Bounding the energy-constrained quantum and private capacities of phase-insensitive Gaussian channels

Kunal Sharma
Quantum Science & Technologies (QST) Group, LSU

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(Kunal **S**harma, Mark M. **W**ilde, Sushovit **A**dhikari, and Masahiro **T**akeoka)

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   - Amplifier channel
   - Additive-noise channel

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Main Results

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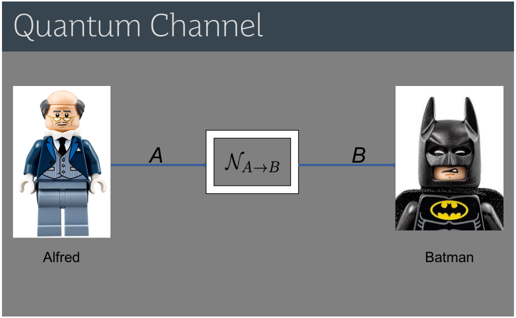
   - Thermal channel
   - Amplifier channel
   - Additive-noise channel
2. Improved achievable rates of private communication over bosonic thermal channels.
3. Optimization problem for the energy-constrained, Gaussian channel discrimination.

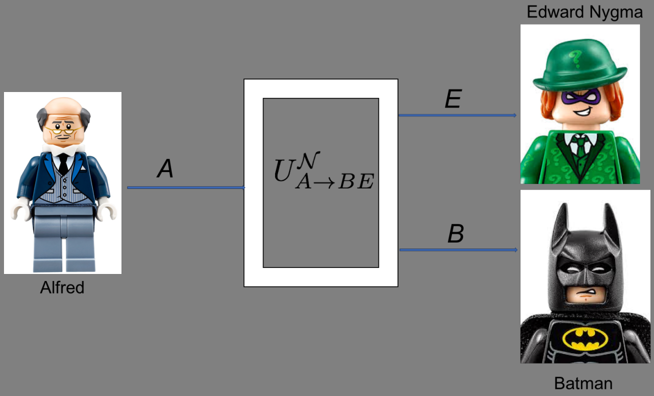
Background: Quantum theory of information



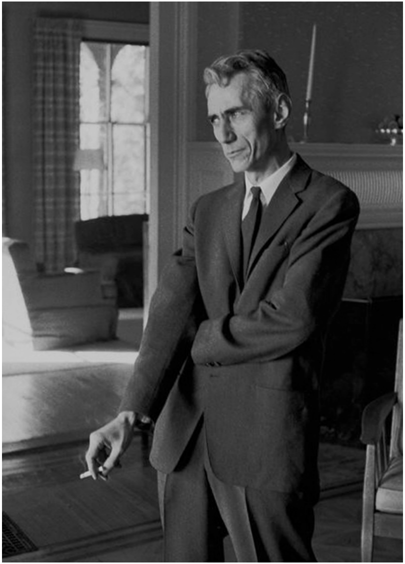
$\mathcal{N}_{A\to B}$ is a **linear, completely positive, trace preserving map**.

Background: Quantum Channel

Extension of a quantum channel



Background: Capacity of a channel



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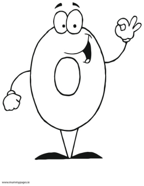

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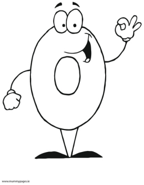
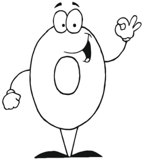


$$\text{Rate} = \frac{\log M}{n}$$

Background: Capacity of a channel

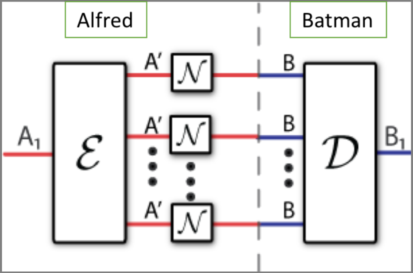


$$\text{Rate} = \frac{\log M}{n}$$

The capacity of a channel $\mathcal{N}$ is the optimal rate at which that data can be transmitted with high fidelity given a large number of uses of $\mathcal{N}$.

