Rocky Mountain Summit on Quantum Information

Achievable rates of Gottesman-Kitaev-Preskill (GKP) codes for Gaussian thermal loss channels

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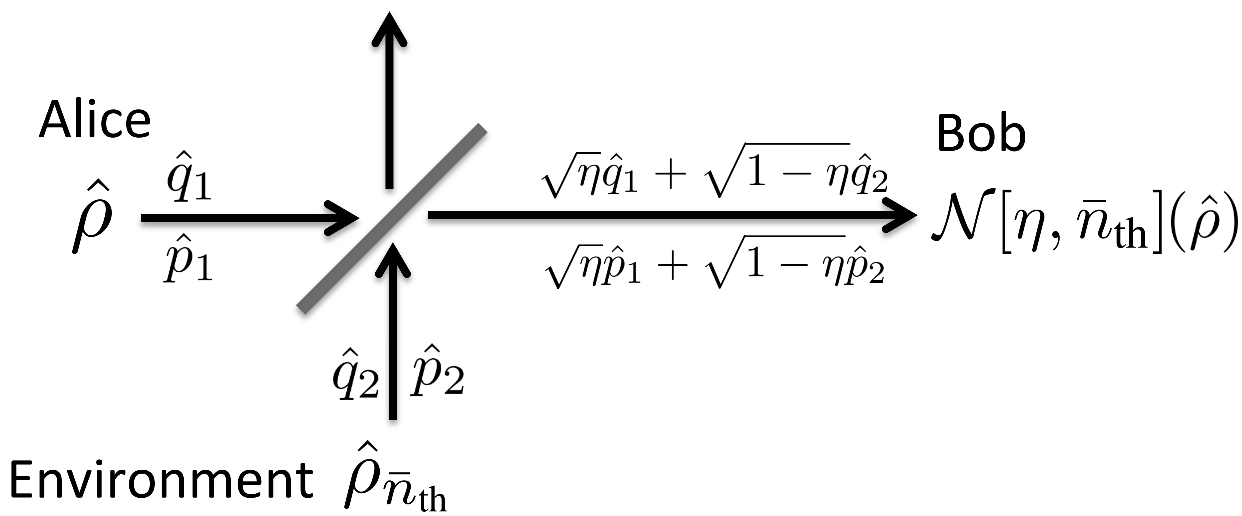
(2018/06/29)

Gaussian thermal loss channels

-Definition C. Weedbrook et al., Rev. Mod. Phys. **84**, 621-669 (2012)



