrow\uparrow} \cdot - \epsilon_{\uparrow\uparrow}\epsilon_{\downarrow\downarrow} \cdot$. Similarly the antisymmetric outer products of left-handed spinors form a left-handed scalar singlet,

$$[\epsilon_{\uparrow\downarrow}, \epsilon_{\downarrow\uparrow}] \cdot = \frac{1}{2}(1 - \varkappa_4) . \tag{104}$$

The symmetric outer products of right-handed spinors form the three right-handed bivectors,

$$\begin{aligned} \{\epsilon_{\uparrow\uparrow}, \epsilon_{\uparrow\uparrow}\} \cdot &= -\gamma_{12} , & \{\epsilon_{\uparrow\uparrow}, \epsilon_{\downarrow\downarrow}\} \cdot &= -\frac{1}{2}(\gamma_{1\bar{1}} + \gamma_{2\bar{2}}) , \\ \{\epsilon_{\downarrow\downarrow}, \epsilon_{\downarrow\downarrow}\} \cdot &= -\gamma_{\bar{1}\bar{2}} , \end{aligned} \tag{105}$$

while the symmetric outer products of left-handed spinors form the three left-handed bivectors,

$$\begin{aligned} \{\epsilon_{\downarrow\uparrow}, \epsilon_{\downarrow\uparrow}\} \cdot &= -\gamma_{1\bar{2}} , & \{\epsilon_{\downarrow\uparrow}, \epsilon_{\uparrow\downarrow}\} \cdot &= -\frac{1}{2}(\gamma_{1\bar{1}} - \gamma_{2\bar{2}}) , \\ \{\epsilon_{\uparrow\downarrow}, \epsilon_{\uparrow\downarrow}\} \cdot &= -\gamma_{\bar{1}2} . \end{aligned} \tag{106}$$

The 8 outer products of basis spinors of opposite chirality map to odd multivectors of the spacetime algebra as follows. Again, the boost and spin charges of the left and right hand sides of each of equations (107)–(108) below match, as they should. The 4 symmetric outer products of right- with left-handed spinors yield the 4 chiral basis vectors,

$$\{\epsilon_{\uparrow\uparrow}, \epsilon_{\downarrow\uparrow}\} \cdot = \frac{1}{\sqrt{2}}\gamma_1 , \quad \{\epsilon_{\downarrow\downarrow}, \epsilon_{\uparrow\downarrow}\} \cdot = -\frac{1}{\sqrt{2}}\gamma_{\bar{1}} , \tag{107a}$$

$$\{\epsilon_{\uparrow\uparrow}, \epsilon_{\uparrow\downarrow}\} \cdot = -\frac{1}{\sqrt{2}}\gamma_2 , \quad \{\epsilon_{\downarrow\downarrow}, \epsilon_{\downarrow\uparrow}\} \cdot = -\frac{1}{\sqrt{2}}\gamma_{\bar{2}} , \tag{107b}$$

while the 4 antisymmetric outer products of right- with left-handed spinors yield the 4 chiral basis pseudovectors,

$$[\epsilon_{\uparrow\uparrow}, \epsilon_{\downarrow\uparrow}] \cdot = \frac{1}{\sqrt{2}}\varkappa_4\gamma_1 , \quad [\epsilon_{\downarrow\downarrow}, \epsilon_{\uparrow\downarrow}] \cdot = -\frac{1}{\sqrt{2}}\varkappa_4\gamma_{\bar{1}} , \tag{108a}$$

$$[\epsilon_{\uparrow\uparrow}, \epsilon_{\uparrow\downarrow}] \cdot = -\frac{1}{\sqrt{2}}\varkappa_4\gamma_2 , \quad [\epsilon_{\downarrow\downarrow}, \epsilon_{\downarrow\uparrow}] \cdot = -\frac{1}{\sqrt{2}}\varkappa_4\gamma_{\bar{2}} . \tag{108b}$$

Appendix C. Other options for the spinor metric tensor

If N is odd, and if the odd algebra is constructed as described in Sect. 4.4 by embedding the odd algebra in one extra dimension and treating either the final (odd) dimension γ_N or the extra (even) dimension γ_{N+1} as a scalar, then there are potentially further options for the spinor metric beyond those described in Sect. 2.4.

The invariance condition (18) need hold only for rotors not involving the scalar dimension γ_N or γ_{N+1} . If the scalar dimension is the odd dimension γ_N , then γ_N can be dropped from the standard spinor metric ε , leaving ε in $N-1$ dimensions. If the scalar dimension is the even dimension γ_{N+1} , then $i\gamma_{N+1}$ can be adjoined to the alternative spinor metric ε_{alt} , giving ε_{alt} in $N+1$ dimensions. The resulting spinor metrics, distinguished with a prime, are

$$\varepsilon'_N = \varepsilon_N \gamma_N = \varepsilon_{N-1}, \quad \varepsilon'_{\text{alt},N} = \varepsilon_{\text{alt},N} i \gamma_{N+1} = \varepsilon_{\text{alt},N+1} \quad (N \text{ odd}). \quad (109)$$