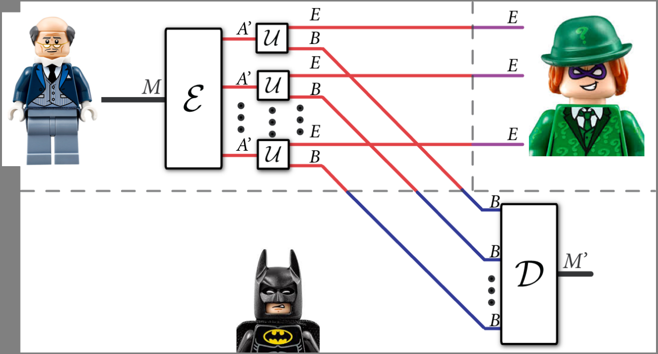
Background: Quantum Capacity of a channel

Quantum Communication



Background - Private Capacity of a channel

Private Communication



Background - Capacities

$$C(\mathcal{N})$$

Background - Capacities

Lower Bound

$$C(\mathcal{N})$$

Upper Bound

Background - Gaussian Quantum Information

m-mode quantum Bosonic system

For $i \in [m]$, let $\hat{x}_i, \hat{p}_i$ denote the position- and momentum-quadrature operators.

Commutation relation

$$\hat{r} = (\hat{x}_1, \ldots, \hat{x}_m, \hat{p}_1, \ldots, \hat{p}_m)^T : [\hat{r}, \hat{r}^T] = i\Omega, \;\; \Omega \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_m$$

Displacement operator

$$D(\xi) \equiv \text{exp}(i\xi^T \Omega \hat{r}),$$

where $\xi \in \mathbb{R}^{2m}$.

Background - Gaussian Quantum Information

Covariance Matrix V_ρ, Mean Vector s_ρ

$$s_\rho = \langle \hat{r} \rangle_\rho = \mathrm{Tr}[\hat{r}\rho],$$
$$V_\rho \equiv \langle \{\hat{r} - s_\rho, \hat{r}^T - s_\rho^T\} \rangle_\rho = \mathrm{Tr}[\{\hat{r} - s_\rho, \hat{r}^T - s_\rho^T\}\rho].$$

Wigner characteristic function and Quantum State

$$\chi_\rho(\xi) \equiv \mathrm{Tr}\{D(\xi)\rho\},\ \rho = \frac{1}{(2\pi)^m} \int d^{2m}\xi\, \chi_\rho(\xi)\, D^\dagger(\xi).$$

Background - Gaussian Quantum Information

Gaussian State

$$\chi_\rho(\xi) = \exp\left(-\frac{1}{4}\left[\Omega\xi\right]^T V^\rho \Omega\xi + \left[\Omega\mu^\rho\right]^T \xi\right).$$

Gaussian Channel

$$\mu^\rho \longmapsto X\mu^\rho + d,$$
$$V^\rho \longmapsto XV^\rho X^T + Y.$$

$$Y + i\Omega - iX\Omega X^T \geq 0.$$

Background - Gaussian Quantum Information

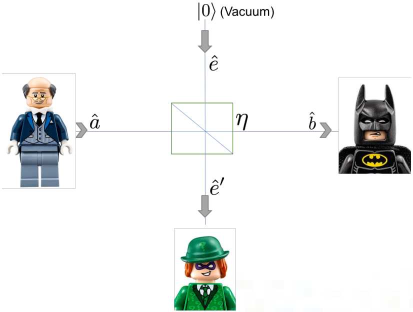
Phase-insensitive Gaussian channels:

$X = \sqrt{\tau} I, Y = \nu I, d = 0.$

- $\tau < 1 \rightarrow$ Thermal bosonic channel
- $\tau > 1 \rightarrow$ Quantum amplifier channel
- $\tau = 1 \rightarrow$ Additive-noise Gaussian channel

where $\nu =$ variance of an additive noise. Moreover, $\nu \geq 0$ and $\nu^2 \geq (1-\tau)^2$.

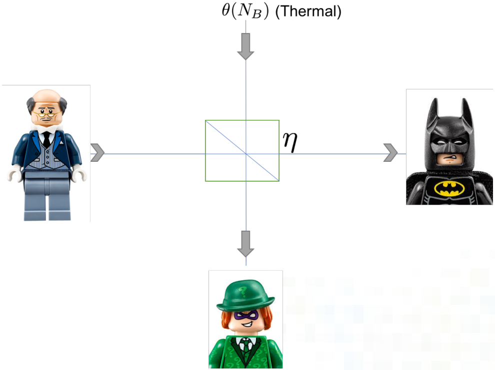
Background - Pure-loss bosonic channel



Input-output relation in Heisenberg picture

$$\hat{b} = \sqrt{\eta}\hat{a} - \sqrt{1-\eta}\hat{e},$$
$$\hat{e}' = \sqrt{1-\eta}\hat{a} + \sqrt{\eta}\hat{e}.$$

