Extreme Points of Unital Quantum Channels

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Overview

- Review: Stinespring, extreme points conds, etc.
- Family of factorizable extreme UCPT maps

  extreme mixed states with max mixed quant marginals
- Extreme points of CPT and UCP with Choi-rank d

  Kraus ops are partial isometries and generalization
- Example for $d = 2\nu + 1$ odd
- Universal example

  Reformulate linear independence as eigenvalue problem

  Associate eigenvectors (lin dep) with irreps of S_n

Complete positivity

Def: $\Phi : M_{d_A} \mapsto M_{d_B}$ is completely positive (CP) if $\Phi \otimes \mathcal{I}_{d_E}$ preserves positivity $\forall\ d_E$. Suffices to consider $d_E = \min\{d_A, d_B\}$

Thm: (Choi) Φ is CP $\Leftrightarrow J_\Phi = \sum_{jk} |e_j\rangle\langle e_k| \otimes \Phi\big(|e_j\rangle\langle e_k|\big) \geq 0$

Quantum Channel: Φ is CP and trace-preserving (CPT)

TP means $\operatorname{Tr} \Phi(A) = \operatorname{Tr} A \quad \forall\ A \in M_{d_A}$

Φ UCP if unital, i.e., $\Phi(I_{d_A}) = I_{d_B}$ and CP

$\Phi : M_{d_A} \mapsto M_{d_B}$ is TP $\Leftrightarrow$ $\widehat{\Phi} : M_{d_B} \mapsto M_{d_A}$ is unital

$\widehat{\Phi}$ adjoint wrt Hilb-Schmidt inner prod. $\operatorname{Tr} [\widehat{\Phi}(A)]^* B = \operatorname{Tr} A^* \Phi(B)$

Choi condition for extremeality

Choi-Kraus CP $\Phi(A) = \sum_k F_k A F_k^*$ F_k not unique but

Choi obtained F_k by "stacking" e-vec of J_Φ with non-zero evals

Thm: (Choi) Φ is extreme in set of CP maps with $\sum_k F_k^* F_k = X$

$\Leftrightarrow$ $\{F_j^* F_k\}$ is linearly independent.

$\Rightarrow \Phi = \sum_k F_k A F_k^*$ extreme CPT map $\Leftrightarrow$ $\{F_j^* F_k\}$ is lin indep.

$\Rightarrow \Phi = \sum_k F_k A F_k^*$ extreme UCP map $\Leftrightarrow$ $\{F_j F_k^*\}$ is lin indep.

Cor: extreme CPT $\Rightarrow$ $d_E \leq d_A$ extreme UCP $\Rightarrow$ $d_E \leq d_B$

Factorizable maps on matrix algebras

Recall Stinespring $\Phi(\rho) = \mathrm{Tr}_E U\big(\rho \otimes |\phi\rangle\langle\phi|\big) U^*$

Factorizable: $\exists$ unitary U such that $\Phi(\rho) = \mathrm{Tr}_E U\big(\rho \otimes \frac{1}{d} I_d\big) U^*$

$\Rightarrow \quad \Phi(\rho) = \sum_k \frac{1}{d} \mathrm{Tr}_E U\big(\rho \otimes |e_k\rangle\langle e_k|\big) U^*$

Factorizable $\Rightarrow$ Not Extreme   Extreme $\Rightarrow$ Not Factorizable

Question: "small environment" two groups showed false $\approx$ 1999

For $\Phi : M_d \mapsto M_d$ can one make environment $d_E \leq d$ if replace $|\phi\rangle\langle\phi|$ by DM γ s. t. $\Phi(\rho) = \mathrm{Tr}_E U\big(\rho \otimes \gamma\big) U^*$

More general: arbitrary γ rather than max mixed $\frac{1}{d} I_d$

More restrictive: $\gamma \in M_d$ rather than higher dim environment

UCPT maps

Question: Are there UCPT maps $\Phi : M_d \mapsto M_d$ not extreme in either UCP or CPT maps, but are extreme in UCPT maps.

Thm: (Landau-Streater) $\Phi : M_d \mapsto M_d$ is extreme in set of UCPT maps $\Leftrightarrow$ $\{A_j^* A_k \oplus A_k A_j^*\}$ linearly independent $\Phi(\rho) = \sum_k A_k \rho A_k^*$

By C-J isomorphism convex set of UCPT maps isomorphic to bipartite states with maximally mixed quantum marginals

Equiv: Are there extreme points in convex set of bipartite states ρ_{AB} with $\rho_A = \rho_B = \frac{1}{d} I_d$ which are not max entang pure states?

