# ANALOG OPTOELECTRONIC INDEPENDENT COMPONENT ANALYSIS FOR RADIO FREQUENCY SIGNALS 

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## ANALOG OPTOELECTRONIC INDEPENDENT COMPONENT ANALYSIS FOR RADIO FREQUENCY SIGNALS

Thesis directed by Professor Dana Z. Anderson

This thesis addresses the problem of blind source separation of signals at radio frequencies. Independent component analysis (ICA), which includes a second-order decorrelation followed by a fourth-order decorrelation, uses signal independence to estimate the original signals from the received mixtures. Until now, ICA has been applied to many applications at or below audio frequencies. The work presented here demonstrates that an optoelectronic implementation using the parallel processing nature of dynamic holography can overcome the computational difficulties associated with algorithmic implementations of ICA.

The holographic nature of a photorefractive crystal combined with the nonlinearity of an electro-optic modulator in a feedback loop can be described by a nonlinear dynamical equation. The dynamics can be cast in the form of Lotka-Volterra equations used to study the dynamics of competing populations of species. Although this analogy with the animal world is interesting, the dynamical equation associated with the fourth-order decorrelation system is fascinating. The statistics associated with the original signals, rather than an external potential, determine the dynamics of the system. In particular, the system is multistable, metastable, or monostable depending on whether the probability density functions of the original signals are sub-Gaussian, Gaussian, or super-Gaussian, respectively. The multistable solution, which occurs for sub-Gaussian signals, provides the winner-takes-all behavior required to separate signals. This ability to separate sub-Gaussian signals is advantageous since signals modulated on a sinusoidal carrier are sub-Gaussian. The fourth-order decorrelation system achieves greater than

40 dB signal separation on 200 MHz single-frequency sine waves and greater than 20 dB signal separation for 10 MHz bandwidth signals. The system performance is degraded by 10 to 20 dB when mixed electronically due to imperfections in the mixing circuitry.

The development of a broadband electro-optic modulator capable of modulating to, at least, twice the half-wave voltage was instrumental to achieving radio frequency blind source separation. This compact $532 \mathrm{~nm} \mathrm{LiNbO}_{3}$ modulator has a 300 MHz bandwidth and a half-wave voltage of less than 16 V . To our knowledge, this is the only free-space modulator capable of this modulation depth.

This thesis also advances the theoretical work in the area of optoelectronic signal processing. Three of the main contributors to signal separation degradation are studied to aid in the characterization and improved performance of the fourth-order decorrelation feedback loop. The fourth-order decorrelation system requires a preprocessor, which orthogonalizes the input signal mixtures. The theoretical framework of an optoelectronic system that performs principal component analysis (PCA), one method of orthogonalizing the signal mixtures, is also presented.

The PCA feedback loop looks identical to the fourth-order decorrelation feedback loop, except the electro-optic modulator is used in its linear regime while the photorefractive gain saturates. Because of the physical similarity of the two optoelectronic feedback loops, our hope is that modular designs will aid in the application of this technology to the telecommunications arena.

## Dedication

To my grandmothers Martha Fisher \& Elizabeth Baylor, whose lives demonstrated that love and perseverance can change the world.

## To my family \& friends,

who love me unconditionally and believed in me when I doubted myself.

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## Chapter 1

## Introduction

### 1.1 Overview

Most telecommunication systems use a coding scheme to separate signals. Examples of these coding schemes include frequency modulation (FM), code division multiple access (CDMA), and wavelength division multiplexing (WDM). These coding schemes are a priori knowledge, which allows the receiver to separate mixtures of received signals.

There is the potential for situations where an array of antennas may receive a mixture of signals that may be of interest, however, the receiver does not have any $a$ priori knowledge about the transmitted signals. An eavesdropping scenario is a prime example of this type of situation. In an eavesdropping situation, an antenna array is placed in a hostile environment where the transmitter does not want the eavesdropper to recover the transmitted information.

As the eavesdropper, you want to be able to recover the original signals from the received mixtures without having foreknowledge of the coding scheme or signal spectrum. This problem is known as blind source separation.

This thesis presents an optoelectronic system capable of solving the problem of blind source separation at radio frequencies (RF). Chapter 1 will provide useful background information for the remainder of the thesis. After the problem of blind source separation is formalized in a mathematical way, one solution to this problem, indepen-
dent component analysis (ICA), is discussed. Next, the dynamical system approach to ICA will be described and compared with DSP implementations. Previous work in the area of optoelectronic signal processing will also be presented along with an outline of the remainder of this thesis. Finally, others who aided in this work and their contribution to the completion of this thesis is presented.

### 1.2 Mathematical Formulation of Blind Source Separation

Blind source separation can be mathematically formulated as follows. There are N transmitted signals $s_{i}(t), i=1 \ldots N$. After propagating some distance, these signals are received by an antenna array with M elements. Each element of the antenna array receives a different linear combination of the original signals given by $\tilde{s_{j}}(t), j=1 \ldots M$, where

$$
\begin{equation*}
\mathbf{s}(t)=\mathbf{A}(\mathbf{t}) \mathbf{s}(\mathbf{t}) \tag{1.1}
\end{equation*}
$$

A is, in general, a slowly time-varying linear $\mathrm{M} \times \mathrm{N}$ mixing matrix. By slow, we mean compared to some characteristic time scale so that $\mathbf{A}$ is essentially constant. Therefore, we will ignore the time dependence of $\mathbf{A}$. Classical blind source separation assumes that $M \geq N$ and that there is no knowledge of $\mathbf{A}$ or $\mathbf{s}$. The problem of blind source separation is to recover the original signals from the measured signals without a priori knowledge of the original signals or how they are mixed together. A cartoon of a possible blind source separation communication scenario is shown in Figure 1.1.

For two sources, we can write

$$
\left[\begin{array}{c}
\tilde{s_{1}}(t)  \tag{1.2}\\
\tilde{s_{2}}(t)
\end{array}\right]=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right] \cdot\left[\begin{array}{l}
s_{1}(t) \\
s_{2}(t)
\end{array}\right] .
$$

In this case, $\tilde{s_{1}}(t)$ and $\tilde{s_{2}}(t)$ are the measured quantities while $s_{1}(t), s_{2}(t), a_{11}, a_{12}, a_{21}$, and $a_{22}$ are unknown.

The solution to blind source separation lies in the ability to determine the inverse matrix. Even with a two-channel system, it is impossible to determine the inverse mixing


Figure 1.1: Cartoon of a communications environment with two transmitted signals.
Each of the transmitted signals is incident upon a receiving array of antennas and mixed through propagation. The detected signal mixtures are the only information available to the ICA processor. The processor uses statistical independence to recover the original signals from the mixtures to within an arbitrary constant.
matrix without making any assumptions. One would like to make the least number of assumptions to make the solution applicable to as many situations as possible. ICA is one such solution, which recovers the original signals based on the assumption that the original signals are statistically independent.

### 1.3 Independent Component Analysis

The power of ICA lies in the fact that it makes the minimum number of assumptions necessary to recover the original signals. These assumptions are as follows:

- the original signals, $s_{i}(t)$ are independent,
- no more than one signal is Gaussian distributed.

It is helpful to discuss each of these assumptions in turn.
The first assumption is satisfied in most cases given the signals are coming from different sources. However, the assumption of signal independence means ICA techniques cannot separate phase-delayed versions of the same signal, e.g., echos. A more extensive discussion of statistical independence is discussed in Section 1.3.1.

The basis of the second assumption is the central limit theorem. A detailed discussion of the central limit theorem and its role in ICA is discussed in Section 1.3.3.

Several other assumptions are generally made that are not required, but ease the computation. First, the mean of the signals is assumed to be zero. If mean of the signals is not zero, the mean can be subtracted from the mixed signals before processing. Second, the PDFs of the unmixed signals are assumed to be symmetric. This is true for all real valued signals. (Asymmetries represent phase shifts or phase delays, which is an interesting subject, but not investigated here.) As a result, odd moments of the signals go to zero, i.e., $\left\langle s_{j}^{n}\right\rangle=0$ for $n=o d d$.

### 1.3.1 Definition of Independence

The formal definition of independence is given by

$$
\begin{equation*}
\left\langle s_{1}{ }^{a} s_{2}{ }^{b}\right\rangle>=\left\langle s_{1}{ }^{a}\right\rangle\left\langle s_{2}{ }^{b}\right\rangle \tag{1.3}
\end{equation*}
$$

for all values of $a$ and $b$. Equation (1.3) says that $s_{1}$ and $s_{2}$ can be decorrelated to all orders, where $a+b$ gives the order of the correlation. This mathematical definition of independence means that we can generate the additional information necessary to recover the original signals. Although an infinite number of values of $a$ and $b$ are possible, up to fourth-order decorrelation is required to perform ICA. This fourth-order decorrelation is usually performed in several steps.

### 1.3.2 ICA Procedure

ICA is performed in three steps, as shown in Figure 1.2. First, a second-order decorrelation is performed. Explicitly, the second-order decorrelation is given by

$$
\begin{equation*}
\left\langle s_{1} s_{2}\right\rangle=\left\langle s_{1}\right\rangle\left\langle s_{2}\right\rangle . \tag{1.4}
\end{equation*}
$$

Several methods for performing this decorrelation are eigenvalue decomposition, singular value decomposition, and principal component analysis (PCA). All of these techniques orthogonalize the received mixtures with the amplitude weighted by the appropriate eigenvalue. PCA is one popular technique for performing this second-order decorrelation step such that the new mixtures lie along the directions of maximum variance. PCA is typically used to decrease the dimensionality of a multidimensional data set by only considering eigenvectors, i.e., directions of variance, above some minimum value.

Once the received mixtures have been orthogonalized, they are normalized to equal length. Together the first two steps, second-order decorrelation and normalization, are called whitening.


Figure 1.2: A schematic of the steps necessary to perform ICA.

Finally, a fourth-order decorrelation is performed to recover the original signals. Explicitly, the fourth-order decorrelation is given by

$$
\begin{align*}
& \left\langle s_{1}{ }^{1} s_{2}{ }^{3}\right\rangle=\left\langle s_{1}{ }^{1}\right\rangle\left\langle s_{2}{ }^{3}\right\rangle,  \tag{1.5}\\
& \left\langle s_{1}{ }^{2} s_{2}{ }^{2}\right\rangle=\left\langle s_{1}{ }^{2}\right\rangle\left\langle s_{2}{ }^{2}\right\rangle, \tag{1.6}
\end{align*}
$$

and

$$
\begin{equation*}
\left\langle s_{1}{ }^{3} s_{2}{ }^{1}\right\rangle=\left\langle s_{1}{ }^{3}\right\rangle\left\langle s_{2}{ }^{1}\right\rangle . \tag{1.7}
\end{equation*}
$$

To build some physical intuition into how the ICA process recovers the original signal, consider the joint distribution between two independent variables that have been mixed in some unknown way so that they result in two mixtures $M 1$ and $M_{2}$. The joint distribution of $M 1$ and $M_{2}$, shown in Figure 1.3, is built up by making many measurements of $M_{1}$ and recording the corresponding value of $M_{2}$. The joint distribution shows that $M 1$ and $M_{2}$ are correlated in the sense that knowledge of $M_{1}$ gives information about the value of $M_{2}$. This is especially true at the maximum and minimum values of $M_{1}$. What we want to do is perform successive operations to end up with signals where information about one signal gives no information about the other signals.

The first operation that moves us in that directions is to compress the mixtures in the direction of greatest variance and expand the signals in the direction of maximum variance. This is exactly the function of the whitening step. The second-order decorrelation, specifically PCA, finds the directions of greatest variance, while the normalization performs the necessary expansion and contraction. The resulting joint distribution due to this whitening step is shown in Figure 1.4, where the new whitened mixtures are give by $M_{1}^{\prime}$ and $M_{2}^{\prime}$. Notice that the whitened mixtures appear as a rotated square. Notice that although the mixtures are orthogonalized, they are still not independent. There are now two equally preferred directions, where information about one signal gives information about the other signal.


Figure 1.3: The joint distribution between mixed variables.


Figure 1.4: The joint distribution between two whitened variables.

It is clear that in order to recover the independent signals, it is necessary to rotate the whitened mixtures until the square is unrotated, as in Figure 1.5. Thus, the fourthorder decorrelation step rotates the whitened mixtures such that the new mixtures, $M_{1}^{+}$ and $M_{2}^{+}$, correspond to the independent signals $S_{1}$ and $S_{2}$.


Figure 1.5: The joint distribution of two independent variables.

The key to this fourth-order decorrelation is knowing how much to rotate the whitened mixtures. The central limit theorem suggests a means of determining the amount of rotation.

### 1.3.3 Measuring Independence

The central limit theorem says the sum of two independent probability density functions (PDFs) looks more Gaussian compared to the original independent PDFs. As an example, the sum of two uniform distributions is a triangle distribution, which looks more Gaussian than the uniform distribution. This suggests that one way to perform the fourth-order decorrelation step is to rotate the whitened signals until resulting signals are the least Gaussian.

Contrary to what most scientists are taught, how Gaussian a signal is can be measured on a continuous scale. One such measure is a quantity called the kurtosis and a related quantity called the excess kurtosis. The kurtosis and excess kurtosis are defined as

$$
\begin{equation*}
\operatorname{Kurt}[S]=\frac{\left\langle S^{4}\right\rangle}{\left\langle S^{2}\right\rangle} \tag{1.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { ExcessKurt }[S]=\frac{\left\langle S^{4}\right\rangle}{\left\langle S^{2}\right\rangle}-3 \tag{1.9}
\end{equation*}
$$

respectively. $S$ is a random variable associated with the temporal signal $s(t)$. Since the Kurtosis of a Gaussian PDF is three, the excess kurtosis is centered around zero. Signals with a kurtosis less than three are called sub-Gaussian. Signals with a kurtosis greater than three are called super-Gaussian. The kurtosis measures the "peakedness" of the PDF at the origin compared to a Gaussian. A super-Gaussian signal has a higher probability near the origin and is more peaked than a Gaussian. An example of a superGaussian PDF is a Laplace distribution. A sub-Gaussian signal has larger probabilities away from the origin compared to a Gaussian. A binary has the lowest kurtosis. The kurtosis of several PDFs is presented in Table 1.1.

By minimizing or maximizing either the kurtosis or excess kurtosis, one can rotate the whitened mixtures the appropriate amount. There is one caveate. The sum of two Gaussian-distributed PDFs is also Gaussian. Thus, it is impossible to separate mixtures of Gaussian signals uniquely. This is why ICA assumes that, at most, one signal can be Gaussian-distributed. The assumption regarding the number of Gaussian signals appears to be very stringent if one considers baseband signals, where many signals have Gaussian distributions. However, once a signal is modulated onto an RF carrier, its PDF is no longer Gaussian.

To see how modulating a signal can change its PDF, let's look at a specific example. For a signal $G(t)$ with a Gaussian-distributed PDF, a random measurement

| Function <br> Name | Kurtosis | Probability Density Function ${ }^{a}$$P(x)$ |  | Characteristic Function ${ }^{b}$ $\phi(v)$ |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Function | Domain |  |
| Laplace | 6 | $\frac{e^{-\sqrt{2}\|x\|}}{\sqrt{2}}$ | $-\infty<x<\infty$ | $\frac{1}{\left(1+\frac{v^{2}}{2}\right)}$ |
| Gaussian | 3 | $\frac{e^{-\frac{x^{2}}{2}}}{\sqrt{2 \pi}}$ | $-\infty<x<\infty$ | $e^{-\frac{v^{2}}{2}}$ |
| Cubic Exponential | 2.42 | $\frac{3}{2} e^{-\|x\|^{3}}$ | $-\infty<x<\infty$ | No Closed Form |
| Uniform | 1.8 | $P(x)= \begin{cases}\frac{1}{2 \sqrt{3}} & \text { if }-\sqrt{3} \leq x \leq \sqrt{3} \\ 0 & \text { Otherwise }\end{cases}$ | $-\infty<x<\infty$ | $\frac{\sin (\sqrt{3} v)}{\sqrt{3} v}$ |
| Harmonic ${ }^{\text {c }}$ | 1.5 | $\frac{1}{\pi \sqrt{2} \sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^{2}}}$ | $-\sqrt{2} \leq x \leq \sqrt{2}$ | $J_{0}(\sqrt{2} v)$ |
| Binary ${ }^{\text {d }}$ | 1 | $\frac{\delta(x-1)+\delta(x+1)}{2}$ | $-\infty<x<\infty$ | $\cos (v)$ |

Table 1.1: Kurtosis and characteristic functions of various probability density functions (PDFs).
${ }^{a}$ All PDFs included haver zero mean and unit variance.
${ }^{b}$ The characteristic function is the Fourier transform of the PDF.
${ }^{c}$ Harmonic refers to the PDF associated with a sinusoidal signal.
${ }^{d} \mathrm{~A}$ binary PDF has the lowest possible kurtosis.
will tend to yield a value close to zero, since that is where the PDF is largest. If our Gaussian signal is now converted to an FM signal, the modulated signal is given by

$$
\begin{equation*}
G_{F M}(t)=C \cos \left(\omega_{c} t+B \int_{0}^{t} G(\tau) d \tau\right), \tag{1.10}
\end{equation*}
$$

where C is the carrier amplitude, $\omega_{c}$ is the carrier frequency, and B is the modulation depth. The modulating signal $G(t)$ changes the frequency of the carrier frequency but not its amplitude. As a result, the PDF of the modulated $\mathrm{G}(\mathrm{t})$ is now that of the sine wave carrier which has a higher probability of measuring the value $+/-\mathrm{C}$ compared to measuring values near zero. Although this is one example, it is clear that any signal modulated onto a carrier takes on the PDF of the carrier.

In addition to measuring kurtosis, there are additional ways of estimating how Gaussian a signal is including entropy, negentropy, and mutual information. These measures form the basis of many algorithms that perform ICA. Because there are multiple ways of estimating independence, there are different approaches to performing ICA. Some of these techniques are discussed in Section 1.3.4.

There are some limits to the ICA procedure. ICA is unable to predict the number of signals that are extracted, the order in which signals are extracted, or the amplitude and sign of the sources. However, these are trivial parameters compared to identifying the original signals themselves. A more substantial drawback of ICA is that it is unable to separate a signal and its echo. Despite this, there are numerous situations where the presence of echos can be ignored. In principle, ICA can be performed on all types of signals. However, only through the use of a dynamical system approach has ICA been realized for RF signals.

### 1.3.4 Historical Context of ICA

Although PCA, originally called the Karhunen-Love Transform, was developed in 1955, applications of PCA and ICA to information processing in the form of neural
networks did not occur until 1986 and 1991, respectively ([36], [41]). In 1994, Comon et al. formulated ICA in today's terms showing applications to ICA beyond neutral networks to applications such as locating astronomical sources, identifying stock market trends, separating body rhythms in biomedical applications, etc. Since then, algorithmic techniques have dominated the implementations of ICA [23].

Algorithmic approaches have emphasized improving the speed and separation performance of ICA. These improvements have amounted to finding clever ways of estimating independence. Some of these approaches include Natural Gradient [1], Maximum Likelihood [13], Infomax [11], Projection Pursuit [42], Negentropy [48], FastICA [37], and Minimum Entropy [68]. All ICA algorithms seek to either maximize or minimize some potential energy function related to the entropy or information content of the signals.

Most recently, the emphasis in the field of ICA has been realizing real-time separation performance. To this end, there are two very different approaches: hardware and optoelectronic. Hardware improvements take advantage of very large system integration (VLSI) and floating point gate array (FPGA). These technologies have enabled real-time separation for algorithmic techniques ([43], [17], [19],[18], [26]). Despite the improvements in algorithmic techniques, real-time separation is still limited to audio frequencies.

The first optoelectronic approach to information processing was demonstrated in 2003. These optoelectronic implementations will be discussed in Section 1.6.

### 1.4 Dynamical Systems

By analyzing the physical system, a dynamical equation can be found that governs the behavior of the system in time. The dynamical equation can be thought of as a fixed rule that describes how a future state of the system evolves from the current state of the system.

Let's consider one particular physical system: a highly damped particle confined to a potential. The forces on the particle are described by

$$
\begin{equation*}
\sum \vec{F}=-\frac{d \vec{x}}{d t}-\vec{\nabla} V(\vec{x}), \tag{1.11}
\end{equation*}
$$

where $\vec{x}$ is the position of the particle and $V(\vec{x})$ is a two dimension potential energy function. We are interested in steady state solutions where the particle is stationary. In this situation the particle seeks to either minimize or maximize its potential energy as seen in the expression

$$
\begin{equation*}
0=-\vec{\nabla} V(\vec{x}) . \tag{1.12}
\end{equation*}
$$

By solving Equation (1.12) it is possible to determine the steady state values $\vec{x}_{s s}$ that make Equation (1.12) true for a particular $V(\vec{x})$. It is also possible to determine under what conditions these solutions are stable.

Suppose that you desire a particular steady state solution because that solution solved a specific problem. If it were possible to construct (or discover) a potential that produced the desired steady state behavior, then the dynamical system could be used to solve that specific problem. This is the idea behind the optoelectronic feedback loop that performs the fourth-order decorrelation step of ICA. The fourth-order decorrelation system is a physical system whose steady state solutions solve the problem of blind source separation.