Background - Thermal channel



Thermal state

$$\theta(\bar{n}) \equiv \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |n\rangle\langle n| \, .$$

Background - Energy-constrained capacities

Energy-constrained quantum capacity [WQ16]

$$Q(\mathcal{N}, H, W) = \lim_{n\to\infty} \frac{1}{n} I_c(\mathcal{N}^{\otimes n}, \overline{H}_n, W),$$

where the energy-constrained coherent information of the channel is defined as

$$I_c(\mathcal{N}, H, W) \equiv \sup_{\rho:\mathrm{Tr}\{\rho H\}\leq W} S(\mathcal{N}(\rho)) - S(\hat{\mathcal{N}}(\rho)),$$

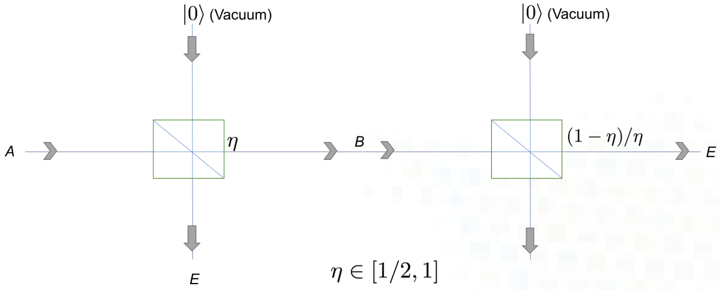
and $\hat{\mathcal{N}}$ denotes a complementary channel of $\mathcal{N}$.

$$\overline{H}_n = \frac{1}{n}(H \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes H).$$

Quantum Capacity of Degradable Channels

Degradable channels

Regularized information quantities are additive!



$\eta \in [1/2, 1]$

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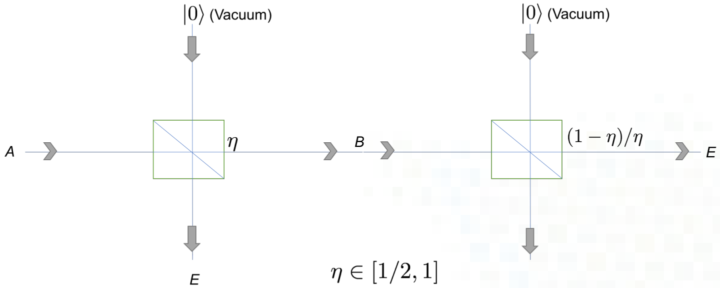
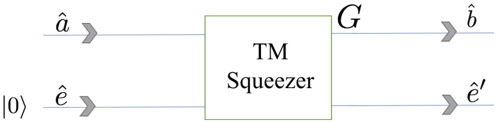


$$Q(\mathcal{L}_{\eta,0}, N_S) = \max[g(\eta N_S) - g((1-\eta)N_S), 0], [WHG12, WQ16]$$

$$g(x) = (x+1)\log_2(x+1) - x\log_2 x.$$

Background - Quantum-limited amplifier channel



Input-output relation in Heisenberg picture

$$\hat{b} = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}^\dagger,$$
$$\hat{e}' = \sqrt{G-1}\hat{a}^\dagger + \sqrt{G}\hat{e},$$

where $G \geq 1$.

Quantum Capacity

Non-degradable channels

It is hard to compute!

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Superadditivity of coherent information

$$nI_c(\mathcal{N}) \; < \; I_c(\mathcal{N}^{\otimes n})$$

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Example: Depolarizing channel.

Quantum Capacity

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Example: Depolarizing channel.

Superactivation of quantum capacity

Channels with zero quantum capacities: $\mathcal{N}_1$ and $\mathcal{N}_2$.
Use these channels together $\rightarrow$ non-zero quantum capacity.
Example: 50% Erasure channel and the entanglement binding channel.

Upper bounds on the energy-constrained quantum capacity

Three different upper bounds:

- Data-processing bound (Q_{U_1})
- ε-degradable bound (Q_{U_2})
- ε-close-degradable bound (Q_{U_3})
- Another data-processing bound (Q_{U_4})

Upper bounds on the energy-constrained quantum capacity

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Data-processing bound can be at most be 1.45 bits larger than a known lower bound.