- Parthasarathy showed NO for $\mathbf{C}_2 \otimes \mathbf{C}_d$.
- Arveson-Ohno gave examples for qutrits, i.e., $\mathbf{C}_3 \otimes \mathbf{C}_3$.

Known results about extreme points of CPT maps

- Qubit channels $\Phi : M_2 \mapsto M_2$

  * Ruskai, Szarek Werner (2002) all extreme points

  * UCPT much earlier, essent conj with I_2 or Pauli matrix

    correspond to max entangled Bells states – tetrahedron

- Parthsarathy ρ_{AB} state on $\mathbf{C}_2 \otimes \mathbf{C}_d$ extreme $\Leftrightarrow$ max entang

- General UCPT $\Phi : M_d \mapsto M_d$ unitary conj are extreme

- Few other results — very special

  * $d = 3$ Werner-Holevo and symmetric variant extreme

    not true for Werner-Holevo when $d > 3$

  * Arveson-Ohno examples – few high rank in low dims

    one low rank family using partial isometries

Family of high rank extreme points of UCPT maps

$$\Phi_{\alpha,\beta}(\rho) = \sum_{k=1}^{4} A_k^* \rho A_k$$ 1-1 correspond with qubit pure states

Def: For $|\alpha|^2 + |\beta|^2 = 1$ let

$$A_1 = \alpha|e_1\rangle\langle e_1| + |e_2\rangle\langle e_3| \qquad A_2 = \beta|e_1\rangle\langle e_3| + |e_3\rangle\langle e_2|$$

$$A_3 = |e_1\rangle\langle e_2| + \overline{\beta}|e_3\rangle\langle e_1| \qquad A_4 = |e_2\rangle\langle e_1| + \overline{\alpha}|e_3\rangle\langle e_3|$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \overline{\beta} & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \overline{\alpha} \end{pmatrix}$$

Observe $U = \begin{pmatrix} A_1 & A_2 \\ -A_3 & A_4 \end{pmatrix}$ is unitary $\in M_3 \otimes M_2$

$\Rightarrow \quad \Phi_{\alpha,\beta}(\rho) = (\mathcal{I}_3 \otimes \mathrm{Tr}\,)\big(U^*\big(\rho \otimes \tfrac{1}{2} I_2\big)U\big)$ factorizable

Sketch proof:

Thm: $\Phi_{\alpha,\beta}$ is an extreme UCPT map for $\alpha, \beta \neq 0, \frac{1}{2}, 1$

corresponds to N and S poles and equator on Bloch sphere

Proof: Can verify that $\{A_k^* A_k \oplus A_k A_k^*\}$ all diagonal and lin indep

Direct calc shows $A_j^* A_k \oplus A_k A_j^*$ for $j \neq k$ splits into 4 disjoint sets

actually two sets disjoint from each other and their adjoints

can verify lin indep $\Leftrightarrow \alpha, \beta, \neq 0, \frac{1}{2}, 1$ calc det of 3×3

How entangled are the corresponding states?

$$\frac{1}{\sqrt{1+|\alpha|^2}}\big(\alpha\,|e_1 \otimes e_1\rangle + |e_2 \otimes e_3\rangle\big) \qquad \frac{1}{\sqrt{1+|\beta|^2}}\big(\beta\,|e_1 \otimes e_3\rangle + |e_3 \otimes e_2\rangle\big)$$

$$\frac{1}{\sqrt{1+|\beta|^2}}\big(\,|e_1 \otimes e_2\rangle + \overline{\beta}|e_3 \otimes e_1\rangle\big) \qquad \frac{1}{\sqrt{1+|\alpha|^2}}\big(|e_2 \otimes e_1\rangle + \overline{\alpha}|e_3 \otimes e_3\rangle\big)$$

$$J_\Phi = \tfrac{1+|\alpha|^2}{6}|\psi_1\rangle\langle\psi_1| + \tfrac{1+|\beta|^2}{6}|\psi_2\rangle\langle\psi_2| + \tfrac{1+|\beta|^2}{6}|\psi_3\rangle\langle\psi_3| + \tfrac{1+|\alpha|^2}{6}|\psi_4\rangle\langle\psi_4|$$

Def: $\mathrm{EoF}(\rho_{AB}) = \inf\Big\{\sum_k x_k E(\psi_k) : \sum_k x_k|\psi_k\rangle\langle\psi_k| = \rho_{AB}\Big\}$