Like a particle in a potential, the feedback loop that performs the fourth-order decorrelation step in ICA has a time-evolution dynamical equation given by

$$
\begin{equation*}
\frac{d \vec{v}(t)}{d t}=\frac{-\kappa R e[\vec{\nabla} \Phi(\vec{v})]-\vec{v}}{\tau} \tag{1.13}
\end{equation*}
$$

The development of this equation will be derived in Section 2.3. For the fourth-order decorrelation feedback loop, the potential energy function is related to the Fourier transform of the PDFs of the unmixed signals. Thus, the state of the system is dependent on the statistics and, more specifically, the PDFs of the signals themselves. Our desire is
to find a steady state solution that recovers the original signals. The solution of interest recovers the original signals and performs blind source separation if $v_{m}=V, v_{m \neq n}=0$. For this winner-takes-all solution, only the amplitude of one of the independent signals is nonzero, where originally all of the signal amplitudes were non-zero indicating a mixture of signals. This is just one of many possible steady state solutions. Which steady state the feedback loop reaches is dependent on the how Gaussian the original unmixed signals are, shown later in this thesis.

As discussed in Section 1.3.3, any signal modulated onto a carrier has a harmonic PDF and is sub-Gaussian. Even if the baseband signal is not sub-Gaussian, the act of modulating the signal onto an RF carrier changes the PDF to sub-Gaussian. The fourth-order decorrelation feedback loop solves the blind source separation problem if at least one of the unmixed signals has a sub-Gaussian PDF. Thus, the optoelectronic ICA system is ideal for telecommunication applications.

The implementation of ICA using a feedback loop is fundamentally different from the algorithmic approach traditionally used in the field of ICA. Algorithmic digital signal processing (DSP) techniques use a step-by-step process that involves the estimation of the inverse-mixing matrix, an assessment of signal independence, and a decision on whether to change the inverse-mixing matrix based on that assessment. Our dynamical system does not implement a step-by-step set of commands. Rather, the system is designed so that the independent signals are extracted from the mixtures as a natural consequence of its operation. The idea is that once the dynamical system reaches steady state (i.e., the $v_{m}$ amplitudes are no longer changing), an independent component has been extracted.

### 1.5 DSP vs Dynamical Systems

Compared to the more than 20 years that DSP has been applied to the field of ICA, a dynamical system approach to ICA is a new development in the field. There
are some aspects these two approaches have in common; however, there are several fundamental differences that makes a dynamical system approach better than DSP techniques for RF applications.

Regardless of whether ICA is implemented algorithmically or using a dynamical system, it is necessary to estimate signal independence. Since there are many techniques for estimating independence (see Section 1.3.4), there are many different algorithms that are implemented using DSP techniques. Each of these algorithms, in some way, estimates the how Gaussian a signal is to estimate independence. In the feedback loop, independence is estimated by the holographic grating written in the photorefractive crystal. The optoelectronic feedback loop does not calculate the fourth-order moment explicitly; rather it performs correlations at all orders, not just the fourth-order.

In both algorithmic and optoelectronic implementations, the original signals are recovered. DSP techniques recover each independent signal on a different channel. Using the dynamical system approach, each feedback loop extracts an independent signal. Once an independent signal is extracted from the loop, electronic subtraction techniques can subtract the recovered signal from the mixtures. Thus, for M signals, $M-1$ feedback loops are required to recover all of the original signals.

ICA is a computationally intensive process. The difficulty in the computation is due to the estimation of the higher-order moments and the sampling rate necessary to estimate the signals themselves. Both DSP and dynamical systems are iterative approaches, and the speed at which each type of implementation is performed is dependent on the speed at which iterations are performed. The speed at which ICA is performed in DSP systems is limited by the processor speed which governs the rate at which operations can be performed. The ICA feedback loop is an analog system, so operations are performed continuously. The speed at which the feedback loop performs ICA is determined by the photorefractive crystal time constant, $\tau$, which is an integration time that indicates the amount of time it takes for the crystal to form a grating, see Section
2.2.4. The grating, i.e., hologram, is formed by the interference pattern of the three beams inside of the photorefractive crystal. In essence, $\tau$ determines the amount of sampling required to identify an independent signal.

Both DSP and dynamical system approaches to ICA effectively find the inverse mixing matrix. In DSP, it is possible to recover the inverse mixing matrix directly and determine how close the algorithm came to estimating the actual mixing matrix. Because the dynamical system is not implementing an algorithm, there is no way to directly recover the inverse mixing matrix. The inverse matrix can be recovered to within an arbitrary constant by performing correlations between the system output and the received mixtures. We prefer to compare the output of the system with the independent signals to measure the amount of signal separation.

One of the major differences between DSP approaches and our dynamical system is the limiting bandwidth that each is able to process. In DSP implementations, the ultimate bandwidth is limited by the speed of the processor used. Nyquist sampling requires sampling the signal of interest at twice its frequency. Moreover, the calculation of the fourth-order moment effectively quadruples the signal bandwidth requirements of the process. These two considerations alone require a processor that is eight times the signal bandwidth. As a result, the signal bandwidth quickly becomes a large fraction of the processor bandwidth. If real-time processing is desired, it is preferable that the processor bandwidth be much greater than eight times the signal bandwidth.

The ICA feedback loop processes signals optically in a parallel fashion. Because the signals are modulated onto a laser, the fractional bandwidth of the microwave signals $\left(10^{8} \mathrm{~Hz}\right)$ compared to the optical carrier $\left(10^{14} \mathrm{~Hz}\right)$ is very small. The fundamental bandwidth limitation of our feedback loop is given by the round trip travel time of the signal around the feedback loop. Thus, the smaller the feedback loop, the larger the system bandwidth. The challenge in reaching infinite bandwidth is the physical size of the optics used in the feedback loop, and our ability to modulate the electronic signals
onto the laser.

### 1.6 Previous Work

Analog optoelectronic techniques for performing information processing offer the opportunity to perform real-time ICA at RF frequencies. In 2003, an all-optical feedback loop was used to perform real-time PCA on 150 MHz intermediate frequency (IF) signals down-converted from a 10 GHz carrier [31]. In 2004, the theory behind an optoelectronic system capable of performing the fourth-order decorrelation step of ICA was published [2]. One year later, the physical system described in [2] is realized at audio frequencies demonstrating real-time signal separation of 20 dB or better [65]. This thesis presents an optoelectronic fourth-order decorrelation system that performs real-time fourth-order decorrelation on radio frequency signals and the theory behind an optoelectronic PCA system, which could be integrated with the fourth-order decorrelation system to form a complete ICA system.

Like the algorithmic approaches, these analog optical-processing techniques minimize a potential energy function to perform their task. However, in our case this potential energy function depends on the statistics of the signals to be separated. Thus, the steady-state performance of the optoelectronic system is dependent on the signals themselves.

### 1.7 Thesis Outline

This thesis presents an optoelectronic implementation of a fourth-order decorrelation system capable of separating 200 MHz signals in real time and the numerical analysis of an optoelectronic feedback loop capable of performing PCA. Chapter 2 gives the principles of operations for the fourth-order decorrelation system. Chapter 3 describes the experimental implementation of the fourth-order decorrelation system. An analysis of how slight modifications from the ideal assumptions presented in Chapter 2
affect the performance of the fourth-order decorrelation feedback loop is presented in Chapter 4. The theoretical framework for an optoelectronic implementation of principal component analysis is presented in Chapter 5. Future areas of improvement and development are presented in Chapter 6.

Several appendices have been included to present detailed information on the development of several components as well as additional background material. Appendix A describes the development and design of a broadband electro-optic modulator (EOM) used to phase modulate the laser at 200 MHz . Next, the mixing circuit used to generate the signal mixtures is presented in Appendix B. Appendix C presents the development and design of a miniaturized carrier suppression module. Finally, Appendix D briefly describes the operator approach to photorefractive coupling, which is used in the numerical analysis of the optoelectronic PCA feedback loop.

### 1.8 Contributors to this Thesis

This section describes those who supported the experimental and theoretical work presented in this thesis.

The theory behind the optoelectronic feedback loop that performs the fourthorder decorrelation step of ICA was first presented in [2] in 2004. I extended this theory by analyzing various nonidealities of the input signals and other aspects of the feedback loop.

Although a theory for all-optical PCA was developed in [29], I developed a theory for optoelectronic PCA that is presented in Chapter 5. Optoelectronic PCA theory was developed through helpful discussions with Dr. Dana Anderson.

The development of the 200 MHz EOM was initiated by Dr. P. C. Smith and K. Erickson, former graduate students in the group. These gentlemen developed the first mounting designs. However, the EOM used in this experiment would not have been realized without the assistance of Terry Brown, a member of JILA's electronic shop.

Terry modeled the EOM based on network analyzer data and suggested improvements to the crystal mounting scheme.

The $\mathrm{BaTiO}_{3}$ and $\mathrm{LiNbO}_{3}$ crystals used in this experiment were processed by Leslie Czaia.

The carrier suppression module was a team effort. I developed the initial optical design and mechanical layouts for the carrier suppression module. Thereafter, I served as a manager of this project. The optical layout, performed in Zemax, was achieved with the assistance of P. Alken, a former graduate student in our group. The mechanical layout was performed by F. Majdeteimouri, an undergraduate working in the group. The assembly and testing of the module was performed by P. Alken. The design of the EOM mount in the carrier suppression module was completed by K. Erickson.

## Chapter 2

## Fourth-Order Decorrelation Principles of Operation

This chapter presents the principles of operation of the fourth-order decorrelation system. Section 2.1 gives an overview of the fourth-order decorrelation system components and the role each component plays in the system dynamics. The theory behind the operation of the optoelectronic feedback loop, presented in [2], is summarized next. The potential energy surface, which is determined by statistics, is plotted for several PDFs to understand the steady behavior of the system. In order to gain insight into the system dynamics, an educational exercise where the closed-loop evolution equations are written in Lotka-Volterra form is presented and discussed.

### 2.1 Overview of Optoelectronic Fourth-Order Decorrelation

The fourth-order decorrelation step of ICA is the step that recovers the original signals. This step effectively rotates the whitened mixtures until the resulting signals are the least Gaussian (see Section 1.3.2). From an algorithmic standpoint, fourthorder decorrelation is the most difficult step in the ICA process because it requires the generation and correlation of higher-order harmonics to gain the required information to recover the signals.

The optoelectronic fourth-order decorrelation system overcomes these computational difficulties using the nonlinearity of an EOM combined with the holographic gain of a photorefractive crystal in a feedback loop configuration.

### 2.2 System Components

The system that performs the fourth-order decorrelation step of ICA is shown in Figure 2.1. Each of the components and their role in optoelectronic fourth-order decorrelation will be discussed in turn.


Figure 2.1: Schematic of the optoelectronic fourth-order decorrelation system.

### 2.2.1 The Input Channels

The input channels change the electronic mixtures into optical signals which are processed by the fourth-order decorrelation feedback loop. There are two channels, one for each electronic mixture. The electronic mixtures are generated from two independent signals that are mixed electronically. Because the fourth-order decorrelation step assumes the input mixtures have already been whitened, the mixing matrix is assumed to be an orthonormal mixture of the independent signals.

These mixed signals are phase modulated onto an optical carrier using an EOM. Because the second-order decorrelation has been performed and the optical carrier is
now common between the two input channels, the laser carrier must be removed. Consequently the photorefractive carrier suppression circuit shown in Figure 2.2 is used to remove the optical carrier.


Figure 2.2: A schematic of one input-channel carrier-suppression setup.

This method of optical carrier suppression replaces a beam combiner in an unbalanced Mach-Zehnder interferometer with a photorefractive crystal. Ideally, the output of one arm of the interferometer contains twice the carrier, while the carrier is removed from the other arm. The photorefractive crystal is able to adapt to drifts in the environment and changing path lengths to produce optical carrier suppression of greater than -70 dB [5]. Although this amount of carrier suppression is not required to obtain reasonable system performance, this method of carrier suppression was chosen because of its long term stability. Roughly 3 dB of the signal is lost using this method of carrier suppression. However, the superior stability of photorefractive carrier suppression makes up for this loss of signal.

### 2.2.2 The Feedback Loop

This feedback loop is half optical and half electronic. The loop consists of an EOM, which phase modulates the loop beam; a photorefractive crystal, which provides linear parametric gain to the loop beam from the input channels; homodyne detection, which converts the phase-modulated loop beam to an amplitude modulated signal that is detected by a photodetector; and electronic amplification, which amplifies the signal and drives the electro-optic modulator. The properties of each of these components and how they are used in the feedback loop are discussed in turn.

### 2.2.3 The Electro-Optic Modulator

The EOM is the only nonlinear element in the feedback loop and phase modulates the loop beam. For large modulation depths, the gain of the EOM begins to saturate. To see this, let's look at the gain of a sinusoidal signal applied to the EOM.

The electric field for a sinsoidal phase modulation is

$$
\begin{equation*}
E=E_{0} e^{-i \omega_{c} t} e^{i v \cos \left(\omega_{m} t\right)} \tag{2.1}
\end{equation*}
$$

where $\omega_{c}$ is the frequency of the laser carrier, $v$ is the modulation depth, and $\omega_{m}$ is the frequency of the modulating signal. Equation 2.1 can be expanded as the sum of Bessel Functions,

$$
\begin{equation*}
E=E_{0} e^{-i \omega_{c} t} \sum_{n=-\infty}^{\infty} i^{n} J_{n}(v) e^{n \omega_{m} t} \tag{2.2}
\end{equation*}
$$

The fundamental of the modulating signal, i.e., $n=1$ and $n=-1$, follows the $J_{1}(v)$ shown in Figure 2.3. The gain of the fundamental is given by the output signal over the input signal and is given by $\frac{J_{1}(v)}{v}$. The gain of the fundamental is shown in Figure 2.4. When the modulation depth is large enough that the gain of the fundamental equals the loss in the feedback loop, the feedback loop reaches steady state. Therefore, the EOM must be driven hard enough to saturate, causing the EOM to generate many harmonics of $\omega_{m}$.


Figure 2.3: $J_{1}(v)$ vs $v$
The fundamental of the phase modulated signal follows the first order Bessel function.


Figure 2.4: $\frac{J_{1}(v)}{v}$ vs $v$
The gain of the phase modulated signal saturates at large values of $v$.

### 2.2.4 The Photorefractive Crystal

Photorefractive crystals are dynamically holographic. These crystals are able to convert a varying intensity pattern into a varying index of refraction that acts as a volume grating inside the material through the photorefractive effect. Because of the protorefractive effect, these crystals provide gain to the extent that the optical beams crossing inside of them are temporally and spatially correlated. These two properties will be discussed in the following paragraphs.

The photorefractive crystal provides gain to for two beams that are temporally correlated. Beams with the same frequency entering the crystal create a fixed interference pattern and establish a strong grating inside of the material. The beams are able to interact with the grating such that one of the beams preferentially receives gain. The beam that receives gain is typically called the signal beam, while the beam that gives up its energy is called the pump beam. This configuration is called two-beam coupling and is discussed in section 3.2.1.

Two beams of different frequencies entering the photorefractive grating write a moving grating. Because the grating is moving, the crystal writes a very weak grating and energy transfer from the pump beam to the signal beam is diminished. The greater the difference in frequency, the weaker the energy transfer.

The consequence of this temporal correlation in the feedback loop is that only the frequencies in the feedback loop (the signal beam) that are temporally correlated with the input channels (the pump beams) receive gain. All other frequencies can be ignored.

In order for the photorefractive crystal to provide gain, the signals must also be spatially correlated. Suppose that two temporally correlated beams write a strong grating, a third beam coming in to the crystal at a slightly different angle cannot read the grating that has been written because plane waves at different angles are spatially
orthogonal. As a practical matter, this means that the two input channels, because they enter the feedback loop at slightly different angles, do not read each other's gratings.

Moreover, because the angle between input channel beams is small, the input channel beams do not write gratings between themselves. Therefore, the pump beams can be analyzed simply as two independent incidents of two-beam coupling with the feedback loop (i.e., each pump individually with the loop beam) rather than as a more complicated three-beam coupling situation.

### 2.2.5 The Homodyne Detection

Homodyne detection is used to convert the phase modulated loop beam to an amplitude modulated signal that can be detected on the photodetector. The difference between phase modulation and amplitude modulation is the laser carrier's phase with respect to the sidebands of the fundamental frequency. By splitting off some of the laser carrier before the feedback loop and reintroducing the carrier into loop beam with the proper phase, the desired signal is recovered by the photodetector. To determine the phase of the homodyne beam, we calculate the intensity of the photodetector $I_{P D}$

$$
\begin{equation*}
I_{P D}=\left|E_{\text {Loop }}(t)+E_{\text {Homodyne }}(t)\right|^{2}=\left|E_{L} e^{i\left(\omega_{c} t-m \sin \omega_{m} t\right)}+E_{H} e^{-i\left(\omega_{c} t+\varphi\right)}\right|^{2} \tag{2.3}
\end{equation*}
$$

Expanding this expression and ignoring the DC terms, we find that the term containing the desired signal is proportional to

$$
\begin{equation*}
I_{P D} \propto E_{L} E_{H} \sin (\varphi) \tag{2.4}
\end{equation*}
$$

To obtain the maximum signal, we require $\varphi=90^{\circ}$.

### 2.2.6 The Electronic Amplifiers

Amplifiers in the electronic portion of the feedback loop insure there is enough electronic gain to drive the EOM into saturation. However, the electronic gain cannot
be too high or the loop will spontaneously oscillate. A variable electronic amplifier (or attenuator) is placed in the loop to control the amount of electronic gain. All amplifiers (or attenuators) are assumed to operate in their linear regime.

### 2.3 Principles of Operation

The principles behind the operation of this feedback loop were first described in [2] and will be described in the next few sections.

By tracing the amplitudes of the independent signals in one round trip through the feedback loop, we can derive an expression for the open-loop signal gain. We start by assuming that the EOM is driven with a linear superposition of the M independent signals, as given by

$$
\begin{equation*}
V(t)=\sum_{m=1}^{M} v_{m}(t) S_{m}(t) \tag{2.5}
\end{equation*}
$$

The $S_{m}(t)$ 's are assumed to be orthogonal and normalized so that $\left\langle S_{m}(t) S_{n}(t)\right\rangle=\delta_{m n}$. We can write the driving signal as only a linear combination of the original signals by looking ahead to the photorefractive gain. Only the components of the loop that are temporally correlated with the input beams will receive gain. Since the input beams only contain linear combinations of the independent signals, these are the only amplitudes of interest. Therefore, we can write the EOM-driving signal in terms of the independent signals because the independent signals form a complete basis set. In particular, this is the basis set of the signals we wish to recover. When all of the amplitudes are zero except for one, then we have recovered one signal from the mixture of signals.

The phase-modulated loop beam is given by

$$
\begin{equation*}
E(t)=E_{0} e^{i V(t)} e^{-i \omega_{c} t}+\text { complex conjugate. } \tag{2.6}
\end{equation*}
$$

The EOM is driven hard enough to produce many harmonics of the driving signals on the loop beam causing saturation of the gain driving signals.

The loop beam is correlated with the input signals inside of the photorefractive crystal. The photorefractive crystal is used in the undepleted pump regime so that it acts as a parametric linear amplifier. As a result, the intensity of the pump beam is assumed to be constant while the intensity of the loop beam grows with the timeaveraged correlation between the loop and pump beams. The photorefractive gain of the $m^{\text {th }}$ signal, $G_{m}^{p r}$, is given by

$$
\begin{equation*}
G_{m}^{p r}=<e^{i V(t)} S_{m}(t)>_{\tau} . \tag{2.7}
\end{equation*}
$$

The homodyne beam converts the phase-modulated loop signal to a detectable amplitudemodulated signal. After electronic amplification, the open-loop gain is given by

$$
\begin{equation*}
A_{m}(\vec{v})=-\kappa R e\left\{\frac{i}{v_{m}}<e^{i V(t)} S_{m}(t)>_{\tau}\right\} \tag{2.8}
\end{equation*}
$$

where $\kappa$ contains all of the linear gain factors. You will notice that the rapidly varying laser carrier term $e^{-i \omega_{c} t}$ has been ignored. The phase of the carrier changes vary quickly and averages out.

The time average due to photorefractive two-beam coupling can be written explicitly as

$$
\begin{equation*}
<e^{i V(t)} S(t)>_{\tau}=\frac{1}{\tau} \int_{-\infty}^{\tau} e^{i V(t)} S(t) d t \tag{2.9}
\end{equation*}
$$

It is reasonable to assume that the statistics of the signals are not changing over the photorefractive time constant. Therefore, we can write the time average as a statistical expectation value

$$
\begin{equation*}
\frac{1}{\tau} \int_{-\infty}^{\tau} e^{i V(t)} S(t) d t \rightarrow \int_{-\infty}^{\infty} e^{i v S} S p(S) d S \tag{2.10}
\end{equation*}
$$

where $S$ is now a random variable associated with the probability density function (PDF) $p(S)$. To reiterate, we are looking at $S$ as a statistical variable rather than as the temporal signal $S(t)$. This can be confusing notation. To designate whether $S$ is a statistical variable or a temporal signal, the time dependence will be stated at $S_{m}(t)$.