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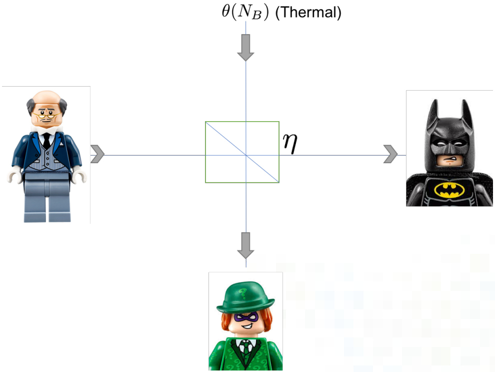
Closeness of these upper bounds with a known lower bound for different parameter regimes.

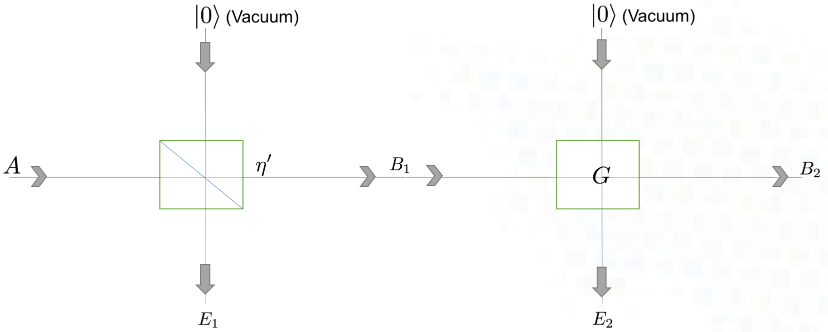
Data-processing bound Q_{U_1}



Data-processing bound Q_{U_1}

$$\mathcal{L}_{\eta,N_B} = \mathcal{A}_G \circ \mathcal{L}_{\eta',0},$$

where $G = (1-\eta)N_B + 1$, and $\eta' = \eta/G$ $[WPG06, SS07, CGH06, GPNBLSC12, KS13]$.



Data-processing bound Q_{U_1}

Theorem

An upper bound on the quantum capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number N_B, and input mean photon number constraint N_S is given by

$$Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta' N_S) - g[(1-\eta')N_S],$$

with $\eta' = \eta/((1-\eta)N_B + 1)$.

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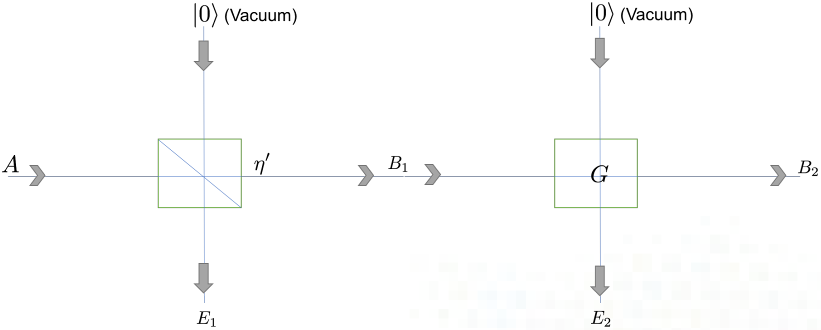
Theorem

$$Q_L(\mathcal{L}_{\eta,N_B}, N_S) \leq Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) \leq Q_L(\mathcal{L}_{\eta,N_B}, N_S) + 1/\ln 2\ .$$

Data-processing bound



Data-processing bound

Proof sketch

$\Delta(\mathcal{L}_{\eta,N_B}, N_S)$ = Q_{U_1} - Q_L. Take limit $N_S \to \infty$.

$$\omega_{B_2E_1E_2} = (\mathrm{id}_{E_1} \otimes \mathcal{V}^G_{B_1\to B_2E_2}) \circ \mathcal{U}^{\eta'}_{A\to B_1E_1}(\theta(N_S))$$
$$Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) = S(B_2E_2)_\omega - S(E_1)_\omega$$
$$Q_L(\mathcal{L}_{\eta,N_B}, N_S) = S(B_2)_\omega - S(E_1E_2)_\omega$$
$$\Delta(\mathcal{L}_{\eta,N_B}, N_S) = S(E_2|B_2)_\omega + S(E_2|E_1)_\omega.$$