$E(\psi_{AB}) = S(\rho_A), \quad \rho_A = \mathrm{Tr}_B|\psi_{AB}\rangle\langle\psi_{AB}|, \quad S(\rho) = -\mathrm{Tr}\,\rho\log\rho$

$$\mathrm{EoF}(\rho_{AB}) = \tfrac{1+|\alpha|^2}{3}h\big(\tfrac{1}{1+|\alpha|^2}\big) + \tfrac{2-|\alpha|^2}{3}h\big(\tfrac{1}{2-|\alpha|^2}\big)$$

$$\tfrac{2}{3} \le \mathrm{EoF}(\rho_{AB}) \le h\Big(\frac{1}{3}\Big) = 0.918296 < 1 = \log 2 < \log 3$$

poles equator

Arveson-Ohno example

$$h(x) = -x \log x - (1-x)\log(1-x) \qquad \text{binary entropy}$$

$$A_1 = |e_1\rangle\langle e_1| \qquad A_2 = |e_1\rangle\langle e_2| + \sqrt{2}\,|e_2\rangle\langle e_3|$$
$$A_3 = \sqrt{2}\,|e_2\rangle\langle e_1| + \sqrt{3}|e_3\rangle\langle e_2| \qquad A_4 = |e_3\rangle\langle e_1| + \sqrt{2}\,|e_1\rangle\langle e_3|$$

$$J_\Phi = \rho_{AB} = \tfrac{1}{12}|e_1 \otimes e_1\rangle\langle e_1 \otimes e_1| + \tfrac{1}{4}|\psi_2\rangle\langle\psi_2| + \tfrac{5}{12}|\psi_3\rangle\langle\psi_3| + \tfrac{1}{4}|\psi_4\rangle\langle\psi_4|$$

$$\begin{aligned}
\psi_2 &= \tfrac{1}{\sqrt{3}}\big(|e_1 \otimes e_2\rangle + \sqrt{2}\,|e_2 \otimes e_3\rangle\big)\\
\psi_3 &= \tfrac{1}{\sqrt{5}}\big(\sqrt{2}\,|e_2 \otimes e_1\rangle + \sqrt{3}\,|e_3 \otimes e_2\rangle\big)\\
\psi_4 &= \tfrac{1}{\sqrt{3}}\big(|e_3 \otimes e_1\rangle + \sqrt{2}\,|e_1 \otimes e_3\rangle\big)
\end{aligned}$$

$$\mathrm{EoF}(\rho_{AB}) \le \tfrac{1}{4}h\big(\tfrac{1}{3}\big) + \tfrac{5}{12}h\big(\tfrac{2}{5}\big) + \tfrac{1}{4}h\big(\tfrac{1}{3}\big) = 0.8637 < 1 = \log 2 < \log 3$$

Extreme points of UCP and CPT maps with rank d

$\{V_1, V_2, \ldots V_d\}$ unitary $\in M_{d-1}$, $\quad S = \sum_k |e_k\rangle\langle e_{k+1}|$ cyclic shift

$$A_m = \tfrac{1}{\sqrt{d-1}}\, S^m \begin{pmatrix} V_m & 0 \\ 0 & 0 \end{pmatrix} S^{d-m}$$

$$A_m^* A_m = A_m A_m^* = \tfrac{1}{d-1}(I_d - |e_m\rangle\langle e_m|)$$

$$\Rightarrow \; \textstyle\sum_m A_m^* A_m = A_m A_m^* = I_d$$

$$\Rightarrow \; \Phi(\rho) = \sum_m A_m \rho A_m^* \text{ is both UCP and CPT}$$

Generalize $\qquad A_m = \tfrac{1}{\sqrt{d-1+t^2}}\, S^m \begin{pmatrix} V_m & 0 \\ 0 & t \end{pmatrix} S^{d-m}$

For $t \in (-1, 1)$, $\quad A_m^* A_m = A_m A_m^* = \tfrac{1}{d-1}\big[I_d - (1 - t^2)\big]|e_m\rangle\langle e_m|)$

Choi rank d which suggests extreme

Almost always extreme

Thm: For $t \in (-1,1)$ fixed and $A_m = S^m \begin{pmatrix} V_m & 0 \\ 0 & t \end{pmatrix} S^{d-m}$ if $\exists$ one example with $\{A_m^* A_n\}$ linearly indep, then for almost every choice of unitary $V_1, V_2, \dots V_d$ the set $\{A_m^* A_n\}$ is lin indep.