The characteristic function is defined as the Fourier transform of the PDF and can be expanded in terms of the statistical moments of the signal as
$\phi(v) \equiv \int_{-\infty}^{\infty} e^{i v S} p(S) d S \approx 1+i v<S>-\frac{1}{2} v^{2}<S^{2}>-i \frac{1}{3!} v^{3}<S^{3}>+\frac{1}{4!} v^{4}<S^{4}>\ldots$,
where $<S^{n}>$ corresponds to a statistical average of $S^{n}$.
Notice that the only difference between Equation (2.10) and the definition of the characteristic function is Equation (2.11) has the addition $S$ in the statistical average of Equation (2.10). By taking the derivative of the characteristic function, we obtain an expression proportional to Equation (2.10)

$$
\begin{equation*}
\frac{d \phi(v)}{d v}=i \int_{-\infty}^{\infty} e^{i v S} S p(S) d S \tag{2.12}
\end{equation*}
$$

When dealing with an ensemble of $M$ signals we need to define the system characteristic function, $\Phi . \Phi(\vec{v})$ is then an M-dimensional Fourier transform. Because the signals are statistically independent, the characteristic functions are separable so the $\Phi(\vec{v})$ can be written as the product of the characteristic functions of the individual signals

$$
\begin{equation*}
\Phi(\vec{v})=\phi_{1}\left(v_{1}\right) \phi_{2}\left(v_{2}\right) \ldots \phi_{m}\left(v_{m}\right) \tag{2.13}
\end{equation*}
$$

Using the system characteristic function, it is possible to rewrite Equation (2.8) as

$$
\begin{equation*}
A_{m}(\vec{v})=-\frac{\kappa}{v_{m}} \operatorname{Re}\left\{\partial_{m} \Phi(\vec{v})\right\} . \tag{2.14}
\end{equation*}
$$

Equation (2.14) shows that the gain of the independent signals is dependent on their statistics. Figure 2.5 shows the open-loop gain associated with three different PDFs. Comparing with Table 1.1, you will notice the lower the Kurtosis of the PDF, the higher the open-loop gain.

The closed-loop dynamics are found by looking at the initial and final signals after one round trip in the feedback loop compared to the integration time governed by


Figure 2.5: Open-loop gain curve for three different PDFs.
$\tau$. Thus, the closed-loop dynamics of the system is given by

$$
\begin{equation*}
\tau \frac{\partial \vec{v}}{\partial t}=-(\vec{v}+\kappa \operatorname{Re}(\vec{\nabla} \Phi(\vec{v}))) \tag{2.15}
\end{equation*}
$$

where $\vec{\nabla}$ is an M-dimensional gradient. Equation (2.15) resembles a particle in a restoring potential. In this case, the restoring potential is governed by the statistics of the independent signals involved.

### 2.4 The Characteristic Function as a Potential Energy Surface

A close look at the closed-loop evolution equation reveals that this expression matches the dynamics of a damped particle confined to a potential. The damping force on a particle is proportional to the velocity of the particle. In Equation (2.15), the damping force is given by

$$
\begin{equation*}
\vec{F}_{\text {damping }}=\tau \frac{d \vec{v}}{d t} \tag{2.16}
\end{equation*}
$$

Moreover, the force on a particle is proportional to the gradient of the potential, $\vec{F}(\vec{x})=-\vec{\nabla} V(\vec{x})$.

In Equation (2.15) there are two potentials to consider. The linear term corresponds to a spring-like potential $V(\vec{x})_{\text {spring }} \propto \vec{x} \cdot \vec{x}$, which provides a linear restoring force. This force seeks to pull our pseudo-particle, the signal amplitudes, back to the origin. This spring force is combined with a force related to the statistics of the signal,

$$
\begin{equation*}
\vec{F}_{\text {statistics }}=-\kappa \operatorname{Re}(\vec{\nabla} \Phi(\vec{v})) . \tag{2.17}
\end{equation*}
$$

Equation (2.17) shows that the system characteristic function serves as a potential energy surface. We can plot the total potential energy surface, the spring potential plus the characteristic function, to obtain a better idea of the potential energy surface the fourth-order decorrelation system is moving in. The full potential, $V(\vec{v})$, is given by

$$
\begin{equation*}
V(\vec{v})=\frac{\vec{v} \cdot \vec{v}}{2}+\kappa \iint \ldots \int \operatorname{Re}(\vec{\nabla} \Phi(\vec{v})) d v_{1} d v_{2} \ldots d v_{M} . \tag{2.18}
\end{equation*}
$$

Because the shape of the potential energy surface depends on statistics, several statistics must be considered to understand the performance of the fourth-order decorrelation system. Five cases will be discussed briefly. The first three cases show the potential of the fourth-order decorrelation system when the two input signals have the same statistics. The last two cases are mixed cases where the two signals have different statistics.

Figure 2.6 shows the potential energy function for two sub-Gaussian input signals. Notice there are several equilibrium points. The origin is an unstable equilibrium point where any perturbation will lead the system further away from that point. There are four stable equilibria, one on each of the axes. Any perturbation away from each of these equilibria will tend to push the system back towards the equilibrium point. Because this equilibrium point is on the axes, the amplitude of one of the signals is at a maximum, while the amplitude of the other signal goes to zero. This is the desired winner-takes-all solution. Because there are two unique solution, this is considered a bistable solution.

The signal selected, i.e., the potential well the system ends up in, depends on the noise in the loop when the system is started. If the noise looks more like one signal than the other so that its amplitude is slightly larger, the signal with the larger amplitude will be selected. Two sub-Gaussian signals with the same Kurtosis will have a $50 / 50$ chance of selection. In the case of two sub-Gaussian signals with different Kurtoses, the signal with the smaller Kurtosis will have a proportionally larger chance of selection.

The potential energy surface for the super-Gaussian vs super-Gaussian case is shown in Figure 2.7. The four stable equilibria are located on the 45 and -45 degree lines. The equilibria on the axes are now unstable saddle points. Any perturbation of the systems away from either axis line will push the system towards the equilibrium points on the 45 and -45 degree lines. Because there is only one unique stable point, this is the monostable solution. This solution is undesirable as the final state of the system consists of equal amounts of each signal.


Figure 2.6: $V(\vec{v})$ vs $\vec{v}$ for two sub-Gaussian signals.
$V(\vec{v}$ is the potential energy surface associated with two sub-Gaussian inputs into the fourth-order decorrelation system. This situation is bistable with one unique stable steady state solution on each axis. This is the winner-takes-all configuration.


Figure 2.7: $V(\vec{v})$ vs $\vec{v}$ for two super-Gausssian signals.
$V(\vec{v}$ is the potential energy surface associated with two super-Gaussian inputs into the fourth-order decorrelation system. This situation is monostable with one unique stable steady state, which corresponds to the equal amounts of each signal oscillating in the feedback loop.

Figure 2.8 shows the potential energy surface for two Gaussian signals. This diagram looks very different from the previous two potential energy surfaces. There is one unstable equilibrium point at the origin and an infinite number of stable solutions. These solutions are confined to a ring of constant power at the base of the central peak. Any initial condition, whether inside or outside of the ring, will push the system towards the ring. Once in the ring, there is no restoring force confining the system to a particular point in the ring. As a result, the system's ability to separate signals will vary unpredictably with time. This is a metastable solution, which is also undesirable. Interestingly, the state space diagram associated with this potential energy surface would look very similar to the PCA state space diagram shown in Figure 5.8.

Now that we have looked at the three pure cases, let's look at two mixed cases. First, consider a system where one input is Gaussian and the other is Sub-Gaussian, see Figure 2.9. The potential energy surface is a melding of the two pure cases. The signal with the sub-Gaussian PDF still results in a stable equilibrium on its axis. There are remnants of the metastable solution at the base of the central peak on the Gaussian axis. It appears that the system could select the Gaussian signal under certain initial conditions. However, the system's ability to maintain its selection of the Gaussian will vary. A perturbation which pushes the system away from the Gaussian axis will likely push the system towards the sub-Gaussian stable solutions. Thus, the equilibrium associated with the Gaussian signal acts as an unstable equilibrium point. One can extrapolate, if there is at least one sub-Gaussian signal, the system will exhibit winner-takes-all behavior and select a sub-Gaussian signal. It will always extract the subGaussian signal. This state space diagram associated with this potential energy surface has similar properties to the PCA state space diagram in Figure 5.9.

One more interesting case is where one input signal is Super-Gaussian and the other signal is Gaussian, as shown in Figure 2.10. In this situation, the system will extract the Gaussian signal. The equilibrium associated with the Super-Gaussian signal


Figure 2.8: $V(\vec{v})$ vs $\vec{v}$ for two Gaussian signals.
$V(\vec{v}$ is the potential energy surface associated with two Gaussian signals. This situation is metastable with an infinite number of solutions confined to a ring corresponding to constant power.


Figure 2.9: $V(\vec{v})$ vs $\vec{v}$ for one Gaussian signal and one sub-Gaussian signal.
$V(\vec{v}$ is the potential energy surface associated with one Gaussian signal and one subGaussian signal. There is one unique stable solution corresponding to the sub-Gaussian signal. This situation shows winner-takes-all behavior, where the sub-Gaussian signal will always be selected.


Figure 2.10: $V(\vec{v})$ vs $\vec{v}$ for one super-Gaussian signal vs one Gaussian signal.
$V(\vec{v}$ is the potential energy surface associated with one super-Gaussian signal and one Gaussian signal. In this case, the Gaussian signal will be selected.
on the vertical axis is a saddle point, which pushes the system towards the solution on the horizontal axis. There is very weak confinement for the stability point associated with the Gaussian signal. As a result, the system's ability to maintain its selection of the Gaussian will vary. However, substantial perturbations will tend to push the system back towards the Gaussian solution. For this particular case, the greater gain of the Gaussian signal over the super-Gaussian signal allows the system to achieve winner-takes-all behavior even though there is no sub-Gaussian signal involved.

The potential energy surfaces presented in this section have assumed the system is operating in the small gain regime. Thus, we have identified the steady state behavior nearest to the origin. With much larger gain, it is possible to find many other steady state solutions. Provided the fourth-order decorrelation system is operated in the small gain regime, one need not worry about these higher-order stability points.

### 2.5 Lotka-Volterra Equations

Although the potential energy surface is an excellent way of observing the steady state behavior of the fourth-order decorrelation system, it is difficult to gain much physical intuition into the dynamics of the fourth-order decorrelation system. As an educational exercise, one can simplify Equation (2.15) into a more easily analyzable form. By expanding $\phi(v)$ up to fourth-order and writing the dynamics in terms of power, the closed-loop evolution equation becomes

$$
\begin{equation*}
\frac{\tau}{\kappa} \frac{d P_{m}}{d t}=\left(\alpha\left\langle S_{m}^{2}\right\rangle-\frac{1}{3}\left\langle S_{m}^{4}\right\rangle P_{m}-\left\langle S_{m}^{2}\right\rangle \sum_{l \neq m}\left\langle S_{l}^{2}\right\rangle P_{l}\right) P_{m} \tag{2.19}
\end{equation*}
$$

where $\tau$ is the photorefractive time constant, $\kappa$ is the small signal gain, $\alpha=2\left(1-\frac{1}{\kappa}\right)$ is the linear gain, and $\left\langle S_{m}^{2}\right\rangle$ and $\left\langle S_{m}^{4}\right\rangle$ are the second- and fourth-order moments of the $m^{t h}$ signal, respectively. For two normalized signals (i.e., $\left\langle S_{m}^{2}\right\rangle=1$ ) with the same PDF,

Equation (2.19) reduces to

$$
\begin{align*}
& \frac{\tau}{\kappa} \frac{d P_{1}}{d t}=\alpha P_{1}-\frac{1}{3}\left\langle S^{4}\right\rangle P_{1}^{2}-P_{2} P_{1}  \tag{2.20}\\
& \frac{\tau}{\kappa} \frac{d P_{2}}{d t}=\alpha P_{2}-\frac{1}{3}\left\langle S^{4}\right\rangle P_{2}^{2}-P_{1} P_{2}
\end{align*}
$$

The coupled equations in Equation (2.20) are conveniently in Lotka-Volterra form.
Lotka-Volterra equations are typically used to describe population dynamics where living species are reproducing and competing with themselves for resources or competing with other species for resources. In the case of the fourth-order decorrelation feedback loop, the signals are competing for gain. Let's discuss each term individually.

The first term is the linear gain term. In the absence of competition, the signal exhibits exponential growth. The second and third terms are competition terms. The second term is the self-saturation term and describes competition of the signal with itself. The third term is the cross-saturation term and describes competition with the other signal.

From the steady-state stability analysis of Equation (2.20), we find that the dynamics of the feedback loop depend on the relationship between the self-saturation and cross-saturation terms. If the self-saturation is greater than the cross-saturation (i.e., $\langle S\rangle / 3>1$ ), then the stable steady state solution occurs when both signals are present in the feedback loop in equal amounts. Since this situation results in one distinct solution, this is the monostable solution. If the self-saturation is equal to the cross-saturation (i.e., $\langle S\rangle / 3=1$ ), then both signals exist with unknown amplitudes. Since there are an infinite number of solutions, this is the metastable solution. Finally, if the self-saturation is less than the cross-saturation (i.e., $\langle S\rangle / 3<1$ ), then only one signal, either signal one or signal two, will exist in the loop. This is the bistable solution since there are two possible distinct solutions. The bistable solution is of particular interest because it is the winner-takes-all solution that recovers the original signals. Thus far, the dynamics can be summarized as follows: the system is bistable, metastable, or monostable depending on whether the fourth-order moment is less than, equal to, or greater than three.

Recall from Equation (1.8) that the fourth-order moment can be used as a measure of how Gaussian a signal is. For normalized signals, $\operatorname{Kurt}[S]=\left\langle S^{4}\right\rangle$. The kurtosis of a signal with a Gaussian-distributed PDF is three. A signal is classified as sub-Gaussian, Gaussian, or super-Gaussian depending on whether its kurtosis is less than, equal to, or greater than a Gaussian. Using this classification, the fourth-order decorrelation feedback loop is bistable, metastable, or monostable depending on whether the kurtosis is sub-Gaussian, Gaussian, or super-Gaussian, respectively, in the case of two normalized signals with the same PDF.

Table 1.1 gives the kurtosis of six different PDFs. Comparing the kurtosis values with the open-loop gain curves shown in Figure 2.5, we see that the lower the kurtosis, the higher the open-loop gain. Sub-Gaussian signals, which have a lower self-saturation, have a higher open-loop gain. Therefore, it makes sense that sub-Gaussian signals achieve the winner-takes-all solution.

In general, the original signals may not have the same PDF. How does the feedback loop behave in this case? The answer depends on the exact PDFs involved. If at least one signal is sub-Gaussian, one sub-Gaussian signal will be extracted from the mixtures. If there is more than one sub-Gaussian signal, then each sub-Gaussian signal has an equal probability of being selected provided their PDFs have the same kurtosis. In general, the signal with the smaller kurtosis will have a higher probability of selection. In the case of one Gaussian and one super-Gaussian signal, the Gaussian-distributed signal will be extracted from the loop as it has the lowest kurtosis. However, mixtures of Gaussian signals cannot be separated.

## Chapter 3

## 200 MHz Optoelectronic Fourth-Order Decorrelation Implementation

### 3.1 Overview

Chapter 3 describes the experimental implementation of optoelectronic fourthorder decorrelation. First, a brief background of the operation regimes of the photorefractive crystal and the EOM is given. Next, the major components to be discussed are the input channels, the optoelectronic feedback loop set-up, and homodyne detection. Finally, signal separation performance data will be given.

### 3.2 Background

### 3.2.1 Photorefractive Crystal Operation Regimes

Figure 3.1 shows a schematic of a typical photorefractive two beam coupling layout. The pump beam, usually the more intense beam, gives up energy to the signal beam. The gain $G$ experienced by the signal beam in a photorefractive crystal of length $L$ assuming no absorption, is given by

$$
\begin{equation*}
G(m)=\frac{1+m}{1+m e^{-\Gamma L}}, \tag{3.1}
\end{equation*}
$$

where $m$ is the ratio of the pump beam intensity to the signal beam intensity when entering the crystal, while $\Gamma$ is the gain per unit length of the crystal. Typical values of $L$ are 1-5 mm . $\Gamma$, which can be as high as $10,000 \mathrm{~cm}^{-1}$, is affected by the orientation of the beams in the crystal, the overlap area of the pump and signal beams, the temporal


Figure 3.1: Schematic of photorefractive two-beam coupling.
The signal beam, denoted by $S$, is the beam that loses energy. The pump beam, denoted by $P$, is the beam that gives up energy to the signal beam. The subscripts in and out refer to the intensity at the input or output of the photorefractive crystal.
and spatial coherence between the pump and the signal beam, as well as the material properties of the crystal itself. A typical gain curve for a photorefractive crystal as derived from Equation (3.1) is shown in Figure 3.2. The photorefractive crystal can act as a linear or non-linear amplifier depending on the ratio of the pump beam to the signal beam. When the ratio of the pump beam to the signal beam is greater than $G_{0}$, the pump beam intensity is, for all purposes, constant even as it gives gain to the signal beam. As a result, the photorefractive crystal gives a constant gain of $10^{G_{0}}$, where $G_{0}=e^{\Gamma L}$. For this reason, the linear regime is also called the undepleted pump regime.

As $m$ approaches and becomes less than $G_{0}$, the pump beam becomes depleted and is unable to provide the maximum gain to the signal beam. This is called the depleted pump or saturation regime. As the pump is depleted, gain of the crystal then drops $10 d B$ per decade down to a gain of 1 or $0 d B$.


Figure 3.2: Typical two-beam coupling gain profile.
$S$ refers to the intensity of the signal beam or the beam that receives gain. $P$ refers to the intensity of the pump beam or the beam that gives gain to the signal beam. $G_{0}$ is the maximum gain of the photorefractive crystal.

### 3.2.2 Electro-Optic Modulator Operation Regimes

The EOM imposes a phase modulation onto the laser beam passing though it. The temporal component of the electric field of the laser after passing through the EOM is given by

$$
\begin{equation*}
E(t)=E_{0} e^{-i\left(\omega_{c} t-m f(t)\right)} \tag{3.2}
\end{equation*}
$$

where $E_{0}$ is the unmodulated electric field of the laser, $\omega_{c}$ is the temporal frequency of the laser carrier, $m$ is the modulation depth, and $f(t)$ is the electronic voltage applied to the EOM. If $f(t)=\sin \left(\omega_{m} t\right)$, then Equation (3.2) can be expanded as a sum of Bessel functions such that

$$
\begin{equation*}
E=E_{0} e^{-i \omega_{c} t}\left(\sum_{n=-\infty}^{\infty} J_{n}(m) e^{i n \omega_{m} t}\right), \tag{3.3}
\end{equation*}
$$

where $\omega_{m}$ is the frequency of the modulating signal.
The EOM can be used as a linear or non-linear modulator depending on whether the $m$ is large or small. If $m$ is small, i.e., $\sin (m) \approx m$, then expansions of Equation (3.2) up to first order in $m$ are kept so that

$$
\begin{equation*}
E \approx E_{0} e^{-i \omega_{c}}\left(1+\frac{m}{2} e^{i \omega_{m} t}+\frac{m}{2} e^{-i \omega_{m} t}+\theta\left(m^{2}\right)\right) . \tag{3.4}
\end{equation*}
$$

The terms in Equation (3.4), are associated with $J_{0}$ (the laser carrier amplitude), $J_{1}$ (the uppersideband amplitude), and $J_{-1}$ (the lower sideband ampltude), respectively.

If $m$ is large, then higher and higher harmonics of $\omega_{m}$ are present. The amplitude of the n-th harmonic follow the $J_{n}{ }^{\text {th }}$ Bessel function.

### 3.3 Input Channels

### 3.3.1 Overview

The input channels serve to take the electrical input mixtures and modulate them onto the laser carrier. Figure 2.2 shows a single input channel. The preparation of the electronic and optical signals are described in turn.

### 3.3.2 Electronic Signals

The electronic signals used as inputs into the optoelectronic feedback loops are mixtures of independent signals. This is a two-channel system, therefore there are two electronic mixtures. These signal mixtures are created using two signal generators and a mixing circuit. The feedback loop has a bandwidth of 20 MHz with a center frequency of 200 MHz . The signal generators were set to generate signals within this bandwidth. In particular, the center frequencies were chosen to be close together. This is because frequency-dependent phase delays adversely affect the signal separation performance of the fourth-order decorrelation feedback loop (see Section 4.4). The signal generator output power is dictated by the mixing circuitry which assumes that the input signals are at -20 dBm .

The mixing circuitry has six inputs as shown in Figure 3.3. Two of the six inputs are the two signals from the signal generator. The other four inputs are the voltages necessary to create the desired mixing coefficients described in Equation (1.2) as $a_{11}$, $a_{12}, a_{21}$, and $a_{22}$. Much care was taken in the design of the mixing circuitry to make sure that unwanted phase delays were not introduced in the mixing process. The values of the DC voltages were set to achieve a specific mixing matrix.

The fourth-order decorrelation feedback loop assumes that whitening has already been performed. Thus, the input mixtures are assumed to be orthogonal and normalized. To accommodate this requirement, the independent signals are mixed using a rotation matrix so that

$$
\begin{align*}
& P C_{1}(t)=s_{1}(t) \operatorname{Cos}(\theta)+s_{2}(t) \operatorname{Sin}(\theta)  \tag{3.5}\\
& P C_{2}(t)=-s_{1}(t) \operatorname{Sin}(\theta)+s_{2}(t) \operatorname{Cos}(\theta)
\end{align*}
$$

$s_{1}(t)$ and $s_{2}(t)$ represent the original signals, $P C_{1}(t)$ and $P C_{2}(t)$ represent the normalized principal components, and $\theta$ represents the angle that the principal component basis makes with the independent signal basis as shown in Figure 3.4.

Although arbitrary angles are possible with the mixing circuitry, there are two

## Mixing Circuitry



Figure 3.3: A schematic of the mixing circuit showing the inputs and outputs.
$S_{1}(t)$ and $S_{2}(t)$ are the independent signals. $V_{A}, V_{B}, V_{C}$, and $V_{D}$ are the applied voltages to obtain the desired mixing coefficients. $\tilde{S}_{1}(t)$ and $\tilde{S}_{2}(t)$ are the signal mixtures that are sent to the input channels.


Figure 3.4: Relationship between principal component basis and the independent signal basis.