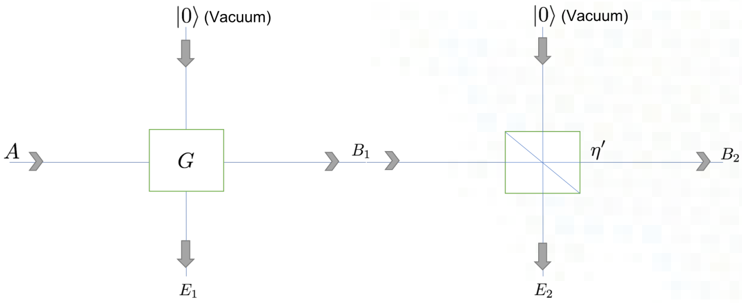
Invariance of conditional entropy under local unitaries, displacement covariance of the channels, and concavity of conditional entropy.

Another data-processing bound Q_{U_4}

Let $\eta > (1-\eta)N_B \to$ not entanglement breaking.

$$\mathcal{L}_{\eta,N_B} = \mathcal{L}_{\eta',0} \circ \mathcal{A}_G,$$

where $G = \eta/\eta'$, and $\eta' = \eta - (1-\eta)N_B$,
$[RMG18, SWAT18, NAJ18]$.



Data-processing bound Q_{U_4}

Theorem

An upper bound on the energy-constrained quantum capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B \geq 0$, such that $\eta > (1-\eta)N_B$, and input mean photon number constraint $N_S \geq 0$ is given by

$$Q_{U_4}(\mathcal{L}_{\eta,N_B}, N_S) \\ = g(\eta N_S + (1-\eta)N_B) - g[(1/\eta' - 1)(\eta N_S + (1-\eta)N_B)],$$

with $\eta' = \eta - (1-\eta)N_B,\ [SWAT18, NAJ18]$.

Comparison of upper bounds

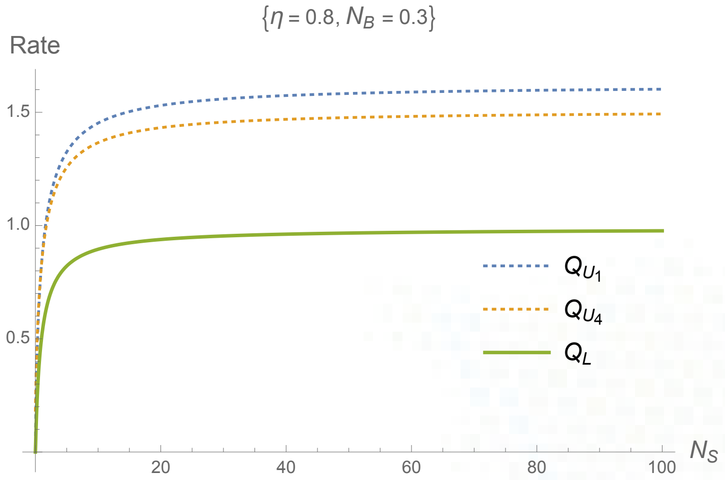
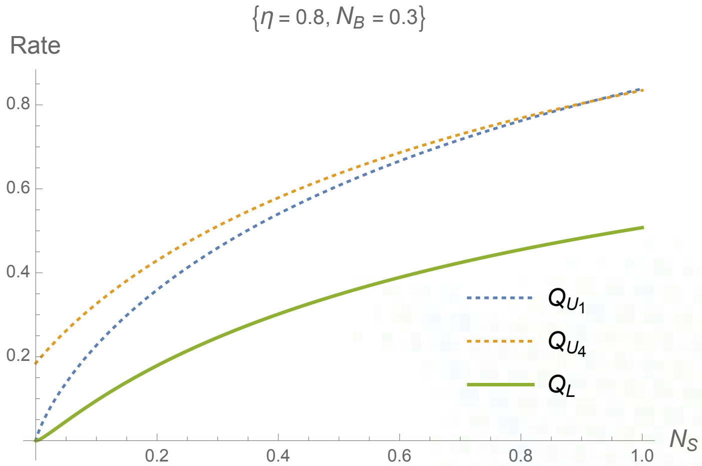


Figure: The figure plots the data-processing bound (Q_{U_1}), the upper bound (Q_{U_4}) and the lower bound (Q_L) on the energy-constrained quantum capacity of thermal channels.