Proof idea: $\{v_j\}$ lin indep iff gram matrix $g_{jk} = \langle v_j, v_k \rangle$ non-sing

For matrices $\{A_m A_n^*\}$ with Hilbert-Schmidt inner prod this is

$$g_{jk,mn} = \mathrm{Tr}\,(A_j A_k^*)(A_m A_n^*)^* = \mathrm{Tr}\, A_j A_k^* A_n A_m^*$$

det G is a poly in elements u_{jk}^m of matrices V_m.

If poly not ident. zero, roots an alg. variety of Haar measure zero

Thm: Same for $\{A_m A_n^*\}$.

Thm: Similar result for $V_1 = V_2 = \dots V_d$

Lin independ of subset with $m = n$

$$A_m = S^m \begin{pmatrix} V_m & 0 \\ 0 & t \end{pmatrix} S^{d-m} \qquad\qquad t \neq \pm 1$$

$$A_m^* A_m = A_m A_m^* = I_d - (1 - t^2)|e_m\rangle\langle e_m|$$

$\Rightarrow \{A_m^* A_m\}$ lin indep and $\;\Rightarrow\; \mathrm{span}\{A_m^* A_m\} = \mathrm{span}\{|e_j\rangle\langle e_j|\}$

$\Rightarrow$ For purpose of determining lin indep of $\{A_m A_n^*\}$ can make arbitrary modifications to diagonal of $A_m A_n^*$ or $A_m A_n^*$

$d = 2\nu + 1$ schematic

$$S_\nu = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ \vdots & & \ddots & \vdots & ⋰ & & \vdots \\ 0 & 0 & \dots & \color{blue}{t} & \dots & 0 & 0 \\ \vdots & & ⋰ & \vdots & \ddots & & \vdots \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix} \qquad S_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \color{blue}{t} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Existence of “fixed point” on skew diagonal seems key

Sketch proof for $d = 2\nu + 1$ odd

$$P_m = \sum_{j=1}^{d} |e_j\rangle\langle e_{2m-j}| \qquad A_m = P_m - (1-t)|e_m\rangle\langle e_m|$$

$P_m = P_m^*$ is perm matrix for ν swaps $(m+k, m-k) \;\Rightarrow\; P_m^2 = I_d$

$A_m A_{m+\ell} = S^{2\ell} - (1-t)\big(|e_{m-\ell}\rangle\langle e_{m+\ell}| + |e_m\rangle\langle e_{m+2\ell}|\big) + \delta_{\ell,0}(1-t)^2|e_m\rangle\langle e_m|$

linear independence of $\{A_m A_n\}$ reduces to lin indep of vectors

$|e_{m-\ell}\rangle\langle e_{m+\ell}| + |e_m\rangle\langle e_{m+2\ell}|$ with ℓ fixed – reduce to prob in $\mathbf{C}_d$

Find $\Phi(\rho) = \sum_m A_m \rho A_m^*$ is extreme in both CPT and UCP maps.

Fixed point $|e_m\rangle\langle e_m| \;\mapsto\; t|e_m\rangle\langle e_m|$ plays central role

For $t = 0$ can make all A_k anti-symmetric.

Does not generalize to even $d = 2\nu$ in natural way

Main example

Notation: $S = \sum_k |e_k\rangle\langle e_{k+1}|$ cyclic shift

$|\mathbb{1}_d\rangle$ denote the vector whose elements are all $d^{-1/2}$.

unitary $V_1 = V_2 = \ldots V_d = V \equiv 2|\mathbb{1}_{d-1}\rangle\langle\mathbb{1}_{d-1}| - I_{d-1}$

$$A_1 = \begin{pmatrix} t & 0 \\ 0 & V \end{pmatrix} \qquad A_m = S^{-m+1} A_1 S^{m-1} = S^{-1} A_{m-1} S$$

Note that $A_m = A_m^* \;\Rightarrow\;$ suffices to consider lin indep of $\{A_m A_n\}$

Thm: For $d \geq 3$ and $t \in (-1, 1)$ and $t \neq -\frac{1}{d-1}$
the set $\{A_m A_n\}$ is linearly independent

Cor: For $d \geq 3$, $t \in (-1, 1)$, $t \neq -\frac{1}{d-1}$ map $\Phi(\rho) = \sum_m A_m \rho A_m^*$
is an extreme point of both the UCP and CPT maps.