A single parameter, $\theta$, is used to describe this relationship.
angles of particular interest because they characterize the system: 0 and 45 degrees. For the zero degree mixture, each synthesizer drives a separate input channel amplifier. For the 45 degree mixture, the mixing circuitry was used with DC input voltages of roughly equal voltage. There is a 180 degree phase shift on one of the DC inputs required to create the orthogonal signals. However, external cable lengths were added to make sure that the outputs were orthogonal. To make sure that the desired mixing matrix was achieved, an oscilloscope $\mathrm{x}-\mathrm{y}$ trace of the two mixtures exiting the circuit was checked (see Figure 3.5). Notice that Figure 3.5 is identical to the joint distribution for two uncorrelated, but not independent, variables shown in Figure 1.5. The specific requirements for normalization and orthogonality are discussed in Chapter 4. A discussion of the process by which the mixing coefficients were is discussed in Appendix B.


Figure 3.5: X-Y oscilloscope trace of 45 degree signal mixture.
The square shape indicates that these signals are orthogonal. The angle that the perpendicular bisector of the sides of the square makes with the axes indicates that the mixing angle is 45 degrees. The equal extent of the square on each axis indicates the mixed-signals have equal power.

Since the mixing circuit was not used for the 0 degree mixture, but was used for the 45 degree mixture, the separation performance at 45 degrees gives an indication of how close the mixing circuit approximates a purely real-valued, orthonormal mixing matrix since the system performance should be independent of mixing angle.

### 3.3.3 Optical Signals

The common laser carrier used in the input channels is an unwanted correlation between the signal mixtures. In other words, the laser carrier looks like an additional signal to the optoelectronic feedback loop. This signal will receive gain and reduce the overall gain of the system. The reduced gain will decrease signal separation at the frequencies of interest. Therefore, carrier suppression is used to remove the optical laser carrier.

A technique for achieving -70 dB of optical carrier suppression has been demonstrated in [5]. Although we do not require nearly this much carrier suppression, this is the technique used in our optoelectronic implementation of fourth-order decorrelation because of its adaptive nature. The photorefractive crystal can adapt to poor wavefront quality and self-compensates for slow path length drifts. Carrier suppression of roughly 20 dB was achieved. The down-side of this technique is that there is a 3 dB loss in the signal power. We feel that the added stability makes up for the loss of signal. Additionally, as long as the optical power in the sidebands is large enough compared to the feedback loop optical power, such that the feedback loop operates in the linear regime and the EOM is modulated linearly, we de not care about the particular intensity in the sidebands.

The electro-optic modulator used in the carrier suppression circuit has a small modulation index in order to produce a linear phase modulation onto the laser carrier. The modulation index is chosen so that the second harmonic is minimal compared to the first harmonic. Since the amplitudes of the harmonics follow Bessel functions, a
power ratio of the second to first harmonic was chosen to be

$$
\begin{equation*}
\left(\frac{J_{2}(m)}{J_{1}(m)}\right)^{2}=0.05 \tag{3.6}
\end{equation*}
$$

where $m$ is the modulation depth. To achieve this ratio, $m$ is 0.54 radians.
The total intensity of the signal portion of the beam after passing through the EOM is given by

$$
\begin{equation*}
I_{s} \approx \frac{I_{0} m^{2}}{4} \tag{3.7}
\end{equation*}
$$

where $I_{0}$ is the intensity of the beam passing through the EOM. In Equation (3.7), I have taken into account the two sidebands and $3 d B$ signal loss due to photorefractive two beam coupling., $m$ is small so that $J_{1}(m) \approx \frac{m}{2}$. The ICA configuration requires that the feedback loop photorefractive crystal be used in its linear regime. As a result, $I_{s} \gg I_{l o o p}$.

### 3.4 Homodyne Detection

A phase modulated beam is not detectable with a photodetector. Therefore, homodyne detection is used to turn the phase-modulated loop beam into an amplitudemodulated beam that is detectable. The difference between phase modulation and amplitude modulation is the phase of the carrier with respect to the sidebands. Thus, homodyne detection requires reintroducing the laser carrier into the loop beam at the correct phase.

The homodyne carrier beam is split off using a polarizing beamsplitter before the loop EOM, as shown in Figure 2.1. A polarizing beam splitter is used in conjunction with a half-waveplate in order to control the intensity ratio between the loop beam and homodyne beam. A half-waveplate is set in the homodyne arm to rotate the vertical polarization to horizontal polarization. One of the alignment mirrors contains a piezoelectric element, which is used to control the phase of the homodyne beam with respect
to the loop beam sidebands. A manually adjusted DC voltage is applied to the piezomirror to adjust the phase. When the phase is adjusted correctly, the system output is at maximum amplitude.

### 3.5 The Optoelectronic Feedback Loop

The feedback loop is a resonator. As with any resonator, if the gain is large enough, it will oscillate strongly in any mode where the gain is larger than one. If the gain is large enough, spontaneous oscillation at all frequencies will occur. We, however, do not want our resonator to oscillate at any frequency. Rather, we desire the system to only oscillate at the frequencies of interest. Oscillations at other frequencies wash out the grating inside the photorefractive crystal reducing the gain at the frequencies of interest. Thus, the electronic gain must be less than unity at all frequencies. Because photorefractive gain only occurs at specific frequencies, if the photorefractive gain is large enough, the gain at the frequencies of interest will be greater than unity and will oscillate in the feedback loop. The feedback loop electronic gain is set to just below unity so that the photorefractive gain required to make the input signal oscillate is reasonable. The open-loop gain for the feedback loop without photorefractive gain is shown in Figure 3.6. The photorefractive gain is approximately 20 dB . Once the frequencies of interest are able to oscillate in the feedback loop, competition can occur and one signal is suppressed.

Signal separation requires that higher harmonics are generated. Therefore, the electronic gain must be large enough to drive the EOM non-linearly. The electronic gain is determined by several parameters: the photodetector gain, the homodyne gain, the bandpass filter, the variable preamplifier or attenuator, and the power amplifier. The gains that can be easily adjusted are the homodyne gain and the variable pre-amplifier.


Figure 3.6: Open-loop gain without photorefractive crystal
This open-loop gain curve for the fourth-order decorrelation feedback loop shows that the gain of the feedback loop is just below unity without the photorefractive gain. This is done so the feedback loop will not oscillate electronically.

The homodyne gain $G_{H}$ is given by

$$
\begin{equation*}
G_{H} \propto \sqrt{I_{L} I_{H}} \sin \phi, \tag{3.8}
\end{equation*}
$$

where $I_{L}$ and $I_{H}$ are the intensities of the loop and homodyne beams respectively, while $\phi$ is the phase of the homodyne beam. To keep the photorefractive crystal in the linear regime, we desire to keep $I_{L}$ small. Therefore $I_{H}$ should be large so that small increases in the loop signal produce large voltages on the EOM. $I_{H}$ is limited by the total laser power available and the saturation level of the detector. The optical intensities used are given in Table 3.1. The variable preamplifier (or attenuator) is used to fill in the gap between the photodetector signal and the input power required to drive the EOM non-linearly.

| Normalized Beam Ratios* | Loop Beam | 1 |
| :---: | :---: | :---: |
|  | Pump Beam | 2,000 |
|  | Homodyne Beam | 100,000 |
| Photorefractive Crystal Gain | $\sim 100$ |  |
| Total Laser Power | 150 mW |  |

Table 3.1: Optical system parameters for the fourth-order decorrelation feedback loop. $*$ Normalized to loop beam power $(160 \mathrm{nW})$.

### 3.6 Experimental Results

The output of the fourth-order decorrelation system was taken from a 10 dB coupler inserted into the loop before the power amplifier. The output of the coupler was viewed using a spectrum analyzer. Several different types of signals were used as inputs into the system. Single frequency sinusoidal signals at 200 MHz and 202 MHz were used as initial starting points. Experiments were also performed with a frequency modulated 202 MHz signal modulated with a 1 kHz sine wave. The peak deviation was 10 MHz , which leads to a modulation index of 1500 . Additionally, one of the synthesizers was swept from 199 to 203 MHz .

Various combinations of these signals were studied at both a 0 degree and a 45 degree mixture. With both spectrum analyzers on, the power of both signals grew at the same rate. This represents the exponential growth portion of the dynamics. As the signals grew, intermodulation peaks with frequency spacing of the input signals also grew as shown in Figure 3.7.

This behavior demonstrates that the EOM is saturating and generating higher harmonics. These harmonics beat with each other generating the intermodulation peaks. As the two signals compete for gain, the power in the inner modulation products fluctuate until one signal wins and the other signal is suppressed. The more one signal is suppressed, the lower the intermodulation peaks. The signal separation was recorded after the feedback loop reached steady state. A photograph of the synthesizer screen
when the system reaches steady state is shown in Figure 3.8
For the narrow-band single frequency inputs (both fixed and swept), signal separation was 20 dB or better. The signal separation with a 0 degree mixture was significantly better where signal separations were on the order of 40 dB or better. This discrepancy between the signal separation at 0 degrees and 45 degrees is due to the mixing circuitry. The system performance should be independent of mixing angle to the extent that the mixing circuitry approximates a purely real, orthonormal mixing circuit.

Signal separation for the 10 MHz bandwidth FM signal competing with a single frequency sine was 10 dB or better. The signal separation for a 0 degree mixing matrix was 20 dB or better, but degraded by 10 dB with the addition of the mixing circuit for the 45 degree mixture. The reduced signal separation for the broadband signal is likely due to the large phase change across the system bandwidth. A $190-210 \mathrm{MHz}$ bandpass filter is used in the feedback loop to set the system bandwidth. The phase of the bandpass filter changes by 180 degrees from the center of the filter to the edges of the filter. This large phase change across the 10 MHz signal bandwidth reduces the gain of the signal an degrades the system performance by roughly 10 dB .

Designing a better bandpass filter and taking more care in general with frequency dependent phase delays is a straightforward way of improving separation performance of the system.


Figure 3.7: Spectrum analyzer display of system before competition starts.
This display shows the output of the feedback loop with one input channel on. The input channel contains one output from the 45 degree mixing circuit. The frequencies input into the system were 202 MHz (third peak from the right) and 200 MHz (fourth peak from the right). The other peaks are intermodulation products of the higher harmonics generated by the EOM saturation. Competition will not start until the other input channel is turned on.


Figure 3.8: Spectrum analyzer display of system in steady state.
In this competition, two single frequency sine waves are input into the fourth-order decorrelation system with a 45 degree mixture. The frequencies input into the system were 202 MHz (third peak from the right) and 200 MHz (fourth peak from the right). The other peaks are intermodulation products of the higher harmonics. In steady state, the 200 MHz signal wins and suppresses the 202 MHz signal by more than 20 dB .

## Chapter 4

## Fourth-Order Decorrelation System Dynamics with Non-ideal Inputs Signal

### 4.1 Overview

The principles of optoelectronic fourth-order decorrelation are derived assuming ideal components. How well can signals be separated if the outputs of the PCA portion of the system are not completely decorrelated? What if automatic gain control in the normalization portion of the system produces principal components that are not identical in amplitude? How do frequency-dependent phase delays affect the performance of the fourth-order decorrelation system. These are just a few questions that must be addressed in an effort to develop performance specifications for various components used to perform ICA.

This chapter presents a detailed analysis of how several nonidealities affect the ability of the fourth-order decorrelation system to separate signals. For each analysis, the problem statement, the derivation of the closed-loop evolution equation, the steadystate analysis, and a discussion of the results is presented. To obtain quantitative answers, a two-channel system is assumed.

### 4.2 Non-Normalized Inputs

### 4.2.1 Problem Statement

The derivation of the fourth-order decorrelation system's dynamical equation assumes its inputs are whitened, that is, normalized and orthogonal. How well would the system separate signals if its input signals were orthogonal, but not normalized? To answer this question, the following method was used. Assuming non-normalized amplitudes, the open-loop gain equation was derived in the principal component basis. Next, I switch to the independent signal basis and derive the new differential equation that governs the behavior of the fourth-order decorrelation feedback loop. Once the differential equation is found, a steady state analysis is performed, which will ultimately give specifications on the normalizations process after PCA is performed.

### 4.2.2 Derivation of Differential Equation

First, it is necessary to derive the open-loop gain equation given non-normalized principal components. Starting with the initial signal applied to the EOM, assume that the voltage applied is

$$
\begin{equation*}
V_{p}(t)=\vec{u}(t) \cdot \hat{Y}(t), \tag{4.1}
\end{equation*}
$$

where $\hat{Y(t)}$ represents the principal components, the " $\wedge$ " symbol represents a normalized vector, and $\vec{u}(t)$ represents the slowly varying signal amplitudes. I should point out that $V_{p}(t)$ must be equal to $V_{s}(t)=v(t) \cdot S(t)$ (see Equation 2.5), since they are just two different bases for expressing the same information. I will be using the equality of these two expressions later in my analysis. The principal components $\hat{Y}(t)$ are linear combinations of the independent signals such that the mixtures are orthogonal. As a result, the principal components are uniquely described by a rotation matrix where the angle $\theta$ represents the angle that one of the principal components makes with the
independent signal basis

$$
\tilde{C}=\left[\begin{array}{cc}
\cos (\theta) & \sin (\theta)  \tag{4.2}\\
-\sin (\theta) & \cos (\theta)
\end{array}\right]
$$

$\tilde{C}$ produces normalized principal components.
To generate non-normalized principal components, $\tilde{B}$ is defined as

$$
\tilde{B}=\left[\begin{array}{ll}
b_{1} & 0  \tag{4.3}\\
0 & b_{2}
\end{array}\right]=\left[\begin{array}{cc}
1+\frac{\varepsilon}{2} & 0 \\
0 & 1-\frac{\varepsilon}{2}
\end{array}\right]
$$

where $\varepsilon$ is the deviation from unit length and is assumed to be small. Combining Equations (4.1, 4.2, and 4.3), the open-loop gain is given by

$$
\begin{equation*}
A_{m}=\left(-\frac{b_{m}}{u_{m}}\left(c_{m 1} \operatorname{Re}\left[d_{1} \Phi(\vec{v})\right]+c_{m 2} \operatorname{Re}\left[d_{2} \Phi(\vec{v})\right]\right)\right) \tag{4.4}
\end{equation*}
$$

where $A_{m}$ is the open-loop gain of the $m^{t h}$ component, $c_{m n}$ is the component in the $m^{\text {th }}$ column and $n^{\text {th }}$ row of $\tilde{C}$.

The resulting closed-loop dynamical equation for the principal components is given by

$$
\begin{equation*}
\tau \frac{d \vec{u}(t)}{d t}=-\kappa \tilde{B} \tilde{C} R e[\vec{\nabla} \Phi(\vec{v})]-\vec{u}(t) \tag{4.5}
\end{equation*}
$$

I am interested in the evolution of the independent components rather than the evolution of the principal components. Therefore, a transformation between the principal components and the independent signals is required. This transformation is given by

$$
\begin{equation*}
\vec{u}(t)=\left(\tilde{C}^{T}\right)^{-1} \vec{v} \tag{4.6}
\end{equation*}
$$

where ' $T$ ' indicates transpose and ' -1 ' indicates inverse. Substituting Equation (4.6) into Equation (4.5) produces the following evolution equation for the independent signals

$$
\begin{equation*}
\tau \frac{d \vec{v}(t)}{d t}=-\kappa \tilde{C}^{T} \tilde{B} \tilde{C} \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})]-\vec{v}(t) \tag{4.7}
\end{equation*}
$$

### 4.2.3 Steady State Analysis

Now that an equation has been derived that describes the dynamics of the feedback loop in the non-normalized principal components case, a steady state analysis is performed to determine the amount of crosstalk present when the winner-takes-all conditions are met.

By multiplying out the leading matrices, it is possible to rewrite Equation (4.7) as

$$
\begin{equation*}
\tau \frac{d \vec{v}(t)}{d t}=-\kappa\left(\tilde{I}+\frac{\varepsilon}{2} \tilde{E}\right) \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})]-\vec{v}(t) \tag{4.8}
\end{equation*}
$$

where $\tilde{E}$ is given by

$$
\tilde{E}=\left[\begin{array}{cc}
\cos (2 \theta) & \sin (2 \theta)  \tag{4.9}\\
\sin (2 \theta) & -\cos (2 \theta)
\end{array}\right] .
$$

The steady state solution is found by setting Equation (4.8) equal to zero and solving for the new steady state solution $v_{s s}$

$$
\begin{equation*}
v_{s s}=-\kappa\left(\tilde{I}+\frac{\varepsilon}{2} \tilde{E}\right) \operatorname{Re}\left[\vec{\nabla} \Phi\left(v_{s s}\right)\right] . \tag{4.10}
\end{equation*}
$$

The steady state solution is composed of two parts

$$
\begin{equation*}
v_{s s}=v_{0}+\delta v, \tag{4.11}
\end{equation*}
$$

where $v_{0}$ is the solution to the unperturbed differential Equation (2.15), and $\delta v$ is the deviation of the steady state solution from the unperturbed solution. Using a Taylor series expansion about $v_{0}$ and assuming $\delta v$ is small such that $\vartheta\left(\delta v^{2}\right)$ can be ignored, $\operatorname{Re}\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{s s}}\right)\right]$ can be expanded as

$$
\begin{equation*}
\operatorname{Re}\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{s s}}\right)\right] \approx \operatorname{Re}\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{0}}\right)\right]+\operatorname{Re}\left[\vec{\nabla}\left\{\vec{\nabla} \Phi\left(\overrightarrow{v_{0}} \cdot \delta \vec{v}\right\}\right] .\right. \tag{4.12}
\end{equation*}
$$

By substituting Equation (4.12) into Equation (4.10), we can identify

$$
\begin{equation*}
\overrightarrow{v_{0}}=-\kappa R e\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{0}}\right)\right], \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\overrightarrow{\delta v}=-\kappa\left(\operatorname{Re}\left[\vec{\nabla}\left\{\vec{\nabla} \Phi\left(\overrightarrow{v_{0}}\right) \cdot \delta \vec{v}\right\}\right]+\frac{\varepsilon}{2} \tilde{E} R e\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{0}}\right)\right]\right) . \tag{4.14}
\end{equation*}
$$

Using the expansion of the characteristic function given in Equation (2.11), $\operatorname{Re}\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{0}}\right)\right]$ can be expressed as

$$
\begin{equation*}
\operatorname{Re}\left[\vec{\nabla} \Phi\left(\overrightarrow{v_{0}}\right)_{n}\right]=-v_{n}+\frac{v_{n} v_{m}^{2}}{2}+\frac{v_{n}^{3}\left\langle S_{n}^{4}\right\rangle}{6} . \tag{4.15}
\end{equation*}
$$

Combining Equation (4.15) with Equation (4.14), we find an expression for the components of $\overrightarrow{\delta v}$,

$$
\begin{equation*}
\delta v_{1}=\frac{\kappa \frac{\varepsilon}{2}\left[\cos (2 \theta)\left(-v_{1}+\frac{v_{1} v_{2}{ }^{2}}{2}+\frac{v_{1}{ }^{3}<S_{1}{ }^{4}>}{6}\right)+\sin (2 \theta)\left(-v_{2}+\frac{v_{2} v_{1}{ }^{2}}{2}+\frac{v_{2}{ }^{3}<S_{2}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{2}{ }^{2}}{2}+\frac{v_{1}{ }^{2}<S_{1}{ }^{4}>}{2}\right)}, \tag{4.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta v_{2}=\frac{\kappa_{2}^{\varepsilon}\left[\sin (2 \theta)\left(-v_{1}+\frac{v_{1} v_{2}{ }^{2}}{2}+\frac{v_{1}{ }^{3}<S_{1}{ }^{4}>}{6}\right)-\cos (2 \theta)\left(-v_{2}+\frac{v_{2} v_{1}{ }^{2}}{2}+\frac{v_{2}{ }^{3}<S_{2}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{1}{ }^{2}}{2}+\frac{v_{2}{ }^{2}<S_{2}{ }^{4}>}{2}\right)} . \tag{4.17}
\end{equation*}
$$

As expected, Equations (4.16) and (4.17) reduce to zero when $\varepsilon=0$.
We are interested in understanding how the signal separation we can achieve is affected by the fact that the input signals are not normalized. Therefore, we want to compute the signal separation when one of the signal wins the competition. For example, let's look at the case where signal 1 wins. When this is the case

$$
\begin{gather*}
v_{01}=\sqrt{\frac{6(\kappa-1)}{\left.<S_{1}^{4}\right)}},  \tag{4.18}\\
v_{02}=0  \tag{4.19}\\
\delta v_{1}=\frac{\kappa \frac{\varepsilon}{2}\left[\cos (2 \theta)\left(-v_{01}+\frac{v_{01}{ }^{3}<S_{1}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{01}{ }^{2}<S_{1}{ }^{4}>}{2}\right)}, \tag{4.20}
\end{gather*}
$$

and

$$
\begin{equation*}
\delta v_{2}=\frac{\kappa \frac{\varepsilon}{2}\left[\sin (2 \theta)\left(-v_{01}+\frac{v_{01}{ }^{3}<S_{1}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{01}{ }^{2}}{2}\right)} \tag{4.21}
\end{equation*}
$$

The signal separation or its inverse, the signal crosstalk, is determined by calculating the ratio of the losing signal to the winning signal's intensity

$$
\begin{equation*}
d B_{\text {crosstalk }}=10 \log \left(\frac{v_{\text {losing }}}{v_{\text {winning }}}\right)^{2}=10 \log \left(\frac{v_{2}^{s s}}{v_{1}^{s s}}\right)^{2} . \tag{4.22}
\end{equation*}
$$

For our specific example, we have

$$
\begin{equation*}
d B_{\text {crosstalk }}=10 \log \left(\frac{v_{2}^{s s}}{v_{1}^{s s}}\right)^{2}=10 \log \left(\frac{\delta v_{2}}{v_{01}+\delta v_{1}}\right)^{2} . \tag{4.23}
\end{equation*}
$$

The crosstalk is a function of several parameters, of which the most interesting is $\theta$. A plot of the crosstalk versus $\theta$ is shown in Figure 4.1 for two sub-Gaussing signals. At two values of $\theta, 0$ and 135 degrees, the crosstalk goes to $-\infty$. This indicates perfect signal separation. To see why this is, recall that $\delta v_{2}$ is proportional to $\sin (2 \theta)$. At 0 and 90 degrees, $\sin (2 \theta)=0$. Therefore, there is no longer any of signal 2 present in the loop even though the amplitude of signal 1 is slightly different than the ideal value.