References

- [1] Baez, J. C., Huerta, J.: The algebra of grand unified theories. In: Bull. Am. Math. Soc. **47**, 483–552 (2010). <https://doi.org/10.1090/S0273-0979-10-01294-2>. arXiv: 0904.1556 [hep-th] (cit. on p. 2)
- [2] Bott, R.: The periodicity theorem for the classical groups and some of its applications. Adv. Math. **4**, 353–411 (1970). [https://doi.org/10.1016/0001-8708\(70\)90030-7](https://doi.org/10.1016/0001-8708(70)90030-7). ((cit. on pp. 2, 9))
- [3] Brauer, R., Weyl, H.: Spinors in n dimensions. Am. J. Math. **57**, 425–449 (1935). <https://doi.org/10.2307/2371218>. ((cit. on pp. 3–5, 11, 24))
- [4] Cartan, É.: Les groupes projectifs qui ne laissent invariante aucune multiplicité plane. In: Bulletin Société Mathématique de France **41**, 53–96 (1913). <https://doi.org/10.24033/bmsf.916> (cit. on p. 1)
- [5] Cartan, É.: Leçons sur la théorie des spineurs. Hermann & Cie, Paris (1938) (cit. on pp. 1, 7, 8, 19)
- [6] Clifford, W.K.: Applications of Grassmann's extensive algebra. Am. J. Math. **1**, 350–358 (1878). <https://doi.org/10.2307/2369379>. ((cit. on p. 3))
- [7] Coleman, S., Mandula, J.: All possible symmetries of the S matrix. Phys. Rev. **159**, 1251–1256 (1967). <https://doi.org/10.1103/PhysRev.159.1251>. ((cit. on p. 2))
- [8] Coquereaux, R.: Modulo 8 periodicity of real Clifford algebras and particle physics. Phys. Lett. B **115**, 389–395 (1982). [https://doi.org/10.1016/0370-2693\(82\)90524-X](https://doi.org/10.1016/0370-2693(82)90524-X). (issn: 0370- 2693 (cit. on pp. 2, 9))
- [9] Doran, C., Lasenby, A.: Geometric algebra for physicists. Cambridge University Press, Cambridge (2003). <https://doi.org/10.1017/CBO9780511807497> (cit. on p. 3)
- [10] Ferrara, S., Sagnotti, A.: Supergravity at 40: reections and perspectives. In: La Rivista del Nuovo Cimento **6**, 279–295 (2017). <https://doi.org/10.1393/ncr/i2017-10136-6>. arXiv: 1702.00743 [hep-th] (cit. on p. 4)

- [11] Grassmann, H.: Die Ausdehnungslehre. Vollständig und in strenger Form begründet. Berlin: Enslin (1862) **(cit. on p. 3)**
- [12] Grassmann, H.: Der ort der Hamilton'schen quaternionen in der ausdehnungslehre. *Math. Ann.* **12**, 375–386 (1877). <https://doi.org/10.1007/BF01444648>. **((cit. on p. 3))**
- [13] Gull, S., Lasenby, A., Doran, C.: Imaginary Numbers are not real- the geometric algebra of spacetime. *Found. Phys.* **23**, 1175–1201 (1993). <https://doi.org/10.1007/BF01883676>. **((cit. on p. 3))**
- [14] Hamilton, A. J. S., McMaken, T.: Unification of the four forces in the Spin(11,1) geometric algebra. *Physica Scripta*, submitted (2022) **(cit. on pp. 2, 4, 6, 26)**
- [15] Hestenes, D.: *Space-Time Algebra*. Gordon & Breach (1966). <https://doi.org/10.1007/978-3-319-18413-5> **(cit. on p. 3)**
- [16] Hestenes, D., Sobczyk, G.: *Clifford Algebra to Geometric Calculus*. D. Reidel Publishing Company (1987). <https://doi.org/10.1007/978-94-009-6292-7> **(cit. on p. 3)**
- [17] Lounesto, P.: *Clifford Algebras and Spinors*, 2nd Edition. Cambridge University Press, London Mathematical Society Lecture Note Series **286** (2001). <https://doi.org/10.1017/CBO9780511526022> **(cit. on p. 3)**
- [18] Mandula, J.E.: Coleman–Mandula theorem. *Scholarpedia* **10**(6), 7476 (2015). <https://doi.org/10.4249/scholarpedia.7476>. **((cit. on p. 2))**
- [19] Pelc, O., Horwitz, L. P.: Generalization of the Coleman–Mandula theorem to higher dimension. *J. Math. Phys.* **38**, 139–172 (1997). <https://doi.org/10.1063/1.531846>. arXiv: hep-th/9605147 **(cit. on p. 2)**
- [20] Peskin, M. E., Schroeder, D. V.: *An Introduction to quantum field theory*. Perseus Books, Reading (1995). <https://doi.org/10.1201/9780429503559> **(cit. on p. 4)**
- [21] Study, E., Cartan, É.: *Nombres complexes*. Editions Jacques Gabay **1908**, 329–468 (2005). **(cit. on pp. 2, 9)**
- [22] Trautman, A.: Clifford and the ‘square root’ ideas. In: *Contemporary Mathematics* **203**, pp. 3–24 (1997). <http://trautman.fuw.edu.pl/publications/Papers-in-pdf/87.pdf>. **(cit. on p. 24)**
- [23] van Nieuwenhuizen, P.: Supergravity as a Yang–Mills theory. In: *50 Years of Yang–Mills Theory*. Ed. by G. 't Hooft. World Scientific, Singapore, 2005, pp. 433–456. https://doi.org/10.1142/9789812567147_0018. arXiv: hep-th/0408137 **(cit. on p. 4)**
- [24] van Nieuwenhuizen, P.: Supergravity. *Phys. Rep.* **68**, 189–398 (1981). [https://doi.org/10.1016/0370-1573\(81\)90157-5](https://doi.org/10.1016/0370-1573(81)90157-5). **((cit. on p. 4))**
- [25] Wilczek, F.: SO(10) marshals the particles. *Nature* **394**, 15 (1998). <https://doi.org/10.1038/27761>. **((cit. on p. 24))**

Andrew J. S. Hamilton
JILA and University of Colorado
Box 440 Boulder CO 80309
USA
e-mail: Andrew.Hamilton@colorado.edu

Received: January 13, 2022.

Accepted: December 16, 2022.