More refined results

a) For $d \geq 3$ and $t = 1$, the sets $\{A_m^2\}_{m=1}^d$ and $\{A_m A_n - A_n A_m\}_{m<n}$ are each separately linearly dependent.

b) For $d \geq 3$, $t = -1$, the set $\{A_m^2\}_{m=1}^d$ is linearly dependent but the set $\{A_m A_n\}_{m \neq n}$ is linearly independent.

c) For $d \geq 4$, $t = \frac{-1}{d-1}$, the set $\{A_m^2\}_{m=1}^d$ is linearly independent, but $\sum_{m \neq n} A_m A_n$ is a multiple of I_d so that $\{A_m A_n\}$ is linearly dependent. Moreover, $\{A_m A_n - A_n A_m\}_{m<n}$ and $\{A_m A_n + A_n A_m\}_{m<n}$ are each linearly dependent.

d) For $d = 3, t = \frac{-1}{d-1}$, the set $\{A_m^2\}_{m=1}^d$ is linearly independent, but $\displaystyle\sum_{m \neq n} A_m A_n = 0 \;\Rightarrow\; \{A_m A_n + A_n A_m\}_{m<n}$ is linearly depend.

Moreover, $\{A_m A_n - A_n A_m\}_{m<n}$ is also linearly dependent.

Return to main example — form of A_m

$$A_1 = \frac{1}{d-1}\begin{pmatrix} t(d-1) & 0 & \dots & & \dots & 0 \\ 0 & d-3 & 2 & \dots & & 2 \\ 0 & 2 & d-3 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ & & & & & \\ 0 & 2 & \dots & & 2 & d-3 \end{pmatrix}$$

$$A_d = \frac{1}{d-1}\begin{pmatrix} d-3 & 2 & \dots & & 2 & 0 \\ 2 & d-3 & 2 & \dots & 2 & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ & & & & & \\ 2 & \dots & & 2 & d-3 & 0 \\ 0 & \dots & & \dots & 0 & t(d-1) \end{pmatrix}$$

Sketch proof for main example

$$V = 2|\mathbb{1}_{d-1}\rangle\langle\mathbb{1}_{d-1}| - I_{d-1}, \qquad A_m = S^{-m}\begin{pmatrix} t & 0 \\ 0 & V \end{pmatrix} S^m$$

$$A_1 A_d = \frac{1}{(d-1)^2}\begin{pmatrix} \tau & b & \ldots & b & \ldots & b & 0 \\ a & & & & & & b \\ \vdots & & & & & & \vdots \\ a & & & \widetilde{V}_{d-2}^2 & & & b \\ \vdots & & & & & & \vdots \\ a & & & & & & b \\ u & a & \ldots & a & \ldots & & \tau \end{pmatrix}$$

$$a = 2(d-3),\ b = 2t(d-1),\ \tau = -t(d-1)(d-3)$$

$$u = 4(d-2), \qquad \widetilde{V}_{d-2}^2 = -4c|\mathbb{1}_{d-2}\rangle\langle\mathbb{1}_{d-2}| + bI_{d-2}$$

First reformulation

$$\sum_m \sum_n A_m A_n = p_d(t)|\mathbb{1}_d\rangle\langle\mathbb{1}_d| + q_d(t) I_d$$

$q(t) \neq 0$ if $d > 3$ $\qquad$ $p(t) \neq 0$ if $t \neq \frac{-1}{d-1}$

$\Rightarrow \{A_m A_n\}$ lin dependent for $t = \frac{-1}{d-1}$

For purpose of linear independ for $d > 3, t \neq \frac{-1}{d-1}$ can replace

$$A_m A_n \mapsto X_{mn} \equiv (d-1)^2 A_m A_n + 4d|\mathbb{1}_d\rangle\langle\mathbb{1}_d|$$

$$= \widehat{u}|e_m\rangle\langle e_n| + \widehat{a} \sum_{j\neq m,n} \big(|e_j\rangle\langle e_m| + |e_n\rangle\langle e_j|\big) + \widehat{b} \sum_{j\neq m,n} \big(|e_m\rangle\langle e_j| + |e_j\rangle\langle e_n|\big)$$

where $\widehat{a} = 2(d-1)$, $\widehat{u} = 4(d-1)$, $\widehat{b} = 2t(d-1) + 4$.