Comparison of upper bounds



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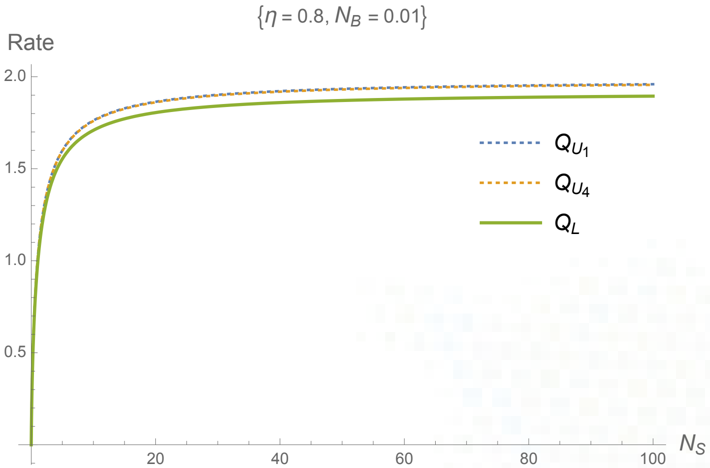

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Approximate Degradable Channel $[SSWR14]$

ε-degradable

A channel $\mathcal{N}_{A\to B}$ is ε-degradable if there exists a channel $\mathcal{D}_{B\to E}$ such that $\frac{1}{2}\left\|\hat{\mathcal{N}} - \mathcal{D}\circ\mathcal{N}\right\|_\diamond \leq \varepsilon$, where $\hat{\mathcal{N}}$ denotes a complementary channel of $\mathcal{N}$.

ε-close-degradable

A channel $\mathcal{N}_{A\to B}$ is ε-close-degradable if there exists a degradable channel $\mathcal{M}_{A\to B}$ such that $\frac{1}{2}\left\|\mathcal{N} - \mathcal{M}\right\|_\diamond \leq \varepsilon$.

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$$
\begin{aligned}
&\|\mathcal{N} - \mathcal{M}\|_{\diamondsuit} \\
&= \max_{|\psi\rangle_{RA}} \left\| (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})\left(|\psi\rangle\langle\psi|_{RA}\right) - (\mathrm{id}_R \otimes \mathcal{M}_{A\to B})\left(|\psi\rangle\langle\psi|_{RA}\right) \right\|_1
\end{aligned}
$$

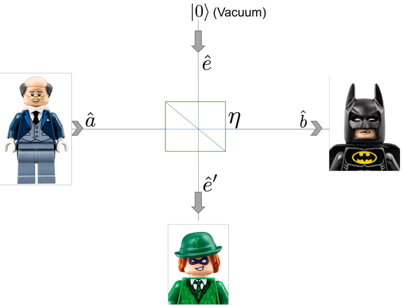
Uniform Continuity Bound (UCB)

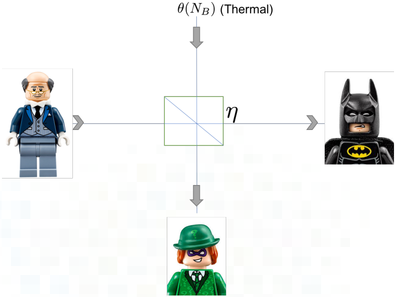
UCB for the conditional quantum entropy with energy constraints $[Win15]$

For a Gibbs observable G, $\frac{1}{2}\left\|\omega_{AB}-\tau_{AB}\right\|_1 \leq \varepsilon < \varepsilon' \leq 1$, Tr$\{(G \otimes I_B)\omega_{AB}\}$, Tr$\{(G \otimes I_B)\tau_{AB}\} \leq W$, where $W \in [0, \infty)$, and $\delta = (\varepsilon' - \varepsilon)/(1+\varepsilon')$, the following inequality holds

$$|H(A|B)_\omega - H(A|B)_\tau| \leq (2\varepsilon' + 4\delta)H(\gamma(W/\delta)) + g(\varepsilon') + 2h_2(\delta)\ .$$