$\eta \in [0,1]$: transmissivity $\gamma \equiv 1-\eta$: loss probability

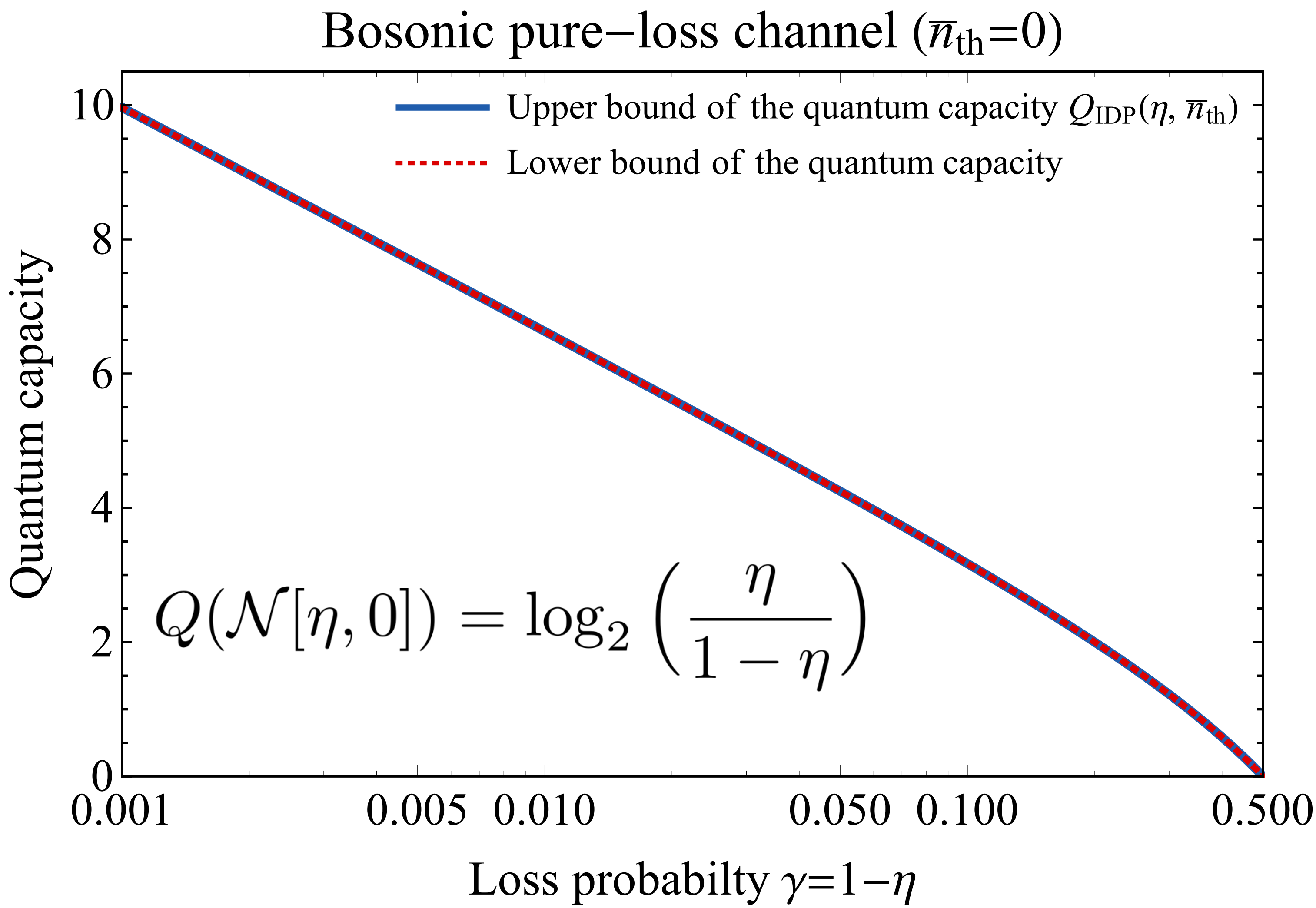
: models (1) practical optical communication channels
and (2) photon loss / gain in microwave cavity modes