Results hold for $d = 3$ but proofs need special handling.

$$
X_{nm} \;=\; \begin{array}{cc}
 & \begin{array}{cccccccc} \phantom{\widehat{b}} & & \phantom{\widehat{b}} & m & \phantom{\widehat{b}} & n & \phantom{\widehat{b}} & \end{array} \\
\begin{array}{c} \\ \\ \\ m \\ \\ \\ \\ n \\ \\ \\ \end{array} &
\left(\begin{array}{cccccccc}
 & & & \widehat{a} & & \widehat{b} & & \\
 & & & \vdots & & \vdots & & \\
 & & & \widehat{a} & & \widehat{b} & & \\
\widehat{b} & \ldots & \widehat{b} & 0 & \widehat{b} & 4 & \widehat{b} & \ldots \\
 & & & \widehat{a} & & \widehat{b} & & \\
 & & & \vdots & & \vdots & & \\
 & & & \widehat{a} & & \widehat{b} & & \\
\widehat{a} & \ldots & \widehat{a} & \widehat{u} & \widehat{a} & 0 & \widehat{a} & \ldots \\
 & & & \widehat{a} & & \widehat{b} & & \\
 & & & \vdots & & \vdots & &
\end{array} \right)
\end{array}
$$

Permutational symmetry

Want to find $C \simeq \{c_{mn}\}$ such that $\sum_{mn} c_{mn} A_m A_n = 0$.

Consider M matrix with rows "unstacked" matrices.

C is "vector" in null space of M.

Moreover, such C form a subspace $\mathcal{N}$ of M_d with properties

- $C \in \mathcal{N} \;\Rightarrow\; C^* \in \mathcal{N}$
- $C \in \mathcal{N} \;\Rightarrow\; P^* C\, P \in \mathcal{N} \quad \forall$ permutation matrices P

Decompose into subspaces $\mathcal{N}_\nu$ assoc with irreducible rep of S_d

Now define $\qquad X^{\pm}_{mn} = X_{mn} \pm X^*_{mn}$

Consider linear indep. of $X_{mn} + X^*_{mn}$ and $X_{mn} - X^*_{mn}$ separately

$$X^{\pm}_{nm}(x) \;=\; \begin{array}{cc} & \begin{array}{cccccccc} & & & m & & n & & \end{array} \\ \begin{array}{c} \\ \\ \\ m \\ \\ \\ \\ n \\ \\ \\ \end{array} & \begin{pmatrix} & & & \pm 1 & & 1 & & \\ & & & \vdots & & \vdots & & \\ & & & \pm 1 & & 1 & & \\ 1 & \cdots & 1 & 0 & 1 & x & 1 & \cdots \\ & & & \pm 1 & & 1 & & \\ & & & \vdots & & \vdots & & \\ & & & \pm 1 & & 1 & & \\ \pm 1 & \cdots & \pm 1 & \pm x & \pm 1 & 0 & \pm 1 & \cdots \\ & & & \pm 1 & & 1 & & \\ & & & \vdots & & \vdots & & \end{pmatrix} \end{array}$$

Can ignore factor of $(\widehat{a} \pm \widehat{b})$ Main interest $x = w_d^{\pm}(t)$

Reformulate as eigenvalue problem

$$w_d^+(t) = \tfrac{2d}{d+1+t(d-1)} \qquad w_d^-(t) = \tfrac{2(d-2)}{(d-3)-t(d-1)}$$

Replace X_{mn} by $\Omega_d^{\pm}(x)$ $\frac{1}{2}d(d-1) \times \frac{1}{2}d(d-1)$ matrix with rows given by elements of $X_{mn}^{\pm}(x)$ above diagonal in lexigraphic order

Elements of $\Omega_d^{\pm}(x)$ are $\begin{cases} x & \text{on diagonal} \\ 0, \pm 1 & \text{otherwise} \end{cases}$

Thm: $\{X_{mn}^{\pm}\}_{m<n}$ lin depend $\Leftrightarrow -w_d^{\pm}(t)$ an eigenvalue of $\Omega(0)$.

Find eigenvals: Mathematica for $d = 3, 4, 5, 6$ then educated guess

Proof: Exhibit lin indep eigenvecs of certain symmetry type with "at least" desired multiplicity

Eigenvalue Summary

Thm: $\{X^{\pm}_{mn}\}_{m<n}$ lin indep $\Leftrightarrow -w^{\pm}_d(t)$ not eigenvalue of $\Omega(0)$.