Figure 4.1: Crosstalk as a function of the mixing angle for non-normalized inputs.
The system achieves perfect signal separation at 0 and 90 degrees. The worst separation occurs at 45 and 135 degrees.

### 4.2.4 Discussion

The behavior for non-normalized inputs makes sense with a little thought. When the rotation angle is 0 or 90 degrees, the principal component axes are aligned with the
independent signal axes. This is what the system is trying to do anyway. Even though we have assumed that the gain for signal 1 and signal 2 is the same, the fact that one input signal is larger, will result in more of that signal on the loop. As a result, the winning signal already has a bias at the beginning of the competition. Thus, the signal with the head-start in the competition, will always oscillate in the feedback loop. This can be seen in the potential energy surface shown in Figure 2.6. A particle moving on the surface away from the 45 and -45 degree lines, tend to move towards the steady state solution on the axis which is closest.

At 45 and 135 degrees, this bias is removed because the total power of each signal is the same even if the principal components do not have the same amplitude. In this situation, the noise in the feedback loop determines which signal will win the competition. Therefore, each signal has a 50-50 chance of being selected. How well the system is able to separate the signals is dependent on the gain of the feedback loop. The higher the small signal gain, $\kappa$, the greater the signal separation.

Experimentally, one can identify biases in the fourth-order decorrelation system by looking at the separation performance and the probability of selecting one signal versus the other at 0 and 45 degree input mixtures.

### 4.3 Non-Orthogonalized Inputs

### 4.3.1 Problem Statement

The purpose of the PCA portion of ICA is to find an orthogonal basis from the received signal mixtures. System noise and non-ideal system components could result in mixtures that are not exactly orthogonal. Assuming this effect is small, what deviation from orthogonal can the fourth-order decorrelation feedback loop tolerate and still recover the original signals?

### 4.3.2 Derivation of Differential Equation

The inputs to the fourth-order decorrelation system are given by the vector $\vec{F}$. The components of $\vec{F}$ are not quite orthogonal and are rotated with respect to the signal space vectors. With two inputs, the components of $\vec{F}$ are shown pictorially in Figure 4.2. Mathematically, $\vec{F}$ can be written as a modified rotation matrix $\tilde{M}$ times


Figure 4.2: Input signals used in the non-orthogonality analysis projected onto the independent signal basis.
$F_{1}$ and $F_{2}$ are the input signals, which are normalized, but deviate from being orthogonal by the angle $\delta . \theta$ represents the nominal angle of the input signals with respect to the independent signal basis.
the independent signals $\vec{S}$

$$
\begin{equation*}
\vec{F}=\tilde{M} \vec{S} \tag{4.24}
\end{equation*}
$$

where $\tilde{M}$ is given by

$$
\tilde{M}=\left[\begin{array}{cc}
\cos \left(\theta+\frac{\delta}{2}\right) & \sin \left(\theta+\frac{\delta}{2}\right)  \tag{4.25}\\
\left.-\sin \left(\theta-\frac{\delta}{2}\right)\right) & \cos \left(\theta-\frac{\delta}{2}\right)
\end{array}\right] .
$$

Using trigonometric identities and small angle approximations for $\delta, \tilde{M}$ can be written in the form

$$
\begin{equation*}
\tilde{M}=\tilde{C}+\frac{\delta}{2} \tilde{E}, \tag{4.26}
\end{equation*}
$$

where

$$
\tilde{C}=\left[\begin{array}{cc}
\cos (\theta) & \sin (\theta)  \tag{4.27}\\
-\sin (\theta) & \cos (\theta)
\end{array}\right]
$$

and

$$
\tilde{E}=\left[\begin{array}{cc}
-\sin (\theta) & \cos (\theta)  \tag{4.28}\\
\cos (\theta) & \sin (\theta)
\end{array}\right]
$$

As in the non-normalized inputs analysis, I will begin the development of the closed-loop differential equation using the principal component. After one round trip, the principal component signal amplitude $\vec{u}^{\prime}$ is given by

$$
\begin{equation*}
\vec{u}^{\prime}=-\kappa \tilde{C} \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})]-\kappa \frac{\delta}{2} \tilde{E} \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})] \tag{4.29}
\end{equation*}
$$

This leads to the differential equation for the principal component amplitudes of

$$
\begin{equation*}
\tau \frac{d \vec{u}}{d t}=-\kappa \tilde{C} \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})]-\kappa \frac{\delta}{2} \tilde{E} \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})]-\vec{u} \tag{4.30}
\end{equation*}
$$

Using the transformation from the principal component basis to the independent signal basis given by Equation (4.6), I find that the independent signal amplitudes are subject to

$$
\begin{equation*}
\tau \frac{d \vec{v}}{d t}=-\kappa R e[\vec{\nabla} \Phi(\vec{v})]-\kappa \frac{\delta}{2} \tilde{E}^{\prime} \operatorname{Re}[\vec{\nabla} \Phi(\vec{v})]-\vec{v} \tag{4.31}
\end{equation*}
$$

where

$$
\tilde{E}^{\prime}=\tilde{C}^{-1} \tilde{E}=\left[\begin{array}{cc}
-\sin (2 \theta) & \cos (2 \theta)  \tag{4.32}\\
\cos (2 \theta) & \sin (2 \theta)
\end{array}\right]
$$

### 4.3.3 Steady State Analysis

The steady state analysis is performed in the same way as the non-normalized inputs scenario discussed in Section 4.2. First, Equation (4.31) is set to zero to solve for the steady state solution $\vec{v}_{s s}$. This leads to

$$
\begin{equation*}
\vec{v}_{s s}=-\kappa \operatorname{Re}\left[\vec{\nabla} \Phi\left(\vec{v}_{s s}\right)\right]-\kappa \frac{\delta}{2} \tilde{E}^{\prime} \operatorname{Re}\left[\vec{\nabla} \Phi\left(\vec{v}_{s s}\right)\right] \tag{4.33}
\end{equation*}
$$

I assume a form of the solution and apply techniques of perturbation theory to simplify the analysis further. I have assumed a steady state solution of the form

$$
\begin{equation*}
\vec{v}_{s s}=\vec{v}_{0}+\delta \vec{v}, \tag{4.34}
\end{equation*}
$$

where $\vec{v}_{0}$ is the steady state solution to the unperturbed steady state equation as shown in Equation (4.14), and $\delta \vec{v}$ represents a small deviation from the unperturbed solution. Using the Taylor series expansion of the characteristic function in Equation (4.12), I obtain the following components for $\delta \vec{v}$

$$
\begin{equation*}
\delta v_{1}=\frac{-\kappa \frac{\delta}{2}\left[-\sin (2 \theta)\left(-v_{1}+\frac{v_{1} v_{2}{ }^{2}}{2}+\frac{v_{1}{ }^{3}<S_{1}{ }^{4}>}{6}\right)+\cos (2 \theta)\left(-v_{2}+\frac{v_{2} v_{1}{ }^{2}}{2}+\frac{v_{2}{ }^{3}<S_{2}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{2}^{2}}{2}+\frac{v_{1}{ }^{2}<S_{1}{ }^{4}>}{2}\right)}, \tag{4.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta v_{2}=\frac{-\kappa \frac{\delta}{2}\left[\cos (2 \theta)\left(-v_{1}+\frac{v_{1} v_{2}{ }^{2}}{2}+\frac{v_{1}{ }^{3}<S_{1}{ }^{4}>}{6}\right)-\sin (2 \theta)\left(-v_{2}+\frac{v_{2} v_{1}{ }^{2}}{2}+\frac{v_{2}{ }^{3}<S_{2}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{1}{ }^{2}}{2}+\frac{v_{2}{ }^{2}<S_{2}{ }^{4}>}{2}\right)} . \tag{4.36}
\end{equation*}
$$

We are interested in understanding how the signal separation that we can achieve is affected by the fact that the input signals are not orthogonal. Therefore, we want to compute the signal separation when one of the signals wins the competition. For example, let's look at the case where signal 1 wins. When this is the case, the components of $\overrightarrow{v_{s s}}$ are as follows

$$
\begin{gather*}
v_{01}=\sqrt{\frac{6(\kappa-1)}{\left.<S_{1}^{4}\right)}},  \tag{4.37}\\
v_{02}=0,  \tag{4.38}\\
\delta v_{1} \frac{-\kappa \frac{\delta}{2}\left[-\sin (2 \theta)\left(-v_{1}+\frac{v_{1}{ }^{3}<S_{1}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{1}{ }^{2}<S_{1}^{4}>}{2}\right)}, \tag{4.39}
\end{gather*}
$$

and

$$
\begin{equation*}
\delta v_{2} \frac{-\kappa \frac{\delta}{2}\left[\cos (2 \theta)\left(-v_{1}+\frac{v_{1}{ }^{3}<S_{1}{ }^{4}>}{6}\right)\right]}{1+\kappa\left(-1+\frac{v_{1}{ }^{2}}{2}\right)} \tag{4.40}
\end{equation*}
$$

A useful way to quantify the performance of the fourth-order decorrelation feedback loop is to look at the crosstalk or how much of the losing signal is mixed in with the winning signal. The expression for the cross talk is calculated in Equation (4.23).

### 4.3.4 Discussion

The crosstalk for nonorthogonal system inputs is graphed in Figure 4.3. The least crosstalk occurs when $\theta$ is 45 or 135 degrees causing $\delta v_{2}$ to go to zero. The most crosstalk occurs when $\delta v_{2}$ is a maximum. This occurs when $\theta$ is 0 or 90 degrees.


Figure 4.3: Crosstalk as a function of mixing angle for non-orthogonal inputs.
The system achieves perfect signal separation at 45 and 135 degrees. The worst separation occurs at 0 and 90 degrees.

With a little bit of thought, the angle dependence of the crosstalk makes sense. When $\theta$ is 0 or 90 degrees, the effect of $\delta$ is to cause a slight rotation of the input matrix so that

$$
\begin{equation*}
\left.\left.\tilde{C}\right|_{0^{\circ}} \approx \tilde{C}\right|_{\delta / 2^{\circ}}, \tag{4.41}
\end{equation*}
$$

as shown in Figure 4.4a. As a result, the total power of each signal is the same. There are no biases in the system and the system achieves its minimum separation performance.

On the other hand, $\theta$ equal to 45 or 135 degrees causes a bias in the system. At these angles, $\delta$ causes the input vector in the signal basis to be closer to one signal axis than the other (see Figure 4.4b). Therefore, the total power of one signal is greater compared to the other signal. This difference in power causes a bias in the system, where one signal initially begins with more gain. This signal dominates the competition and wins.


Figure 4.4: Diagram of nonorthogonal inputs at 0 and 45 degrees.
(a) Diagram of nonorthogonal inputs for $\theta=0$ degrees. (b) Diagram of nonorthogonal inputs for $\theta=45$ degrees.

### 4.4 Frequency-Dependent Phase Delays in the ICA Feedback Loop

### 4.4.1 Problem Statement

ICA assumes that the optical and electronic phase delays are perfectly matched and frequency independent. In an effort to buy off-the-shelf, inexpensive electronic components, it is probable that frequency-dependent phase delays are present in the electronic portion of the feedback loop. The consequence of these phase delays is the input signals will not be properly correlated with their corresponding signals in the feedback loop resulting in less energy transfer than expected. This analysis looks at how this decrease in signal gain affects the separation performance of the fourth-order decorrelation feedback loop.


Figure 4.5: Conceptual set-up used to analyze frequency dependent phase delays in the ICA feedback loop.

### 4.4.2 Problem Set-Up

A simplified model is used to understand how frequency dependent phase delays affect the performance of the feedback loop. Figure 4.5 shows two beams, the loop beam and one signal beam, interacting inside the photorefractive crystal in the feedback loop. Each beam contains a single signal with the same two frequencies, $\omega_{1}$ and $\omega_{2}$, present. However, on the loop beam, a phase shift $\varphi$ is present at frequency $\omega_{2}$. To describe this, the signal beam $S(t)$ is given by

$$
\begin{equation*}
S(t)=\cos \left(\omega_{1} t\right)+\cos \left(\omega_{2} t\right), \tag{4.42}
\end{equation*}
$$

while the loop beam is given by

$$
\begin{equation*}
S(t)^{\prime}=\cos \left(\omega_{1} t\right)+\cos \left(\omega_{2} t+\varphi\right) \tag{4.43}
\end{equation*}
$$

It is possible to write $S(t)$ in terms of $S(t)^{\prime}$ as

$$
\begin{equation*}
S(t)=S(t)^{\prime}-\cos \left(\omega_{2} t+\varphi\right)+\cos \left(\omega_{2} t\right) . \tag{4.44}
\end{equation*}
$$

### 4.4.3 Derivation of Differential Equation

The derivation of the open-loop gain in this analysis is identical to the derivation performed in Section 2.3 except that $\mathrm{V}(\mathrm{t})$ is given by Equation (4.43). The gain
amplitude in this case is

$$
\begin{equation*}
A=-\operatorname{Re}\left\{\frac{i}{v}<e^{i v S^{\prime}(t)} S(t)>_{\tau}\right\} \tag{4.45}
\end{equation*}
$$

where $v$ is the slowly varying amplitude. Substituting Equation (4.44) into the above equation and strategically grouping terms, we can write

$$
\begin{equation*}
A=-\frac{1}{v} \operatorname{Re}[\partial \phi(v)]+\operatorname{Re}\left\{\frac{i}{v}<e^{i v S_{\prime}(t)}\left[\cos \left(\omega_{2} t+\varphi\right)-\cos \left(\omega_{2} t\right)\right]>_{\tau}\right\}, \tag{4.46}
\end{equation*}
$$

where $\phi(v)$ is the characteristic function of $S^{\prime}(t)$. Equation (4.46) can be broken down into two components such that $A=A_{0}+\delta A . A_{0}$ is the unperturbed gain amplitude similar to Equation (2.14). $\delta A$ is the modification to the gain amplitude due to the phase delay of one frequency. $A_{0}$ and $\delta A$ are given by

$$
\begin{equation*}
A_{0}=-\frac{1}{v} \operatorname{Re}[\partial \phi(v)], \tag{4.47}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta A=\operatorname{Re}\left\{\frac{i}{v}<e^{i v S \prime(t)}\left[\cos \left(\omega_{2} t+\varphi\right)-\cos \left(\omega_{2} t\right)\right]>_{\tau}\right\} . \tag{4.48}
\end{equation*}
$$

The $\left[\cos \left(\omega_{2} t+\varphi\right)-\cos \left(\omega_{2} t\right)\right]$ term represents the correlation between $S(t)$ and $S^{\prime}(t)$. This correlation will range in magnitude from 0 to 1 . Since $\delta A$ is the deviation term, we expect that Equation (4.48) will either be 0 when $\varphi=0$ or cancel Equation (4.47) exactly when $\varphi=\pi$. Thus, I can reasonably rewrite Equation (4.48) as

$$
\begin{equation*}
\delta A=\operatorname{Re}\left\{\frac{i}{v}<e^{i v S^{\prime}(t)} S^{\prime}(t) \sin ^{2}\left(\frac{\varphi}{2}\right)>_{\tau}\right\} . \tag{4.49}
\end{equation*}
$$

It is necessary to look at $A$ when $\varphi$ is 0 and $\pi$ to insure that the expected behavior is achieved. When $\varphi$ is 0 , there is no frequency dependent phase delay so that we expect no deviation from the expected solution. We can see that when $\varphi=0$, then $\delta A=0$ and $A=A_{0}$, as expected. In the case that $\varphi$ is $\pi, S(t)$ and $S^{\prime}(t)$ are uncorrelated and do not write a grating inside the photorefractive crystal. Therefore, their gain amplitude is zero. Looking at Equation (4.49) when $\varphi=\pi$, we see that $\sin ^{2}\left(\frac{\pi}{2}\right)=1$ and $\delta A=-A_{0}$. Thus, $A=0$ as expected.

The closed-loop differential equation is determined from the gain amplitude. The final signal amplitude $v_{f}$ is given by $v_{f}=v \kappa A$ where $\kappa$ is the small signal gain. The closed-loop differential equation is calculated as

$$
\begin{equation*}
\frac{\partial v}{\partial t}=\frac{v_{f}-v}{\tau} \tag{4.50}
\end{equation*}
$$

Algebraic manipulation and trigonometric substitutions lead to the following closed-loop equation

$$
\begin{equation*}
\tau \frac{\partial v}{\partial t}=-\kappa \cos ^{2}\left(\frac{\varphi}{2}\right) \operatorname{Re}[\partial \phi(v)]-v \tag{4.51}
\end{equation*}
$$

### 4.4.4 Steady State Analysis

The steady state equation is found by setting the closed-loop differential equation to zero and solving for $v_{s s}$, the steady state solution. The steady state equation is given by

$$
\begin{equation*}
0=-\kappa \cos ^{2}\left(\frac{\varphi}{2}\right) \operatorname{Re}\left[\partial \phi\left(v_{s s}\right)\right]-v_{s s} \tag{4.52}
\end{equation*}
$$

From the expansion of the characteristic function found at Equation (2.11), $\operatorname{Re}[\phi(v)]$ in the small signal regime is given by

$$
\begin{equation*}
\operatorname{Re}[\partial \phi(v)]=-v<S^{\prime 2}>+\frac{v^{3}}{6}<S^{\prime 4}> \tag{4.53}
\end{equation*}
$$

Substituting Equation (4.53) into Equation (4.52) and solving for $v_{s s}$ we find that

$$
\begin{equation*}
v_{s s}^{2}=\frac{6\left(\kappa \cos ^{2}\left(\frac{\varphi}{2}\right)<S^{\prime 2}>-1\right)}{\kappa \cos ^{2}\left(\frac{\varphi}{2}\right)<S^{\prime 4}>} . \tag{4.54}
\end{equation*}
$$

When $\varphi=0, v_{s} s$ reduces to the expected value of $v_{0}$ for the ideal system.

### 4.4.5 Discussion

The goal of this analysis was to determine how frequency-dependent phase delays affect the gain of the signals in the feedback loop. One way to determine this effect is
to compare the steady state value of the signal in the ideal case with the steady state value in this perturbed case. Mathematically, I can write the deviation $\operatorname{Dev}(\varphi)$ in $d B$ from the ideal case as

$$
\begin{equation*}
\operatorname{Dev}(\varphi)=10 \log \left(\frac{v_{0}^{2}-v_{s s}(\varphi)^{2}}{v_{0}^{2}}\right) \tag{4.55}
\end{equation*}
$$

If we assume a sub-Gaussian signal $\kappa=1.1$, and normalized signals, then we can plot $\operatorname{Dev}(\varphi)$ vs $\varphi$ as shown in Figure 4.6. Figure 4.6 is symmetric about 180 degrees. When $\varphi$ is near 0 or 360 degrees, the deviation from ideal is very small since $v_{s} s$ is very small. However, as the system approaches 180 degrees the deviation from ideal increases. The deviation is largest at 180 degrees because the signal in the loop is uncorrelated with the input signal; so, no grating is written. Therefore, the deviation is maximized.


Figure 4.6: $\operatorname{Dev}[\varphi]$ vs $\varphi$
This graph plots the deviation of the new steady state value from the ideal steady state value for a frequency-dependent phase delay of $\varphi$ in the fourth-order decorrelation feedback loop.

According to the graph, for the deviation to be less than 20 dB , any frequency dependent phase delays must be less than roughly 20 degrees. For many off-the-shelf components, the phase delays are less than 10 degrees. Thus, this does not appear to
be a major issue in the operation of the fourth-order decorrelation feedback loop. This is encouraging. However, this is a simplified version of the feedback loop.

This analysis of phase delay has assumed that there is one input into the feedback loop; however, there are actually two inputs. Next is to see how the dynamics are affected with two inputs. Since the net effect of the phase difference is to change the amplitude of the signal into the feedback loop, this analysis boils down to the input signals not having the same normalization. So, I expect the same results as those of the non-normalization analysis performed previously.

### 4.5 Summary

This chapter has looked at some of the major contributors to the degradation of system performance in the fourth-order decorrelation system, where the inputs are assumed to be orthogonal and normalized. Non-normalized input mixtures result in a bias that causes better separation of unmixed signals compared to maximally mixed signals. Slightly non-orthogonal input mixtures cause better separation for maximally mixed signals compared to unmixed signals. Frequency-dependent phase delays decrease signal competition, which decreases signal separation.

For signal separation of 20 dB or better, these deviations from ideal must be small. For example, assuming two signals with an harmonic PDF and a linear gain near 1 (i.e., $\kappa=1.1$ ), the deviation from normalized must be less than 24 parts in 1000 (i.e., $\varepsilon \leq 0.024$. There are similar requirements for $\delta$ and $\varphi$. As $\kappa$ increases, these requirements are lessened. However, $\kappa$ is limited because we do not want the system to spontaneously oscillate. We must carefully balance the system gain, with the normalization, orthogonality, and phase requirements.

In the ideal case, the signal separation should be constant with the mixing angle. The analyses provided here are useful diagnostics to determine reasons for degraded signal separation in the fourth-order decorrelation system.