ε-close-degradable





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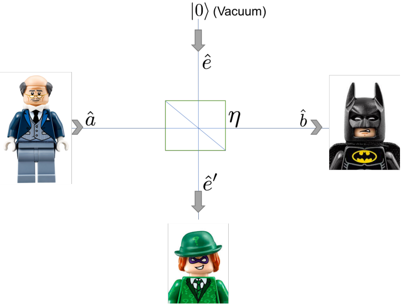
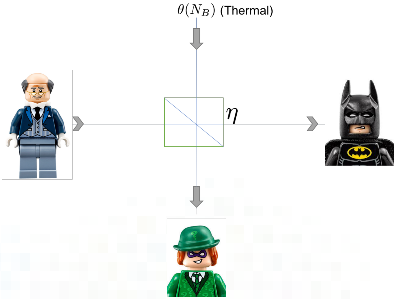




$$\frac{1}{2}\left\|\mathcal{L}_{\eta,N_B}-\mathcal{L}_{\eta,0}\right\|_\diamond \leq \frac{N_B}{N_B+1}\,.$$

ε-close-degradable bound

Theorem

An upper bound on the quantum capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number N_B, and input mean photon number constraint N_S is given by

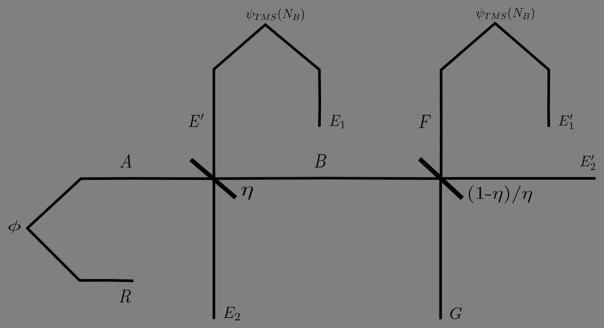
$$Q(\mathcal{L}_{\eta,N_B}, N_S) \leq Q_{U_3}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S) - g[(1-\eta)N_S] + (4\varepsilon' + 8\delta)g[(\eta N_S + (1-\eta)N_B)/\delta] + 2g(\varepsilon') + 4h_2(\delta) ,$$

with $\varepsilon = N_B/(N_B+1)$, $\varepsilon' \in (\varepsilon, 1]$ and $\delta = (\varepsilon' - \varepsilon)/(1+\varepsilon')$.

ε-degradable

Recall: $\frac{1}{2}\left\|\hat{\mathcal{N}} - \mathcal{D} \circ \mathcal{N}\right\|_{\diamond} \leq \varepsilon.$

Degrading channel

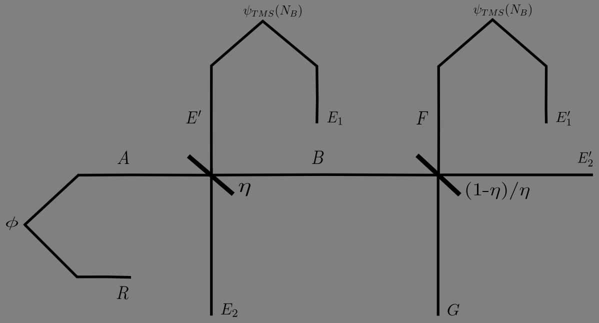


ε-degradable

Recall: $\frac{1}{2}\left\|\hat{\mathcal{N}} - \mathcal{D} \circ \mathcal{N}\right\|_\diamond \leq \varepsilon$.

$\mathrm{id}_R \otimes \hat{\mathcal{N}}_{A \to E_1 E_2}$, $\mathrm{id}_R \otimes (\mathcal{D} \circ \mathcal{N})_{A \to E_1' E_2'}$

Degrading channel



ε-degradable bound

Theorem

Fix $\eta \in [1/2, 1]$. Let $\mathcal{L}_{\eta,N_B}$ be a thermal channel with transmissivity η, and let $\mathcal{D}_{(1-\eta)/\eta,N_B}$ be a degrading channel. Then

$$\frac{1}{2}\left\|\hat{\mathcal{L}}_{\eta,N_B} - \mathcal{D}_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B}\right\|_{\diamond} \leq \sqrt{1-\eta^2/\kappa(\eta,N_B)}\,,$$

with

$$\kappa(\eta,N_B) = \eta^2 + N_B(N_B+1)[1+3\eta^2 - 2\eta(1+\sqrt{2\eta-1})]\,.$$

ε-degradable bound

Theorem

An upper bound on the quantum capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number N_B, and input mean photon-number constraint N_S is given by

$$Q_{U_2}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S + (1-\eta)N_B) - g(\zeta_+) - g(\zeta_-)$$
$$+ (2\varepsilon' + 4\delta)g([(1-\eta)N_S + (1+\eta)N_B]/\delta) + g(\varepsilon') + 2h_2(\delta)\ ,$$

with

$$\varepsilon = \sqrt{1 - \eta^2 / \left(\eta^2 + N_B(N_B+1)[1 + 3\eta^2 - 2\eta(1 + \sqrt{2\eta - 1})]\right)}\ ,$$