Thm: The eigenvalues of $\Omega^{+}_d(0)$ are

- $2(d-2)$ non-degenerate $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & & & & d \\ \hline \end{array}$
- $d-4$ with multiplicity $d-1$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline k \\ \cline{1-1} \end{array}$
- -2 with multiplicity $\binom{d-1}{2} - 1 = \frac{1}{2}d(d-3)$ $\qquad t = \frac{-1}{d-1}$

Thm: The eigenvalues of $\Omega^{-}_d(0)$ are

- $d-2$ with multiplicity $d-1$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline k \\ \cline{1-1} \end{array}$ $\qquad t = 1$
- -2 with mullt $\frac{1}{2}(d-2)(d-1)$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline j \\ \cline{1-1} k \\ \cline{1-1} \end{array}$ $\qquad t = \frac{-1}{d-1}$

Symmetric eigenvectors

- $x = 2(2-d)$ $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & & & & d \\ \hline\end{array}$.

  $C = |\mathbb{1}_d\rangle\langle\mathbb{1}_d| \qquad \sum_{m \neq n} X^+_{mn} = 0 \;$ but $\; x = 2(2-d) \neq w^+_d(t)$

- $x = 4 - d \neq w^+_d(t)$ for $t \in [-1, 1]$ unless $d = 3$.

  $C_{k\ell} = \sum_{j \neq k,\ell} \big(|e_k\rangle\langle e_j| + |e_j\rangle\langle|e_k| - |e_\ell\rangle\langle e_j| - |e_\ell\rangle\langle e_j|\big) \qquad k \neq \ell$

  $C_{1k}, \;\; k = 2, 3, \ldots d \;$ lin indep $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline k \\ \cline{1-1}\end{array}$

  $$\sum_{j \neq k,\ell} A_j A_\ell + A_\ell A_j - A_j A_k - A_k A_j = \begin{cases} Diag & d \geq 5 \\ 0 & d = 3 \end{cases}$$

Symmetric eigenvectors for $t = \frac{-1}{d-1}$

- $x = 2 = w_d^+\left(\frac{-1}{d-1}\right)$ eigenvecs $C_{jk,mn} = B_{jk,mn} + B^*_{jk,mn}$

$$B_{jk,mn} \equiv |e_m\rangle\langle e_j| - |e_m\rangle\langle e_k| - |e_n\rangle\langle e_j| + |e_n\rangle\langle e_k|$$

$\{B_{2k,1n} : 3 \le n < k \le d\} \cup \{B_{2k,13} : k = 4, 5 \ldots d\}$

lin indep $\Rightarrow$ $\frac{1}{2}d(d-3)$ lin indep eigenvecs $C_{jk,mn}$

Young tableaux

1	2		...	
n	k			

and

1	3		...	
2	k			

$$C_{24,13} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \qquad C_{34,12} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$$A_mA_j + A_jA_m - A_mA_k - A_kA_m - A_nA_j - A_jA_n + A_nA_k + A_kA_n = 0$$

lin dep A_mA_n not directly trans. to X^+_{mn} for $t = \frac{-1}{d-1}$ but still OK

Skew symmetric eigenvectors

- $x = 2 - d = w_d^-(1)$ multiplicity $d - 1$ $\qquad t = 1$

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & & & & & \\
\hline
k \\
\cline{1-1}
\end{array}$$

$$C_1 = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \sum_{k=1}^{d-1} C_k = C_d$$

$$C_k = \sum_{j \neq k} \big(|e_k\rangle\langle e_j| - |e_j\rangle\langle|e_k| \big) \text{ lin indep for } k = 1, 2, \dots d - 1$$

$$\Rightarrow \text{ linear dep} \quad \sum_{m \neq n} \big(A_m A_n - A_n A_m \big) = 0 \qquad n = 1, 2 \dots d$$

Skew symmetric eigenvectors (cont)

- $x = 2 = w_d^-\left(\frac{-1}{d-1}\right)$ multiplicity $\binom{d}{2}$ $t = \frac{-1}{d-1}$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline j \\ \cline{1-1} k \\ \cline{1-1}\end{array}$$

lin indep $1 < j < k \leq d$

$$C_{124} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & \ldots \\ -1 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \vdots & 0 & \ddots & \ldots \\ \vdots & \vdots & & & \vdots & \ddots \end{pmatrix}$$

$$C_{jk\ell} = |e_j\rangle\langle e_k| - |e_k\rangle\langle e_j| - |e_j\rangle\langle e_\ell| + |e_\ell\rangle\langle e_j| + |e_k\rangle\langle e_\ell| - |e_\ell\rangle\langle e_k|$$

$$\Rightarrow\ A_\ell A_j - A_j A_\ell - A_\ell A_k + A_\ell A_k - A_j A_k + A_j A_k = 0.$$

with j, k, ℓ distinct for $t = \frac{-1}{d-1}$.