## Chapter 5

## Optoelectronic Principal Component Analysis

The previous three chapters have focused on the theory behind the dynamics of the fourth-order decorrelation system, the actual implementation of the fourth-order decorrelation, and the modified system dynamics due to non-ideal system inputs. Chapter 5 shifts to a very different subject. Moving from a focus on the fourth-order decorrelation step in the ICA process, this chapter focuses on the second-order decorrelation step. The second-order decorrelation step, the orthogonalization step in ICA, can be performed in several ways. One popular method is principal component analysis (PCA).

The first optoelectronic PCA system was described in [31, 29]. This system used an all-optical feedback loop to extract the first principal component from a mixture of signals. The bandwidth of the feedback loop is given by the round trip time in the feedback loop. The advantage of the all-optical feedback loop is that it can be made small resulting is signal bandwidths greater than 1 GHz . One potential disadvantage is that this all-optical feedback loop cannot be easily modified to perform the fourth-order decorrelation step. If one wants to perform ICA on greater numbers of mixed signals, a modular system would be advantageous.

The optoelectronic feedback loop that performs the fourth-order decorrelation step can in principle also perform PCA without any physical modifications to the system. Instead, the transformation between fourth-order decorrelation and PCA occurs by arranging for the loop EOM to operate as a linear amplifier, while the gain of the
photorefractive crystal saturates. Which regime each of these devices operates within depends on the electronic driving power in the case of the EOM and the optical power ratio of the two input channel beams and the loop beam as described in Sections 3.2.1 and 3.2.2. Therefore, by adjusting the loop beam power and the gain of the preamplifier (or attenuator) one can switch between PCA and fourth-order decorrelation. Since the two physical systems are identical, one can imagine that it would be possible to manufacture large quantities of identical modules (e.g., carrier suppression modules and feedback loop modules) to build a complete ICA system for more than two channels.

This chapter presents a theoretical framework for optoelectronic principal component analysis (PCA) to show that indeed the optoelectronic fourth-order decorrelation system can be transformed to perform PCA. Section 5.1 will give a brief introduction to PCA and whitening. Next, a numerical analysis of the PCA feedback loop is performed. Several particular cases of interest are studied to understand the operation of the feedback loop. State space diagrams are used to aid in visualizing the operation of the feedback loop. Finally, assumptions and limitations of this analysis are discussed.

### 5.1 Overview of PCA and Whitening

PCA is frequently used to reduce the dimensionality of a multidimensional data set by finding an orthogonal basis set such that the data is projected onto axes that point along the directions of greatest variance. The contribution of an axis to the variance is determined by the associated eigenvalue. The larger the eigenvalue, the larger the variance of the data along a particular axis. The direction of greatest variance is called the first principal component. The direction of second greatest variance is the second principal component, and so on.

Prior to performing PCA, the data is centered. The data is centered by calculating the mean of the data and subtracting the mean from the data set. Once the data is centered, techniques such as Singular Value Decomposition or eigenvalue decompo-
sition are used to transform the data to the new basis set. Both techniques create an orthonormal basis set, but only eigenvalue decomposition will be discussed here.

In eigenvalue decomposition, a vector of independent signals or random variables, $\vec{S}$, is mixed by a matrix $\tilde{A}$. The received signals, $\vec{x}$, are given by

$$
\begin{equation*}
\vec{x}=\tilde{A} \vec{S} \tag{5.1}
\end{equation*}
$$

The covariance matrix of the received signals is given by

$$
\begin{equation*}
\operatorname{Cov}(\vec{x})=x x^{T} \tag{5.2}
\end{equation*}
$$

where ${ }^{T}$ represents the transpose. Principal components are defined as the normalized eigenvectors of $\operatorname{Cov}(\vec{x})$ or $E\left\{x x^{T}\right\}$. The eigenvectors are ordered by their eigenvalues so that the eigenvector associated with the largest eigenvalue is the first principal component.

Whitening occurs when mixed signals $\vec{x}$ are projected into the principal component basis set such that the new mixed signals $\vec{x}^{\prime}$ are now orthogonal and normalized. The whitening process is performed by

$$
\begin{equation*}
\vec{x}^{\prime}=\tilde{D}^{-1 / 2} \tilde{E}^{T} \vec{x} \tag{5.3}
\end{equation*}
$$

where $\tilde{E}$ is the column matrix of the ordered eigenvectors of the $\cos (\vec{x}), \tilde{D}$ is a diagonal matrix containing the associated eigenvalues, and $\tilde{D}^{-1 / 2}$ is the diagonal elements of $\tilde{D}$ to the $-1 / 2$ power.

The covariance matrix is analogous to the two-beam coupling density matrix $\rho$ defined in the photorefractive operator approach [24]. A brief description of relevant aspects of photorefractive operator theory is presented in Appendix D. The covariance matrix and $\rho$ differ by a multiplicative constant which affects the magnitude of the eigenvalues, but not the eigenvectors or their order.


Figure 5.1: Schematic of the optoelectronic principal component analysis system.
Schematically, this feedback loop is identical to the fourth-order decorrelation feedback loop. In PCA, the photorefractive crystal is used as a nonlinear amplifier, while all other components of the loop are linear.

### 5.2 Optoelectronic PCA Theory of Operation

Figure 5.1 shows the layout of the optoelectronic PCA system. The feedback loop contains an electro-optic phase modulator, a photorefractive crystal, a homodyne phase-to-amplitude modulation converter, a photodetector, and an amplifier. The inputs into the feedback loop are the mixed signals described by Equation (5.1). The feedback loop optical intensities and electronics are arranged so that the photorefractive crystal is used as a nonlinear amplifier that saturates before any other component of the feedback loop.

The saturation of the photorefractive crystal is described by this set of coupled differential equations

$$
\begin{align*}
\frac{d E_{s}^{*}}{d z} & =\gamma \Omega E_{p}^{*}  \tag{5.4}\\
\frac{d E_{p}}{d z} & =-\gamma \Omega^{*} E_{s} \tag{5.5}
\end{align*}
$$

and

$$
\begin{equation*}
\frac{d \Omega}{d t}=\frac{E_{p} E_{s}^{*}}{I_{p}+I_{s}} \tag{5.6}
\end{equation*}
$$

where $E_{s}$ is the amplitude of the signal beam (i.e., the loop beam), $E_{p}$ is the amplitude of one of the pump beams, $I_{s}$ and $I_{p}$ are the intensities of the signal and pump beams, respectively, $\Omega$ is the grating, and $*$ indicates a complex conjugate. These coupled differential equations are sufficiently complicated that it is difficult, and in general impossible, to find a closed-form analytical equation. Thus, numerical integration techniques were used to analyze the behavior of the PCA feedback loop.

Since the only nonlinear component in the feedback loop is the photorefractive gain, the PCA feedback loop can be simplified as shown in Figure 5.2. All of the linear components and the system gain and loss are combined into the parameter $\beta$, which is the linear gain.

### 5.2.1 Definition of Variables

The numerical integration of the feedback loop was performed using Mathematica code created by Anderson et al. [3]. The code uses the operator approach to photorefractive dynamics described briefly in Appendix D. The inputs into the numerical integration are the input field vectors, the loss, the interaction length of the photorefractive crystal, and the number of iterations around the loop. Each of these inputs will be discussed in turn, followed by a discussion of how the numerical integration was modeled.

The input field vector describes how much of a given temporal signal is on a given spatial port. In the loop described in Figure 5.2, there are three ports: one plus port (i.e., the loop beam) that receives gain from the pump beams and two minus ports (i.e., the two pump beams). There are two temporal signals denoted by $\omega_{1}$ and $\omega_{2}$. These signals can be either a single frequency or a group of frequencies associated with temporally independent signals. Each temporal signal is described by a different field


Figure 5.2: Schematic of simplified PCA feedback loop used in the numerical model.
The initial loop beam field amplitude, $E\left(\omega_{1}, \omega_{2}\right)$, receives photorefractive gain from the two pump beams. Next a multiplicative gain factor, $\beta$, modifies the overall gain of the loop. After one roundtrip, the electric field is $E^{\prime}\left(\omega_{1}, \omega_{2}\right)$.
vector, where each field vector has three components. In general, we can write these field vectors in the form

$$
\left(\begin{array}{c}
E_{+}  \tag{5.7}\\
E_{1-} \\
E_{2-}
\end{array}\right)_{\omega}
$$

where $E_{+}$is the electric field amplitude on the plus port, $E_{1-}$ and $E_{2-}$ correspond to the electric fields on the first and second plus ports, respectively, and $\omega$ denotes which temporal component the field vector is associated with.

For a two-channel system, there are two temporal components. The field vectors
for each of the temporal components is represented by the following variables

$$
\vec{E}_{i}=\left(\begin{array}{c}
E_{\omega i p}  \tag{5.8}\\
E_{\omega i m 1} \\
E_{\omega i m 2}
\end{array}\right)
$$

where $i$ is the $i^{\text {th }}$ temporal signal.
$\beta$ accounts for all the linear gain and loss in the feedback loop and can take on any positive real value. However, in order for the feedback loop to reach a steady state value, $\beta$ must be less than one. $\beta=0.7$ was a typical value used in these analyses.

The interaction length of the crystal, $L$, affects the photorefractive gain. The photorefractive gain is given by $e^{\Gamma L}$, where $\Gamma$ is the coupling constant in units of inverse length. The larger the value of $L$, the larger the photorefractive gain of the crystal. $\Gamma$ can be quite large, e.g., $20 \mathrm{~cm}^{-1}$.

The number of iterations, $N$, around the loop is chosen to be large enough so that the loop can reach a steady state value. To some extent, the number of iterations required to reach steady state is determined by $L$. The larger the value of $L$, the fewer iterations required for the loop to reach steady state. Typical values of L and the number of iterations were 5 and 100, respectively.

### 5.2.2 Numerical Integration Approach

Using the photorefractive two-beam coupling Mathematica package, I choose some initial field vectors at the input to the photorefractive crystal. After one traversal of the feedback loop, the initial field vectors have been modified. I assume that the signal intensity for each temporal component on the plus port after one trip around the loop is immediately re-incident on the photorefractive crystal again. The minus port amplitudes just before the photorefractive crystal are from an external source and held constant. Thus, the plus port values change over time, while the minus ports maintain their initial
values. The goal of the numerical integration is to determine under what circumstances, if any, does the feedback loop extract the principal component.

### 5.2.3 Cases of One Principal Component in Each Pump Beam

I start this numerical integration by assuming that only one principal component is present on each minus port. The initial field vectors are of the form

$$
\vec{E}_{1}=\left(\begin{array}{c}
E_{\omega 1 p}  \tag{5.9}\\
E_{\omega 1 m 1} \\
0
\end{array}\right) \quad \text { and } \quad \vec{E}_{2}=\left(\begin{array}{c}
E_{\omega 2 p} \\
0 \\
E_{\omega 2 m 2}
\end{array}\right)
$$

Since the feedback loop initially starts with noise, there is a tiny amount of each principal component on the plus port ( $E_{\omega 1 p}$ and $E_{\omega 2 p} \ll 1$ ). The minus port amplitudes are much greater than the loop amplitude. Thus, $E_{\omega 1 m 1}$ and $E_{\omega 2 m 2}$ are $\approx 1$. I have chosen $L=5$, $\beta=0.7$, and $100 \leq N \geq 1000$. Three different cases were investigated to understand the performance of the PCA feedback loop. Each case will be discussed in turn.

### 5.2.4 Case 1: Minus Port Intensities Equal

In this analysis, $E_{\omega 1 m 1}=E_{\omega 2 m 2}=1$. The initial amplitude of each principal component on the plus port ( $E_{\omega 1 p}$ and $E_{\omega 2 p}$ ) is varied. This situation did not create the winner-takes-all situation. The ratio of the intensities of each of the principal components in steady state was equal to the ratio of intensities on the plus port initially. Mathematically,

$$
\begin{equation*}
10 \log \left(\frac{I_{\omega_{1} p}^{s s}}{I_{\omega_{2} p}^{s s}}\right)=20 \log \left(\frac{E_{\omega_{1} p}^{i n i t a l}}{E_{\omega_{2} p}^{i n i t a l}}\right), \tag{5.10}
\end{equation*}
$$

where $s s$ denotes steady state value and initial denotes starting conditions.
Figure 5.3 is a sample plot from the numerical integration. The simulation parameters were $L=5, \beta=0.7, N=100$, and $E_{\omega_{1} p}^{\text {inital }} / E_{\omega_{2} p}^{\text {inital }}=0.5$. The graph shows that once the feedback loop reaches steady state, the ratio of the principal component intensities is 0.25 or $\left(E_{\omega_{1} p}^{i n i t a l} / E_{\omega_{2} p}^{i n i t a l}=0.5\right)^{2}$.


Figure 5.3: Case 1: Minus port intensities equal - intensity vs iterations.
This plot shows the intensity of each temporal component on the loop beam (i.e., plus port) as a function of the number of iterations around the PCA feedback loop for the case of a single principal component on each minus port. Each principal component has the same minus port intensities. The steady-state intensity ratio equals the starting intensity ratio of the principal components on the loop beam.

### 5.2.5 Case 2: Plus Port Intensities Equal (Winner-Takes-All)

In this analysis, $E_{\omega_{1} p}^{\text {inital }}=E_{\omega_{2} p}^{\text {inital }}=0.001$ for the plus port amplitudes. The initial intensity of each principal component on the minus port ( $E_{\omega 1 m 1}$ and $E_{\omega 2 m 2}$ ) is varied; however, their amplitudes are not allowed to be the same. This situation creates the winner-takes-all situation. Whichever principal component has the larger minus port intensity is the principal component present in the feedback loop in steady state. The intensity of the other principal component is zero.

Figure 5.4 shows a sample numerical integration for Case 2. The simulation parameters were $L=5, \beta=0.7, N=100$, and $E_{\omega 1 m 1=0.9}$ and $E_{\omega 2 m 2=1}$. The closer the values of $E_{\omega 1 m 1}$ and $E_{\omega 2 m 2}$ the more iterations required for the feedback loop to reach a steady state.


Figure 5.4: Case 2: Plus port intensities equal - intensity vs iterations.
This plot shows the intensity of each temporal component on the loop beam (i.e., plus port) as a function of the number of iterations around the PCA feedback loop for the case of a single principal component on each minus port. Each principal component has the same plus port intensity, but different minus port intensities. In steady state, only the principal with the larger minus port intensity remains in the loop. This is the winner-takes-all scenario where the PCA loop extracts the first principal component.

### 5.2.6 Case 3: Both Plus and Minus Port Intensities Unequal

This analysis is a mixture of Cases 1 and 2 and exhibits winner-takes-all behavior. Winner-takes-all occurs because the intensities on the minus ports are not equal. The principal component with the larger intensity on the minus port will be the winner regardless of which principal component has the larger starting intensity on the plus port. The starting intensity of each principal component on the minus port affects how much the "losing" principal component grows before it diminishes to zero.

Figure 5.5 is a sample plot from the numerical integration. The simulation parameters were $L=5, \beta=0.7, N=100, E_{\omega 1 m 1}=0.9, E_{\omega 2 m 2}=1, E_{\omega_{1} p}^{\text {inital }}=0.001$ and $E_{\omega_{2} p}^{\text {inital }}=0.005$.


Figure 5.5: Case 3: Initial intensities random - intensity vs iterations.
This plot shows the intensity of each temporal component on the loop beam (i.e., plus port) as a function of the number of iterations around the PCA feedback loop for the case of a single principal component on each minus port. The principal components have different plus and minus port intensities. In steady state, only the principal with the larger minus port intensity remains in the loop, regardless of the plus port intensity. Again, this is the winner-takes-all scenario where the PCA loop extracts the first principal component.

### 5.2.7 Summary and Implications of Cases 1, 2, and 3

Whether the PCA feedback loop extracts a principal component depends on the whether the principal component inputs into the feedback loop have the same or different powers. If the principal components have the same power, how well the system extracts a single principal component is dependent on the noise in the feedback loop at the beginning of the competition. If the principal components do not have the same power, then the loop will extract the principal component with the most power.

Recall that principal components are essentially eigenvectors. Eigenvectors have eigenvalues, which describe the relative importance, and, in our case, the relative strength of a particular eigenvector. Therefore, as long as the eigenvalues are not degenerate, the PCA feedback loop should extract the first principal component (i.e., the principal component with the largest eigenvalue).

### 5.3 Case of Random Input Mixtures

The signal inputs into the feedback loop are not as described with each principal component on a separate minus port. In actuality, the input signals are a "random," but real, mixture of several independent signals. This random mixture is described by the mixing matrix $\tilde{A}$ in Equation (5.1). Thus, the field vectors are actually as shown in Equation (5.11)

$$
\vec{E}_{1}=\left(\begin{array}{c}
E_{\omega 1 p}  \tag{5.11}\\
a_{11} \\
a_{21}
\end{array}\right) \quad \text { and } \quad \vec{E}_{2}=\left(\begin{array}{c}
E_{\omega 2 p} \\
a_{12} \\
a_{22}
\end{array}\right)
$$

where $a_{i j}$ are the elements of $\tilde{A} . \tilde{A}$ has a set of principal components associated with it that we assume have the form of the inputs described in Equation (5.9), where $E_{\omega 1 m 1}=$ $\lambda_{1}$ and $E_{\omega 2 m 2}=\lambda_{2}$.

To determine if the PCA feedback loop is extracting the principal components of the mixed signal, we must first compute the principal component from the covariance matrix. The covariance matrix is created from the mixed signals $\vec{x}$ by taking a time average and assuming that the original signals are independent. (As stated previously, the principal components can be computed equally well by finding the minus port density matrix). The eigenvectors of the covariance matrix are traditionally called the principal components. By this definition, the feedback loop does not extract the first principal component, but rather the feedback loop output points in the direction of the first whitened component associated with the largest principal component eigenvalue. As a result, we refer to the whitened components as the principal components since they are the orthogonalized representation of the received input mixtures. (There is precedent for this designation, see the discussion on bottom of page 6 in [39]).

Here is a sample calculation using the following mixing matrix in Equation (5.1):

$$
\tilde{A}=\left[\begin{array}{ll}
2 & 3  \tag{5.12}\\
2 & 1
\end{array}\right]
$$

The associated covariance matrix (or $\rho_{\text {minus }}$ ) of the measured variable $\vec{x}$ is

$$
\rho_{\text {minus }} \propto \operatorname{Cov}(\vec{x}) \propto\left[\begin{array}{cc}
13 & 7  \tag{5.13}\\
7 & 5
\end{array}\right] .
$$

The eigenvectors of the covariance matrix form the columns of $\vec{E}$. The vectors are placed in order of largest eigenvalue

$$
\vec{E}=\left[\begin{array}{cc}
0.196 & 0.981  \tag{5.14}\\
0.981 & -0.196
\end{array}\right]
$$

The associated eigenvalues are 17.06 and 0.94 , respectively. The whitened components $\vec{x}^{\prime}$ can be found from Equation (5.3) using

$$
\tilde{D}^{-1 / 2} \tilde{E}^{T} \tilde{A}=\left[\begin{array}{cc}
0.662 & 0.750  \tag{5.15}\\
0.750 & -0.662
\end{array}\right]
$$

The first row contains the coefficients for the first-whitened signal or principal component. The ratio of these coefficients matches the ratio of the steady state electric field amplitudes of the frequencies in the PCA feedback loop. This correspondence indicates that the output of the PCA feedback loop extracts a mixture of $S_{1}$ and $S_{2}$ that points in the direction of the first principal component, but is not necessarily normalized.

To summarize the performance of the PCA feedback loop with random input mixtures, we must consider two situations: degenerate and non-degenerate eigenvalues. If the eigenvalues are degenerate, then this situation is identical to Case 1 presented in 5.2.4. The loop extracts a linear combination of the principal components equal to the ratio of the principal components initially in the feedback loop. If the eigenvalues are not degenerate, then the situation is described in Section 5.2.6, and the feedback loop extracts the first principal component.

### 5.4 State Space Diagrams

State space diagrams are a useful tool for understanding the dynamics and steady state behavior of physical systems. Typically, one plots momentum as a function of position. However, for the optoelectronic feedback loop, this does not make physical sense. Instead we plot the rate in change of the electric field of each frequency, $\Delta E_{\omega 1 p}$ and $\Delta E_{\omega 2 p}$, as a function of the current electric field, $E_{\omega 1 p}$ and $E_{\omega 2 p}$. For a given set of initial plus port values, the "trajectory" of the feedback loop is recorded at each iteration. A sample single trajectory is shown in Figure 5.6. The state space diagram is built up by plotting system trajectories for many different initial values of $E_{\omega 1 p}$ and $E_{\omega 2 p}$, while the minus port intensities are held constant.

A sample state space diagram is shown Figure 5.7. The horizontal axis represents $E_{\omega 1 p}$, while the vertical axis represents $E_{\omega 2 p}$. At each point a vector is drawn. The length of the vector and the direction it points represent the magnitude of and direction of $\Delta E_{\omega 1 p}$ and $\Delta E_{\omega 2 p}$ at that particular value of $E_{\omega 1 p}$ and $E_{\omega 2 p} . \Delta E_{\omega 1 p}$ is the horizontal component of the vector, and $\Delta E_{\omega 2 p}$ is the vertical component of the vector. This state space diagram was created for the case where the minus port intensities are equal, as described in Section 5.2.4. You will notice that the state space diagram is axially symmetric. Thus, by looking at the first quadrant, we gain enough information to analyze this situation. To gain a better understanding of what is occurring, let's zoom into the first quadrant.