$\varepsilon' \in (\varepsilon, 1]$, and $\delta = (\varepsilon' - \varepsilon)/(1 + \varepsilon')$, and $\zeta_\pm$ are well behaved non-negative functions of η, N_B and N_S

Comparison of upper bounds

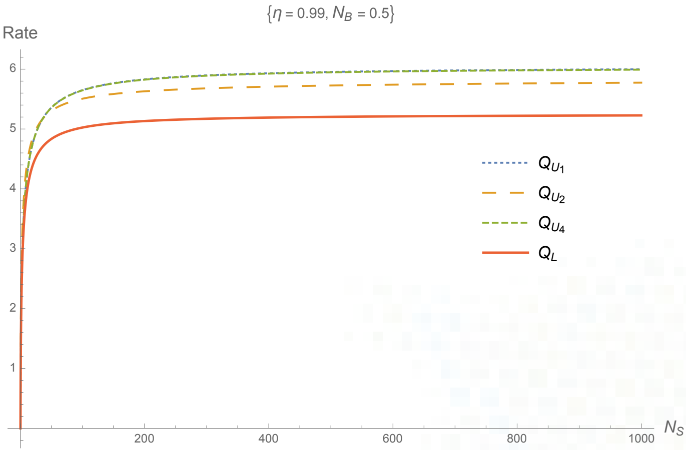


Figure: The figure plots the data-processing bound (Q_{U_1}), the ε-degradable bound (Q_{U_2}), the upper bound (Q_{U_4}), and the lower bound (Q_L) on the energy-constrained quantum capacity of thermal channels.

Conclusion

- A systematic study of upper bounds on the energy-constrained quantum and private capacities of non-degradable Gaussian channels.
- Established several upper bounds on energy-constrained quantum and private capacities of all phase-insensitive quantum Gaussian channels.
- Our results establish strong limitations on any potential superadditivity of coherent information of a thermal channel in the low-noise regime.

Improved lower bound on the energy-constrained private capacity

$$P^{(1)}(\mathcal{N}, H, W) \equiv \sup_{\bar{\rho}_{\mathcal{E}_A} : \text{Tr}\{H \bar{\rho}_{\mathcal{E}_A}\} \leq W} [S(\mathcal{N}(\bar{\rho}_{\mathcal{E}_A})) - S(\hat{\mathcal{N}}(\bar{\rho}_{\mathcal{E}_A})) - \int dx\, p_X(x) [S(\mathcal{N}(\rho_A^x)) - S(\hat{\mathcal{N}}(\rho_A^x))]] ,$$

Strategy

$$\mathcal{E} \equiv \{p_{N_S^1}(\alpha), D(\alpha)\, \theta(N_S^2)\, D(-\alpha)\},$$

$$p_{N_S^1}(\alpha) = \frac{1}{\pi N_S^1} \exp(-|\alpha|^2 / N_S^1),$$

where $N_S^1 + N_S^2 = N_S$, which is the mean number of photons input to the channel.

Improved lower bound on the energy-constrained private capacity

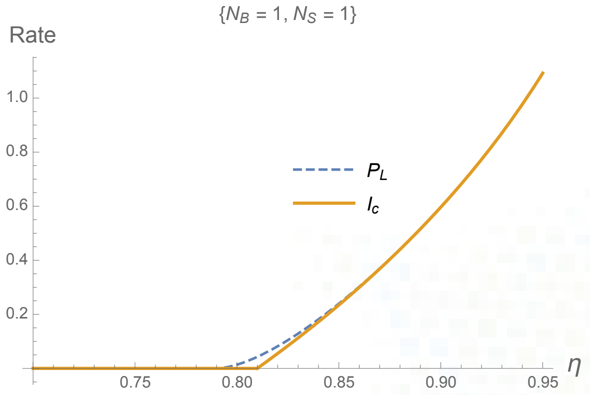


Figure: The figures plot the optimized value of the lower bound on the private information $P_L(\mathcal{L}_{\eta,N_B}, N_S)$ (dashed line) and coherent information $I_c(\mathcal{L}_{\eta,N_B}, N_S)$ (solid line) of a thermal channel versus transmissivity parameter η.