$A_m A_n \mapsto X_{mn}$ not needed. $X_{mn}^- = A_m A_n - A_n A_m$ direct cancel

Are New Channels Subadditive for min $S[\Phi(\rho)]$?

Recall Werner-Holevo for $d = 3$ Equiv to $\mathrm{Tr}_E|\Psi_{BE}\rangle\langle\Psi_{BE}|$

with Ψ_{BE} in anti-sym subspace $\mathbf{C}_4 \wedge \mathbf{C}_4$.

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 0 & 0. & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Subadditive for Renyi entropy with $p > 4.73$?

BUT $$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 0 & 0. & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

always additive – output is too pure – state with e-vals $\frac{2}{3}, \frac{1}{3}, \frac{1}{3}$

Universal Example $V = 2|\mathbb{1}_{d-1}\rangle\langle\mathbb{1}_{d-1}| - I_{d-1}$

$$A_1 = \frac{1}{d-1}\begin{pmatrix} t(d-1) & 0 & \dots & & \dots & 0 \\ 0 & d-3 & 2 & \dots & & 2 \\ 0 & 2 & d-3 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ & & & & & \\ 0 & 2 & \dots & & 2 & d-3 \end{pmatrix}$$

$$\Phi\Big(|\mathbb{1}_d\rangle\langle\mathbb{1}_d|\Big) = \frac{1}{d-1+t^2}\Big[(d-2+2t)|\mathbb{1}_d\rangle\langle\mathbb{1}_d| + \frac{(1-t)^2}{d} I_d\Big]$$

largest e-val $\quad \frac{(d+1-t)^2}{d(d-1+t^2}$

$t = 0 \qquad 1 - \frac{1}{d} \quad$ non-deg $\qquad \frac{1}{d(d-1)}$ mult $d-1$

$t = 1 \qquad 1, 0, \dots, 0 \qquad$ i.e. pure output

Moral: Highly symmetric channels don't violate additivity

Interesting New Channel

$$2|\mathbb{1}_4\rangle\langle\mathbb{1}_4| - I_4 = \frac{1}{2}\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \qquad X = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A_1 = \frac{1}{2}\begin{pmatrix} X & Y & 0_2 \\ Y & X & 0_2 \\ 0_2 & 0_2 & 0_2 \end{pmatrix} \quad A_2 = \frac{1}{2}\begin{pmatrix} X & 0_2 & Y \\ 0_2 & 0_2 & 0_2 \\ Y & 0_2 & X \end{pmatrix} \quad A_3 = \frac{1}{2}\begin{pmatrix} 0_2 & 0_2 & 0_2 \\ 0_2 & X & Y \\ 0_2 & Y & X \end{pmatrix}$$

Similar to symmetric WH channel $0 \mapsto X = -X^*, \ 1 \mapsto Y = Y^*$

$$A_1 = \frac{1}{2}\begin{pmatrix} X & Y & 0 \\ -Y & X & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A = \frac{1}{2}\begin{pmatrix} X & 0 & Y \\ 0 & 0 & 0 \\ -Y & 0 & X \end{pmatrix} \quad A_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & X & Y \\ 0 & -Y & X \end{pmatrix}$$

$A_k^* = -A_k$ BUT turns out to be additive.

Additivity Violations

Moral: Don't look for needle in haystack of min output entropy

Look for superadditivity of Holevo capacity — it suffices to show

$$\exists \quad \Psi \quad \text{s.t} \quad H\Big(\Phi^{\otimes N}(|\Psi\rangle\langle\Psi|),\ [\Phi(\rho_{\rm av})]^{\otimes N}\Big) > N\, C_{\rm Holv}(\Phi)$$

Don't need to find actual capacity or output average for $\Phi^{\otimes N}$

Suffices to find entangled Ψ s.t. Expect "haystack" 2^N

$$H\Big(\Phi^{\otimes N}(|\Psi\rangle\langle\Psi|),\ [\Phi(\rho_{\rm av})]^{\otimes N}\Big) \approx N C_{\rm Holv}(\Phi)$$

BUT that's another talk.

BIRS 2019 Workshops in Banff

The Many Faceted Connes Embedding Problem

14-19 July 2019 (19w5163)

M.B. Ruskai, M. Junge, V. Paulsen, C. Palazuelos

Algebraic and Statistical ways into Quantum Resource Theories

21-26 July 2019 (19w5120)

G. Gour, F. Buscemi, E. Chitambar