Figure 5.8 shows the first quadrant of the state space diagram for the case where the minus port intensities are equal. In the inner region of the diagram, the arrows point radially outward. The arrows in the outer region point radially inward. In total, the system trajectories all point to a circular ring of constant field magnitude. If the system starts or ends up at a position along this ring, there is nothing to confine the system to a particular point on the ring. As a result, any location on the ring is a viable


Figure 5.6: Single numerical integration of the PCA feedback loop.
This figure shows a sample trajectory of the PCA feedback loop. The horizontal and vertical axes are $E_{\omega 1 p}$ and $E_{\omega 2 p}$, respectively. At each point, $\Delta E_{\omega 1 p}$ and $\Delta E_{\omega 2 p}$ are plotted as a vector. The length and direction of the vector correspond to the magnitude and direction of $\Delta E_{\omega 1 p}$ and $\Delta E_{\omega 2 p}$, respectively. The trajectory starts in the bottom right corner of the plot. The arrow points to the starting point of the next iteration of the loop. One hundred iterations have been plotted. The end of the trajectory occurs in the top left of the plot where additional iterations produce no change in the values of $E_{\omega 1 p}$ and $E_{\omega 2 p}$.


Figure 5.7: PCA state space diagram for minus port intensities equal.
This figure shows a state space diagram for the case where both minus port intensities are equal, as presented in 5.2.4. The horizontal and vertical positions represent $E_{\omega 1 p}$ and $E_{\omega 2 p}$, respectively. The arrow at each position represents the trajectory of the system at the specific values of $E_{\omega 1 p}$ and $E_{\omega 2 p}$. Specifically, the length and direction of the vector describe the magnitude and direction of and $\Delta E_{\omega 1 p}$, respectively. Note: the initial step, which is radially outward, has been removed for the sake of clarity.


Figure 5.8: PCA state space diagram for minus port intensities equal - first quadrant.
This figure shows the first quadrant of the state space diagram for the case where one principal component is on each of the minus ports and the minus port intensities are equal. The horizontal and vertical positions represent $E_{\omega 1 p}$ and $E_{\omega 2 p}$, respectively. This is a metastable case. Note: the initial step, which is radially outward, has been removed for the sake of clarity.
steady state position. This is generally described as a metastable solution.
The other situation of interest for the PCA feedback loop is the case where the minus port intensities are not equal, as described in Section 5.2.6. Figure 5.9 shows the state space diagram in this case where the starting intensity of $E_{\omega 1 p}$ is less than $E_{\omega 2 p}$. You will notice that near the origin, the arrows tend to point radially outward, while near the outer edges, the arrows point radially inward. This is similar to the metastable case. However, as the trajectories approach the interior of the diagram, they turn towards a point on the vertical, or $E_{\omega 2 p}$, axis. Once the system reaches this point on the vertical axis, any deviations from this point produce a trajectory that returns the system to this point (see Figure 5.10 for a magnified view). This point is a stable equilibrium, where only one principal component is present on the loop beam.

There appears to be another equilibrium point on the horizontal axis. However, from the magnified perspective shown in Figure 5.11, we see this is not a stable equilibrium, but rather a saddle point. Trajectories along the horizontal axis go towards this equilibrium point. However, any deviation away from horizontal pushes the system towards the vertical axis. Thus, the equilibrium position on the horizontal axis is an unstable saddle point.

If $E_{\omega 2 p}$ is less than $E_{\omega 1 p}$, the figure would be rotated by 90 degrees so that the stable point was on the horizontal axis. This is the winner-takes-all situation where regardless of the initial conditions, the system ends up selecting one principal component. In particular, it selects the first principal component because it has the largest eigenvalue and therefore the largest amplitude. Because there is a single unique steady state solution, this is considered a monostable solution.

### 5.5 Assumptions and Limitations of PCA Numerical Analysis

Several assumptions have been made in the numerical analysis that limit the conclusions we can draw from this analysis of the PCA feedback loop. One assumption is


Figure 5.9: PCA state space diagram for minus port intensities unequal - first quadrant.
This figure shows the first quadrant of the state space diagram for the case where one principal component is on each of the minus ports, and the minus port intensities are different. The horizontal and vertical positions represent $E_{\omega 1 p}$ and $E_{\omega 2 p}$, respectively. In particular, minus port 1 has a slightly smaller intensity than channel 2 . This results in winner-takes-all behavior.


Figure 5.10: PCA state space diagram for minus port intensities equal.
This figure zooms in on the equilibrium position on the vertical axis of Figure 5.9. The horizontal and vertical positions represent $E_{\omega 1 p}$ and $E_{\omega 2 p}$, respectively. Because all of the arrows point toward this equilibrium, it is a stable equilibrium in that any disturbance will push the system back to the equilibrium position.


Figure 5.11: PCA State Space Diagram for minus port intensities unequal.
This figure zooms in on the equilibrium position on the horizontal axis of Figure 5.9. The horizontal and vertical positions represent $E_{\omega 1 p}$ and $E_{\omega 2 p}$, respectively. Although many arrows initially point towards a single point on the axis, small deviations in the positive or negative vertical directions push the system farther away from this point. This is an unstable saddle point.
that the grating inside of the photorefractive crystal arises and changes instantly. In actuality, the feedback loop has gone through many iterations before a grating starts to arise in the photorefractive crystal. The grating is created from the interference pattern of the beams crossing in the crystal. As the grating grows over many iterations, it changes the loop beam. As a result, the grating must adapt to the changes caused by the changes in the loop beam. Given the straightforward nature of the coupling differential equations describing two-beam coupling, ignoring the slow evolution of the photorefractive grating probably does not change the global dynamics of the PCA system.

Another limitation of this analysis is it does not describe the stability of the steady state points. Although we can infer stability from the state space diagram, a more formal way of proving the stability of the steady state points is desired. Creating a Lyapunov function is one way to study the stability of the steady state solutions without a closed form dynamical equation.

## Chapter 6

## Future Work

This thesis has presented an optoelectronic implementation of two components of the ICA process at 200 MHz . This is a giant leap towards the application of blind source separation techniques at RF frequencies. However, there are many issues that must be addressed before the optoelectronic ICA can be readily applied to the RF domain. A few areas of future work will be discussed here. First hardware development, then theoretical development will be discussed.

The most immediate next step is to integrate the two-channel optoelectronic PCA and fourth-order decorrelation systems into one optoelectronic ICA system. To do this, a replica of the current system, which was used for both the PCA and fourth-order decorrelation experiments, must be built: one to perform PCA and one to perform fourth-order decorrelation.

A means of recovering the second principal component will be required as an input into the ICA system. Although optical techniques could be used for this purpose, the most straightforward way to recover the second principal component would be to subtract the first principal component from one of the input mixtures using an AGC circuit such that the output is uncorrelated with the first principal component. (Note: this technique can also be used to recover the second independent component after the fourth-order decorrelation system). Once the second principal component is recovered, the principal components can be normalized with yet another AGC loop.

One of the major issues in an integrated ICA system is the available electronic and optical power, especially if the system is to be portable. Using electronic subtraction between the PCA and fourth-order decorrelation systems means that two separate lasers, rather than one massive laser, can be used for each portion of the system.

In addition to integrating the ICA system, another area of interest is creating an optoelectronic ICA system that can separate more than two signals, a capability required for any real-world application. Although optoelectronic ICA is probably not reasonable for applications requiring the separation of hundreds of signals, there are potential applications where a four-to-ten channel optoelectronic ICA system would be appropriate. Let's look at the requirements for a four channel system.

To separate four signals, three replicas of the PCA and fourth-order decorrelation systems are required. There are many components that PCA and fourth-order decorrelation systems share in common. For example, a four channel system would require roughly 10 carrier suppression circuits and 6 replicas of the feedback loop. Even if clever methods could reduce these numbers by $50 \%$, the number of repetitive parts would advocate for a modularized system. The carrier suppression circuit described in Appendix C is one step in that direction.

Additional modules not required for the two channel system would have to be developed. For example, once a principal component or independent component has been recovered, it must be subtracted from the received mixtures before application of the next ICA system. These component removal modules can be either optical or electronic. From a power and size stand point, electronic techniques are probably the most straightforward method for both the component removal and normalization steps.

Another area of interest is miniaturization of the ICA system. This would not only allow the application of ICA to higher bandwidths, but also it would add to system stability, portability, and extendibility. Modularizing components of the ICA system using software packages such as Zemax and Solidworks can substantially shrink the size
of the system. For example, the modularized carrier suppression circuit is about $25 \%$ of its tabletop counterpart. Other ways to substantially miniaturize the system would include the use of fiber optics or even direct-write photopolymers.

Thus far, I have given three possible future directions of experimental research. There are also areas in the theory of optoelectronic ICA that can be explored.

This work presented an analysis of PCA based on a numerical integration of a simplified PCA system. Although this was adequate to obtain a cursory understanding of optoelectronic PCA, a more rigorous development of PCA, including a closed-form dynamical equation, is desired. Although, in general, it is not possible to obtain a closed-form evolution equation for the coupled-mode photorefractive equations, there are reasons to suspect one might exist for this case. The full density matrix, which governs the two-beam-coupling dynamics, has several zeros. This indicates that the field vectors do not span the space and that one of the eigenvectors has zero length. Therefore, it may be possible to reduce the dimensionality of the analysis to aid in the development of a closed-form evolution equation.

Another area of interest has to do with the ability of the fourth-order decorrelation system to separate signals. We know that if at least one of the original signals is subGaussian, the system will extract the signal from the feedback loop. In a two-channel system, it is possible to recover the other signal regardless of its statistics. However, if there are more than two channels and only one signal is sub-Gaussian, it may be impossible to recover the remaining signals.

There are two ways to approach this problem, neither of which has been investigated. One is to see if it is possible to develop an optoelectronic ICA system that can separate super-Gaussian signals. The other is to see if there is a linear sub-Gaussian modulation scheme that can be applied to the received mixtures such that the received mixtures can be separated into their component signals regardless of their individual PDFs or modulation schemes. The idea would be to modulate the received mixtures
so the PDF of each composite signal is now sub-Gaussian; recover all of the original signals, demodulate the applied modulation, and recover the original signal regardless of its PDF. If such a modulation scheme were possible, it would greatly expand the applicability of our optoelectronic ICA system.

One of the most exciting aspects of the ICA system from a telecommunications standpoint is that ICA can separate signals within a beamwidth for a given size and number of elements in an antenna array. This feature limits mass and power requirements by reducing the number of antenna elements required to separate signals. What is the distance between sources that ICA can tolerate and still achieve reasonable signal separation? Ideally, this distance would be close to zero. Unfortunately, signal-to-noise issues reduce this substantially. At a minimum, ICA requires at least the same number of antenna elements as sources to be separated. However, more antenna elements increase the signal-to-noise ratio. What is the tradeoff between number of antenna elements and the smallest distance between resolvable sources?

All of the future directions presented here lead very naturally from the work presented in this thesis. I leave it to future generations of graduate students to pursue these and other interesting questions in the field of holographic signal processing. As for me, I am going whitewater rafting.

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## Appendix A

## 200 MHz Electro-Optic Modulator

## A. 1 Overview

The nonlinearity of the electro-optic modulator (EOM) is crucial to the fourthorder decorrelation portion of the optoelectronic ICA system's ability to perform ICA. Although the EOMs on the input channels and in the PCA feedback loop operate in the EOM's linear regime, the EOM in the fourth-order decorrelator feedback loop operates in the nonlinear regime. The nonlinear regime allows us to generate the higher-order harmonics that are correlated with the input mixtures by the photorefractive crystal in the feedback loop.

The maximum voltage that the ICA loop EOM must provide is the voltage necessary for the feedback loop to reach steady state. The feedback loop reaches steady state when the open-loop gain equals the loss. From the open-loop gain curves shown in Figure 2.5, the open-loop gain associated with a binary signal requires the largest voltage for the system to reach steady state, assuming the loss is greater than 0 . To get an estimate of the maximum voltage required to drive the EOM, the binary signal's open-loop gain goes to zero at 1.56 times the zero of the harmonic signal. The open-loop gain of the harmonic signal goes to zero at a modulation depth of 3.83 radians. Thus the gain of the binary signal goes to zero when $m=4.43$ radians, or $1.41 \pi$. Thus, to insure that we can apply enough voltage for the feedback loop EOM to reach steady state, we estimate that the modulation depth required is at least $1.41 V_{\pi}$.

Because EOMs that operate at this modulation depth for green lasers are not available off the shelf, it was necessary to design a custom EOM.

## A. 2 Design Considerations

## A.2.1 Crystal Design

There are various layouts for an EOM. In essence, they are all a crystal inside of a capacitor. A transverse modulator, as shown in Figure A.1, was the basis of the EOM used in this experiment. The voltage required for a $\pi$ phase shift is dependent on the crystal geometry, electro-optic properties, and the wavelength of light to be modulated. In general, the half-wave voltage is given by

$$
\begin{equation*}
V_{\pi}=\frac{d \lambda_{0}}{L n^{3} r}, \tag{A.1}
\end{equation*}
$$

where $d$ is the distance between the capacitive plates, $\lambda_{0}$ is the optical wavelength in vacuum, L is the length over which the field if applied, n is the index of refraction of the dielectric at the wavelength of interest, and $r$ is the Pockels, or linear electro-optic, coefficient. To minimize the power requirements of the EOM, a low Vpi is desired. $V_{\pi}$ is minimized when the plate area is large and the plate separation is small. In addition to the aspect ratio issues that would make the crystal difficult to manufacture and handle, there are other considerations that affect the dimensions of the crystal. These issues will be discussed in turn.

The plate separation cannot be arbitrarily small. Because it is necessary to get the laser beam through the crystal, a smaller plate separation requires a smaller beam size. The diffraction-limited spot size, $D$, is given by

$$
\begin{equation*}
D=\frac{1.22 f}{a} \tag{A.2}
\end{equation*}
$$

where f is the focal length of the lens, and $a$ is the beam waist of the laser. Ideally we would prefer to use off-the-shelf elements, which should not prevent us from achieving


Figure A.1: Schematic of a transverse EOM.
small spot sizes. Unfortunately, smaller spot sizes result in shorter crystal lengths as I will discuss in a moment.

In addition to the crystal thickness, the length and width of the plate also must be considered. Because only the portions of the electric field that the laser passes through affect the modulation of the beam, it is not necessary for the width to be much larger than the plate separation. However, a slightly larger width eases the optical alignment.

The length of the crystal is restricted as well. We want the area of the beam that interacts with the dielectric to be as constant as possible. Thus, the beam should be collimated as it passes through the crystal. The confocal parameter, $b$, is the distance over which a laser is considered collimated. The confocal parameter is given by

$$
\begin{equation*}
b=2 z_{0}=\frac{2 \pi w_{0}{ }^{2} n}{\lambda} \tag{A.3}
\end{equation*}
$$

where $z_{0}$ is the rayleigh range, $w_{0}$ is the beam waist, $n$ is the index of refraction of the crystal, and $\lambda$ is the optical wavelength. Equation (A.3) shows that the tighter the focus, the smaller the confocal parameter. To increase the interaction length of the crystal, we must increase the beam focus, which increases the width of the crystal. As previously stated, to minimize Vpi, we desire the smallest possible plate separation. Thus, there needs to be a balance between the width of the crystal and the length of the crystal.

The crystal dimensions we have chosen are $30 \times 0.3 \times 1 \mathrm{~mm}$. In order for the laser to be collimated over the 30 mm length of the crystal, the beam waist at the focus should be roughly $24 \mu \mathrm{~m}$, which incidentally minimizes the mode volume of the beam in the dielectric. Assuming the focus is placed at the center of the crystal, the beam waist at the crystal face is $48 \mu m$, which easily fits within the crystal's $0.3 \times 1 \mathrm{~mm}$ optical cross-section.

## A.2.2 Mount Design and Electrical Considerations

The mount serves not only as a means of physically holding the crystal in the correct location, but also is responsible for making electrical connections to the crystal.

In order for the laser to pass through the crystal without hitting the sides, the mount must allow for optical access to the crystal along the laser beam path, height adjustment, and vertical tilt adjustments. Visual access to view the top edge of the crystal was also incorporated to aid in alignment.

Electronically, the mount must provide access to the ground and signal sides of the dielectric. These paths must be electrically short, i.e., much less than a wavelength, and the crystal must make good electrical connections. Figures A. 5 though A. 8 provide drawings of the EOM mount.

To provide the ground plane connection, the gold-coated LiNbO3 crystal is pressed against a thin piece of copper that is simultaneously pressed against the ground plane on the back of the circuit board and aluminum shelf. The SMA connector is screwed into the aluminum shelf. Thus, the ground of the SMA is securely connected to the ground of the circuitboard and the LiNbO3 crystal. Metal screws go through the front ground connection of the circuit board, the back ground plate, the copper, and into the shelf to insure the ground connections are secure.

A cantilever design is used to hold the crystal in place. An "L-shaped" block is made out of an electrically conducting material and is used as the connection between the signal path of the circuit board to the top of the crystal. Simultaneously, the block presses the back of the crystal against the ground plane. This allows for a very good ground connection for the back of the crystal. Nylon screws are used to adjust the pressure of the cantilever against the crystal. Nylon screws are used because they penetrate through the ground plane and into the aluminum shelf without making an electrical connection. Since the cantilever is making the signal connection to the crystal,
the cantilever must be insulated from the ground plane.
Although LiNbO3 is very strong, it will crack under differential pressure. Unless the L-shaped block is specially designed, it will apply more pressure to one edge of the crystal than to the other. This pressure difference would potentially break the crystal. To ease the tolerances on the machining of the cantilever, indium is used as a conductive gasket material between the cantilever and the signal connection to the crystal.

The circuit board is designed to be versatile so that it can accommodate a variety of matching circuits if necessary. A drawing of the circuit board is shown in Figure A.2. The signal path is a $50 \Omega$ track that connects to the SMA signal pin on one end and intersects a pad that runs the length of the top of the circuit board. On either side of the signal track lies a ground connection. Together, the $50 \Omega$ track and side ground planes are designed to allow the addition of series and parallel elements of a matching-circuit design. The circuit board material is FR4 and contains a ground plane on the back of the circuit board.


Figure A.2: EOM circuitboard drawing.

## A. 3 Finalized EOM Design

## A.3.1 LiNbO3 Crystal Dimensions

The crystal dimensions are $d=300 \mu m, h=1 \mathrm{~mm}$, and $L=30 \mathrm{~mm}$. The optical faces $d \times h$ have an antireflection coating. The crystal's electrical faces $L \times h$ are plated with a $100-150 \mathrm{~nm}$ layer of gold over a titanium adhesion layer. Given these crystal dimensions, $r=30 \mathrm{pm} / V, n=2.31$, and $\lambda=532 n m, V_{\pi}$ is calculated to be $14.4 V_{0 P}$ or $10.2 V_{R M S}$.

## A.3.2 Radio Frequency Voltage Delivery

Before assessing the best way of applying voltage across the crystal, we analyze the electrical properties of the crystal in its mount. We are most interested in the power delivered to the crystal as a function of frequency. Using a network analyzer, it is possible to easily measure the S11 parameter of the EOM. The S11 parameter gives the amount of incident power on the EOM that is reflected. Figure A. 3 shows the magnitude and phase of the S11 parameter. The EOM reflects most of the incident power at all frequencies, except a comb of frequencies spaced by roughly 12 MHz . These resonances correspond to acoustic modes in the crystal itself.

The acoustic resonance in the crystal can be calculated in a straightforward manner. The speed of sound in a solid is given by

$$
\begin{equation*}
v_{\text {solid }}=\sqrt{\frac{\epsilon}{\rho}}, \tag{A.4}
\end{equation*}
$$

where $\epsilon$ is Young's modulus of the material, and $\rho$ is the density of the material. For LiNbO3, $\epsilon=59.5 G P a$, and $\rho=4640 \mathrm{~kg} / \mathrm{m}^{2}$. These values yield $v_{L i N b O 3}=3580 \mathrm{~m} / \mathrm{s}$. From this speed, we calculate the resonant frequency spacing for a crystal of thickness $d=300 \mu m$ to be

$$
\begin{equation*}
f=\frac{v_{L i N b O 3}}{d}=12 \mathrm{MHz} . \tag{A.5}
\end{equation*}
$$

## S11 Parameter For Crystal In Mount



Figure A.3: Plot showing the S11 parameter magnitude and phase for the LiNbO3 crystal in its mount with no matching circuit.

These acoustic resonances appear to have no effect on the electrical operation of the modulator.

This value matches the spacing of the resonance in Figure A.3. In addition to looking at the S11 parameter, it is helpful to have a model for the EOM. We use a series RLC circuit to model the EOM. Using the network analyzer, it is possible to directly measure the resistance of the EOM. At 200 MHz , the resistance, R , is $300 \mathrm{~m} \Omega$. Moreover, by measuring the reactance at two different frequencies it is possible to determine the inductance, L , and the capacitance, C , of the EOM. In this case, $f_{1}=190 \mathrm{MHz}$, $X_{1}=-18.97 \Omega, f_{2}=210 \mathrm{MHz}$, and $X_{2}=-15.96 \Omega$, yield $C=33.6 \mathrm{pF}$ and $L=5 \mathrm{nH}$.

In designing electronics to drive an EOM, the standard approach is to design a matching circuit that delivers the maximum power to the EOM over the bandwidth of interest. It is not possible to deliver power to a purely reactive load. An EOM is largely capacitive with a tiny real resistance. Because R is tiny, designing a matching circuit to maximize the power delivered with the necessary bandwidth requirements is difficult. However, we want to maximize the voltage across the EOM, not the power delivered to
the EOM. Thus, we use a different approach.
Our EOM does not contain a matching circuit and is broadband. As a result, the power delivered to the EOM is reflected back towards the amplifier. To protect the amplifier, an electronic isolator was used. Although all of the power dumped onto the EOM is reflected, what is ultimately important is the voltage applied to the EOM. Because the impedance of the modulator is not equal to the 50 Ohm source, only a portion of the voltage applied is delivered to the modulator. It is possible to calculate the amount of voltage delivered to the EOM.


Figure A.4: Schematic and equivalent circuit for EOM driving electronics.
(a) A schematic of the EOM driving electronics. (b) The equivalent circuit for the EOM driving electronics.

Figure A. 4 shows a physical layout of the EOM driving circuitry and its effective electrical schematic. The reflection coefficient, $\Gamma$, for the EOM terminated by a $50 \Omega$ load is $\Gamma=0.67-i 0.46$. This means that $33 \%$ of the incident voltage is reflected and that $67 \%$ of the incident voltage is applied to the EOM. The remainder goes into heating the crystal. Thus we expect that we need to apply roughly $1.5 \times\left(1.41 V_{\pi}^{\text {calculated }}\right)$ to reach the zero for the binary-signal open-loop-gain zero point.

## A. 4 Layout Overview

This section gives a pictorial overview of the EOM mount. A Compac ${ }^{T M}$ box is used as the basis of the mount. Holes in this box provide optical access to the LiNbO3 crystal and viewing access to the top of the crystal. Screw holes in the back of the box match the spacing of slots in a bracket, which allows vertical height adjustment. The modulator itself is mounted to the front lid of the box.


Figure A.5: The complete EOM mount design.


Figure A.6: An expanded view of the EOM mount.


Figure A.7: The actual EOM portion of the mount.


Figure A.8: An expanded view of the EOM.

## A. 5 Measurement of $V_{\pi}$

A characteristic parameter of the EOM is its half-wave voltage. The half-wave voltage was determined from the open-loop gain curve of the fourth-order decorrelation system with a sinusoidal signal. Figure A. 9 shows the set-up used to measure $V_{\pi}$.


Figure A.9: Experimental set-up to measure $V_{\pi}$.
A modified open-loop gain measurement is used to measure the half-wave voltage of the EOM. By changing the power incident on the EOM and measuring the photodetector signal, it is possible to determine the half-wave voltage of the EOM.

A 200 MHz signal of varying power enters the EOM Driver. The power of the detected 200 MHz signal is measured using a photodetector. Figure A. 10 show the open-loop gain data and the mathematical fit based on Equation (2.14). For a single sinusoidal input, we expect

$$
\begin{equation*}
A(\vec{v})=-\frac{\kappa}{v} \operatorname{Re}\{\partial \phi(\vec{v})\}=A_{0} \frac{J_{1}(v)}{v} \tag{A.6}
\end{equation*}
$$

where $v=\frac{m V_{\pi}}{\pi}$, and $m$ is the modulation depth. From the graph, we see that the zero occurs at $18.8 V_{\text {rms }}$. For a sinusoidal signal, the open-loop gain goes to zero at $m=1.22 \pi$. Thus $V_{\pi}=15.4 V_{r m s}$, which is 1.5 times greater than the $10.2 V_{r m s}$ calculated in section A.3.1. This is exactly the factor expected because of the impedance mismatch between the EOM and the driver.


Figure A.10: $V_{\pi}$ calibration data.
This data is used to calculate $V_{\pi}$ for the EOM. The zero of the open-loop gain curve occurs at the zero of $J_{1}$ for a sinusoidal signal, which occurs at $1.22 V_{\pi}$. This data leads to a measured $V_{\pi}$ of 15.4 V .

## A. 6 Partslist

| Components of Electro-optic Modulator | Number Required |
| :--- | :---: |
| Mg-doped LiNbO3 Crystal | 1 |
| Shelf | 1 |
| Circuitboard | 1 |
| Metal Plate | 1 |
| Indium | 1 |
| AlN | 1 |
| SMA Jack (Amphenol RF Products 901-9891-RFX) | 1 |
| Compac Box 1"x1.5"x0.75" RFT Series R51085 | 1 |
| Bracket | 1 |
| $\# 2-56,1 / 8 "$ Nylon screws (Metal Plate to Shelf) | 2 |
| $\# 2-56,1 / 8 "$ Metal screws (Front Lid to Shelf) | 3 |
| $\# 2-56,1 / 8 "$ Metal screws (SMA Lid to Shelf) | 2 |
| $\# 2-56,1 / 8 "$ Metal screws (Circuitboard to Shelf) | 2 |
| $\# 2-56,1 / 8 "$ Metal screws (Lid to Box) | 2 |
| $\# 2-56,1 / 4 "$ Metal screws (Bracket to Box) | 2 |

Table A.1: Partslist for Electro-Optic Modulator.

## A. 7 CircuitBoardLayout



Figure A.11: EOM Circuitboard Layout
This Circuitboard is made out of FR4 and contains a ground plain on the back side. The circuit is conductive material is copper coated with tin.

## A. 8 Mechanical Drawings

The mechanical drawings for the EOM mount is found on the pages to follow.





$-1 \times 0.3 \mathrm{~mm}$ Optical faces AR coated for 532 nm
$-1 \times 30 \mathrm{~mm}$ faces have $30-40 \mathrm{~nm}$ Tita nium
ad hesion layer and $100-200 \mathrm{~nm}$ Gold


Material: LithiumNiobate
Material: (Mg Doped)
Units: Millimeters

|  |  |  |  |
| :--- | :--- | :--- | :--- |


















## Appendix B

Mixing Circuitry

## B. 1 Overview

The mixing circuit is used to generate the electronic mixtures that are input into the fourth-order decorrelation feedback loop. The mixing circuit takes in four DC voltages and two radio frequency (RF) signals and outputs two mixed signals as shown in Figure 3.3.

## B. 2 Conceptual Framework

## B.2.1 Overview

This section presents the conceptual design that the actual circuit was based on. The input DC voltages are used as coefficients of the mixing matrix. Automatic gain control (AGC) circuits are used to maintain these voltages. Although the fourthorder decorrelation system specifically requires sine and cosine values, the mixing circuit allows flexibility by allowing the user to input voltages individually or to tie voltages together to generate the sine-cosine values. A box that allows one to adjust the matrix coefficients between some minimum and maximum value or to tie each of two matrix elements together is shown in Figure B.1.

Since it is important that the outputs of the mixing circuit are orthogonal and normalized for the fourth-order decorrelation system, switches that separately control


Figure B.1: Circuit for mixing matrix coefficients.
This circuit is used to generate four voltages from a single DC supply. Suggested values for V and R are $V=2$ Volts and $R=1 k \Omega$. In the mixing circuit schematic $V_{A}=\operatorname{NormCos}(\mathrm{x}), V_{B}=\operatorname{NormSin}(\mathrm{x}), V_{C}=\operatorname{OrthoSin}(\mathrm{x})$, and $V_{D}=\operatorname{OrthoCos}(\mathrm{x}) . V_{A}$ and $V_{D}$ can be tied together and $V_{B}$ and $V_{C}$ can be tied together.
an orthogonalization circuit and a normalization AGC circuit are provided.

## B.2.2 Schematics




## B. 3 Circuit Schematics

## B.3.1 Specifications

The DC inputs each accept $+/-5 \mathrm{~V}$. Although the notation in the circuit schematic suggests that the DC inputs correspond to sin and cos, in actuality the user determines the input voltages, which are arbitrary. The maximum allowable input RF signal is 0 dBm . The suggested input power is -15 dBm . The RF bandwidth is roughly 1 MHz to 500 MHz . The output power is roughly 17 dBm .

## B.3.2 Schematics






## B. 4 Tuning the Mixing Circuit

The $\mathrm{x}-\mathrm{y}$ display on the oscilloscope was instrumental in tuning the mixing circuit at 45 degrees. Two synthesizers are used to generate signals at two different frequencies near 200 MHz . The outputs of the mixing circuit go to two inputs on the spectrum analyzer that are set to x -y display. Make sure the DC level has been set to the zero line.

Turn on each synthesizer separately. For the synthesizer that is on, remove one input to the oscilloscope. This makes the x-y display produce a line. Adjust the DC knob that controls this input amplitude to the desired input $+/$ - output level. I chose $+/-1$ V. Do not adjust this knob once it has been set. Perform this same operation for the other channel on the oscilloscope for the same synthesizer. To make sure that the phase between the two channels is correct, look at the x-y display for both oscilloscope channels connected. For a 45 degree mixture, the display should be a line at 45 degrees. If the display shows an ellipse, then there is a phase delay between the two channel and the circuit may need to be adjusted. To eliminate this, phase delays within the circuit must be removed.

Now repeat these same operations for the other synthesizer.
Once all the DC voltages have been set, connect both outputs of the mixing circuit to the oscilloscope. The output should be a square rotated by 45 degrees.

## B. 5 Future Improvements

The phase control and the normalization controls on the box were unstable and were disabled. The phase control was intended to allow adjustment of the orthogonalization loop set point when the phase switch was on, meaning that the orthogonalization AGC loop was on.

The normalization control was intense to allow adjustment of the normalization
set point when the normalization switch was, on meaning that the normalization AGC loop was on.

Another suggestion would be to design a separate box to test the fourth-order decorrelation system versus the PCA system that met each system's specific needs.

## Appendix C

## Carrier Suppression Module

## C. 1 Overview

Although the advantages of optoelectronic ICA are evident to telecommunications applications, the large size of a table-top system still makes the application of this technology to real-world systems prohibitive. Optoelectronic ICA and PCA are identical from an optical component standpoint. In particular, the carrier suppression portions of the loops and the feedback loops themselves are identical from a component standpoint. Therefore, a modular approach to miniaturization is appropriate. To this end, the carrier suppression portion of the ICA and PCA systems have been miniaturized. This appendix details the design of the carrier suppression module.

## C. 2 Carrier Suppression Module Design and Layout

For a modular design, it is important to decide on a collimated beam size and beam height that will be uniform for every module. In this case, the input beam and exit beam are 1 mm collimated and the beam height is 1 mm above the baseplate. Moreover, the entrance and exit beams are assumed to be horizontally polarized. The entrance and exit paths are along the same optical line so that the carrier suppression module is a black box with a co-linear input and output.

The footprint of the carrier suppression module, $13.5 \times 8.5 \mathrm{~cm}$, was designed to be compact, without lots of specialized optics. Most of the optical components are
available off-the-shelf. The mounts have enough degrees of freedom to allow for excellent alignment of the optic, without allowing too many degrees of freedom.

Two-beam coupling is required to perform carrier suppression. In particular, two beams of roughly equal intensity are required to cross in the photorefractive crystal. Both of these beams must be polarized along the crystal axis, in our case horizontally polarized, before entering the photorefractive crystal.

Typically, the two beams are split off before the EOM. One of the beams passes through the EOM and is phase modulated in the EOMs linear regime. The other beam is the laser carrier. This technique was not used in the carrier suppression module.

The EOM used in the carrier suppression module is an earlier version compared to the design described in Appendix A. This earlier version suffered from significant thermal expansion because of resistive heating of the electrical contacts. As a result, the beam passing through the EOM was thermally unstable compared to the unmodulated beam. To alleviate this problem, the separate beams are not split before the EOM. Instead, a half-waveplate with the fast axis at 22.5 degree is used to rotate the incoming polarization to 45 degree. The EOM is designed to modulate only the horizontally polarized portion of the beam. This leaves the vertical polarization unmodulated. The modulated and unmodulated beams are separated using a polarizing beamsplitter after the EOM as shown in Figure C.1. The modulated and unmodulated beams are recombined in the photorefractive crystal at an angle of roughly 17 degree. The carrier suppressed beam is recollimated before exiting the module along the same line as the incoming beam. For detailed mechanical drawings, please contact Dr. Dana Anderson at JILA/University of Colorado.

The measured carrier suppression was $20 d B$ or better for a $350 \mu W$ entrance beam. This is adequate performance for the feedback loops to perform PCA or fourthorder correlation.


Figure C.1: Schematic of carrier suppression module.


Figure C.2: Picture of actual carrier suppression module.

## C. 3 Alignment of Carrier Suppression Module

Courtesy of Patrick Alken

(1) Mount all mirrors on mounts, lenses in v-grooves, waveplates, PBS, photorefractive, and EOM. Use a very thin layer of wax for these mounts by heating up the wax and mount and then using a brush to apply wax to mount.
(2) Create a 1 mm collimated laser beam at a height of 12.66 cm above the table ( $=4$ inch post height+baseplate height+optical beam height above baseplate).
(3) Mount the baseplate on 4 inch posts and clamp it to the posts (I used screws and washers as clamps). Move the baseplate so that the incoming beam will hit the center of the first set of lenses. Clamp the posts to the table.
(4) Install the following mounts onto the baseplate:
(a) First lens set in its v-groove.
(b) First two mirrors before EOM.
(c) Waveplate after second mirror.
(d) PBS and waveplate right after EOM.

Do not install the EOM yet.
(5) We will first align the lenses.
(a) Place the second lens (closest to the mirror) about 15 mm away from the mirror.
(b) Get the laser beam roughly level and through the center of the PBS after the EOM.
(c) Put an observation screen 8.7 cm from the center of the PBS (this is roughly the position of the PR crystal).
(d) Now put the first lens about 9.5 mm from the first lens and move it around slightly to minimize the size of the dot on the observation screen (we want the beam size in the crystal to be as small as possible). This should also guarantee that the beam sizes at the EOM location are as small as possible.
(e) Also look for back reflections from the optical components to make sure everything is along the incoming beam.
(6) Next we will put in the EOM.
(a) Now that the lenses are aligned, make a perfectly level beam which goes through the center of the PBS using the first two mirrors.
(b) Adjust the EOM mount so that the center of the crystal is 1 cm above the bottom of the mount.
(c) Put the EOM on the baseplate and it should be fairly straightforward to get a beam through it. You can adjust the mirrors to get the beam through the EOM since the crystal will probably not be perfectly horizontal.
(d) If you can get a level beam at the right height after the EOM. Adjust the mirrors before the EOM to get the best output beam you can. You should have a single bright dot afterwards. If you see more than one dot then the beam is probably clipping the crystal and you need more adjustments on the mirrors.
(e) If you get a really big downward deflection you can try changing the tilt of the crystal. I found a hammer and file worked well for this. Loosen both screws on the EOM slightly, put the file under the second screw and tap it with a hammer to bring the back end of the crystal up. If you want to move the back end down, put the file on top of the screw and tap with the hammer. Never bring the file near the actual crystal in case you slip and
break it.
(f) Ideally you should have a level output beam at the right height ( 12.66 cm with respect to the table) and a back reflection from the EOM along the incoming beam. There are mirrors after the EOM designed to adjust for beam misalignments due to crystal deflections.
(7) Once the EOM is aligned, put in the 3 mirrors inside the interferometer and try to get the beams in the same plane with an intersection at the location of the photorefractive crystal. I tried making the arm with one mirror as close to level and at the right height as possible. This may not be exact if the EOM isn't perfect but it should be pretty close. Then the arm with two mirrors can be put into the same plane as the other beam.
(8) Put in the photorefractive crystal and get the two beams to couple in the right direction (the 2 mirror arm gives up energy to the 1 mirror arm). With perfect coupling, you should see the 2 mirror arm beam disappear completely and then come back when blocking the other beam. Also when positioning the photorefractive, put in the next mirror after the PR to make sure the beam hits it. Don't actually screw in the mirror since you'll want to study the coupling output, but just put it on top of the hole so you can be sure the beam will hit it. Otherwise you'll have to reposition the PR and make the intersection point different.
(9) The 2 mirror beam is our output.
(10) Next is the lens set after the crystal. Use wax to mount them in their v-groove and position it so that the first lens is about 10 mm from the crystal. Adjust the second lens until the output far away is as small a beam as possible. Tweezers seem more useful than fingers for moving the lenses around by small amounts.
(11) The final two mirrors and PBS are straightforward to align.

## C. 4 Partslist

| Components of Carrier Suppression Module | Number Required |
| :--- | :---: |
| V-Groove (1.8 mm) | 1 |
| V-Groove (1.9mm) | 1 |
| Ball Bearing Mirror Mounts | 9 |
| Mount for single Half Wave Plate | 1 |
| Mount for Beamsplitter and Half-Wave Plate | 1 |
| Mount for Single Beam Splitter | 1 |
| Mount for the Crystal | 1 |
| Mount for the EOM | 1 |
| Clamps | 4 |
| Base Plate | 1 |
| Adapters (6mm) | 3 |
| Adapters (8mm) | 1 |
| Lens (Rolyn 22.1030) | 1 |
| Lens (Edmund 32987) | 1 |
| Lens (Linos 31-4419) | 1 |
| Lens (Thorlabs AC080-016-A1) | 1 |
| Mirrors (beveled) | 9 |
| Half-Wave plates | 2 |
| EOM crystals | 1 |
| Beamsplitters | 2 |
| Gold bar Photorefractive Crystals | 1 |
| Screws for V-Grooves (4-40 x 1/2in) | 2 |
| Small screws for v-grooves (2-56 x 1/8in) | 2 |
| Washers for V-Grooves | 2 |
| Top screws for mirror mounts (4-40 x 1/2in) (same as for v-grooves) | 9 |
| Bottom screws for mirror mounts (4-40 x 3/8in) | 9 |
| Washers for mirror mount tops | 9 |
| Washers for mirror mount bottoms (same as for v-grooves) | 9 |
|  |  |

Table C.1: Partslist for one carrier suppression module.

## Appendix D

## Photorefractive Operator Theory

The photorefractive operator approach is a theoretical construct that allows one to determine the dynamics of photorefractive two-beam coupling without the direct use of coupled mode equations. The operator approach is presented in [24]; a brief summary is presented here.

The operator approach begins by defining fieldvectors, which contain both spatial and temporal information. Each temporal component is represented by a different fieldvector. Each fieldvector is a column vector, where each row represents a different optical beam or "spatial port" involved in two-beam coupling. The first row contains the electric field amplitude of the plus port. The plus port is the spatial beam which receives gain. In couple mode theory, this is the signal beam. The subsequent rows contain the electric field amplitudes of the minus ports. Minus ports give up their energy to the plus port and are traditionally called the pump beams. A sample field vector might appear as follows

$$
\vec{E}_{\omega}=\left(\begin{array}{c}
E_{+}  \tag{D.1}\\
E_{1-} \\
E_{2-} \\
E_{3-}
\end{array}\right) .
$$

The sample fieldvector represents a situation where four beams are interacting inside of the photorefractive crystal. There is one signal beam and three pump ports.

Fieldvectors are used to create the density matrix, $\rho$, which describes the temporal correlation between each port with itself and the other ports. $\rho$ is defined as

$$
\begin{equation*}
\rho=\frac{1}{I} \sum_{\omega} \vec{E}_{\omega} \otimes \vec{E}_{\omega}^{\dagger}, \tag{D.2}
\end{equation*}
$$

where dagger is defined as hermitian conjugate, $\otimes$ is the outer product, and $I$ is defined as

$$
\begin{equation*}
I=\sum_{\omega} \vec{E}_{\omega} \cdot \vec{E}_{\omega}^{\dagger} . \tag{D.3}
\end{equation*}
$$

Two beam coupling is represented by a rotation of the fieldvectors in an $N$ dimensional space, where $N$ is the number of spatial ports. The rotation or transfer matrix, $T$, is and $N \times N$ matrix. For actual two-beam coupling. The transfer matrix is straightforwards to write down

$$
T(\beta)=\left(\begin{array}{cc}
\cos (\beta) & \sin (\beta)  \tag{D.4}\\
-\sin (\beta) & \cos (\beta)
\end{array}\right)
$$

where $\beta$ is the coupling angle, which depends on $\rho$. Each temporal fieldvector undergoes the same rotation. Thus, each fieldvector after the two-beam coupling interaction $E_{\omega}^{\prime}$, is related to the initial fieldvector by

$$
\begin{equation*}
E_{\omega}^{\prime}=T(\beta) E_{\omega} . \tag{D.5}
\end{equation*}
$$

The transfer matrix evolves with the interaction length, z , through the crystal. The evolution of the transfer matrix can be written in terms of the density matrix

$$
\begin{equation*}
\frac{d T}{d z}=\frac{1}{4} e^{i \gamma \sigma_{3}}\left[\sigma_{3}, \rho\right] T \tag{D.6}
\end{equation*}
$$

$\gamma$ is the coupling constant, which is 0 for purely real coupling and $\pi / 2$ for purely imaginary coupling. $\sigma_{3}$ is a diagonal matrix whose diagonal elements correspond with the rows of the fieldvectors. The diagonal element is a 1 , for a position corresponding to a plus port. The element is a -1 is the position corresponds to a minus port. [ $\left[\sigma_{3}, \rho\right]$ is the traditional commutator relation

$$
\begin{equation*}
[A, B]=A B-B A . \tag{D.7}
\end{equation*}
$$

The expression

$$
\begin{equation*}
e^{i \gamma \sigma_{3}}=\cos (\gamma)+i \sigma_{3} \sin (\gamma) \tag{D.8}
\end{equation*}
$$

seems a little daunting. However for purely real coupling, this expression reduces to $e^{i \gamma \sigma_{3}}=1$. With this simplification, Equation (D.6) reduces to

$$
\begin{equation*}
\frac{d T}{d z}=\frac{1}{4}\left[\sigma_{3}, \rho\right] T \tag{D.9}
\end{equation*}
$$

This equation is easy enough to integrate numerically even if $d T / d z$ cannot be directly integrated. It is interesting to note that the dynamics of two-beam coupling are determined solely by the density matrix. Equation (D.9) is the basis of the photorefractive numerical integration package used in the PCA feedback loop analysis discussed in Chapter 5.