N. Ofek et al., Nature **536**, 441–445 (2016)

-Special case

$\bar{n}_{\mathrm{th}} = 0$: bosonic pure-loss channel

Quantum capacity of Gaussian thermal loss channels

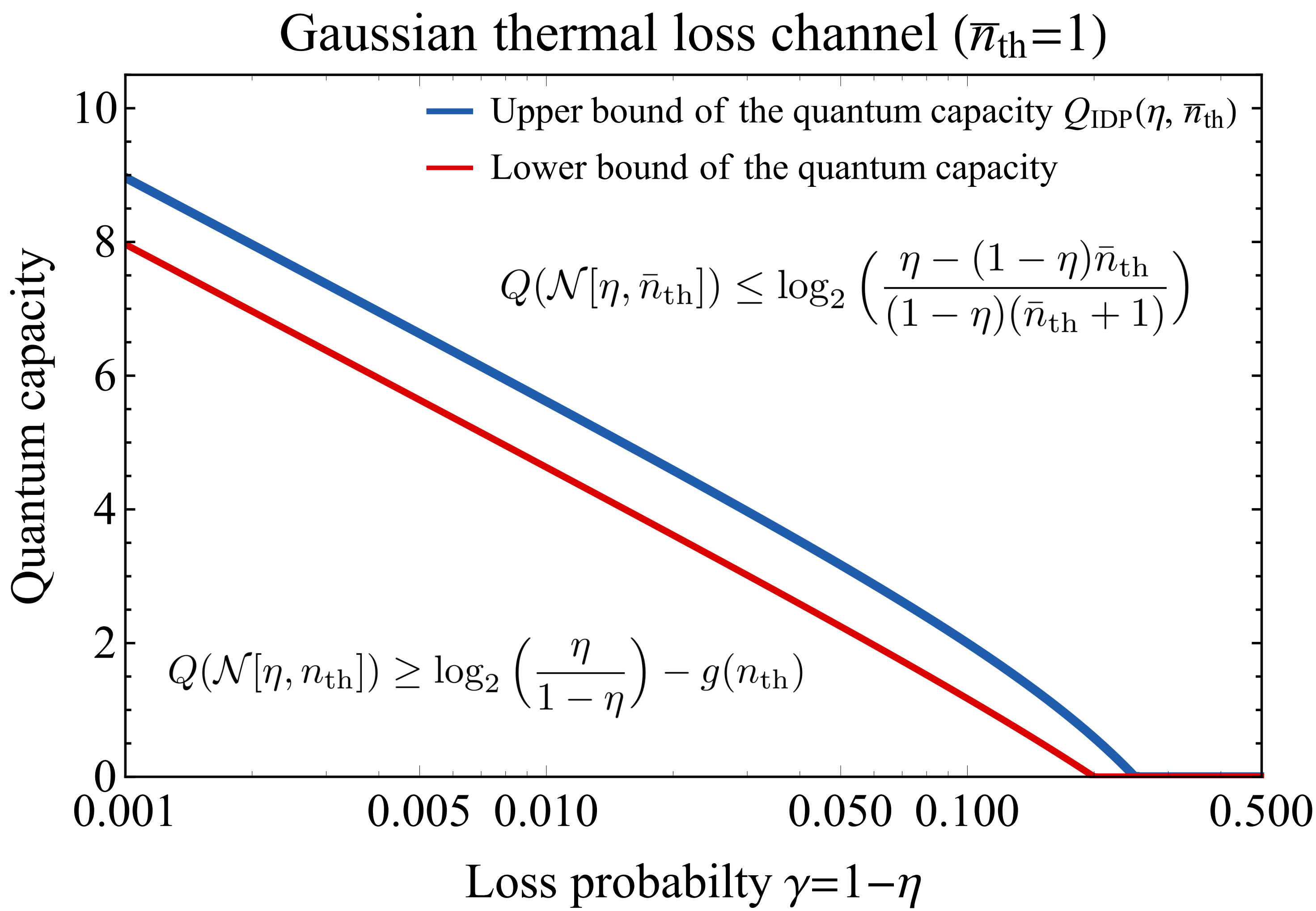


A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001) : Lower bound

M. M. Wolf et al., Phys. Rev. Lett. **98**, 130501 (2007) : Upper bound

M. M. Wilde and H. Qi, arXiv:1609.01997 (2016) : Upper bound

Quantum capacity of Gaussian thermal loss channels



A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001) : Lower bound

K. Sharma, M. M. Wilde, S. Adhikari, and M. Takeoka, arXiv:1708.07257v2 (2018) : Upper bound

M. Rosati, A. Mari, and V. Giovannetti, arXiv:1801.04731v2 (2018) : Upper bound

KN, V. V. Albert, and L. Jiang, arXiv:1801.07271v2 (2018) : Upper bound

Achieving the one-shot coherent information

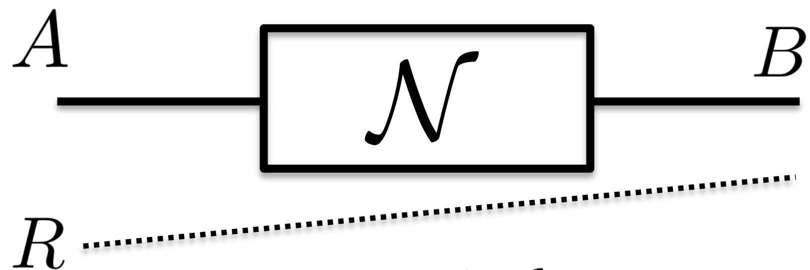
$\hat{\rho}_A$ Thermal state

purification

$|\Phi_{\hat{\rho}}\rangle_{AR}$

Two-mode squeezed vacuum

Gaussian thermal loss channel



$$\hat{\omega}_{BR} \equiv (\mathcal{N}_{A\to B} \otimes \mathrm{id}_R)(|\Phi_{\hat{\rho}}\rangle\langle\Phi_{\hat{\rho}}|_{AR})$$

Some remote noisy (Gaussian) entangled state

R_1 ········ B_1

$\hat{\omega}_{B_1 R_1}$

⋮

R_n ········ B_n

$\hat{\omega}_{B_n R_n}$

Entanglement distillation

$n \to \infty$

$nI_c(\hat{\omega})$ perfect Bell pairs

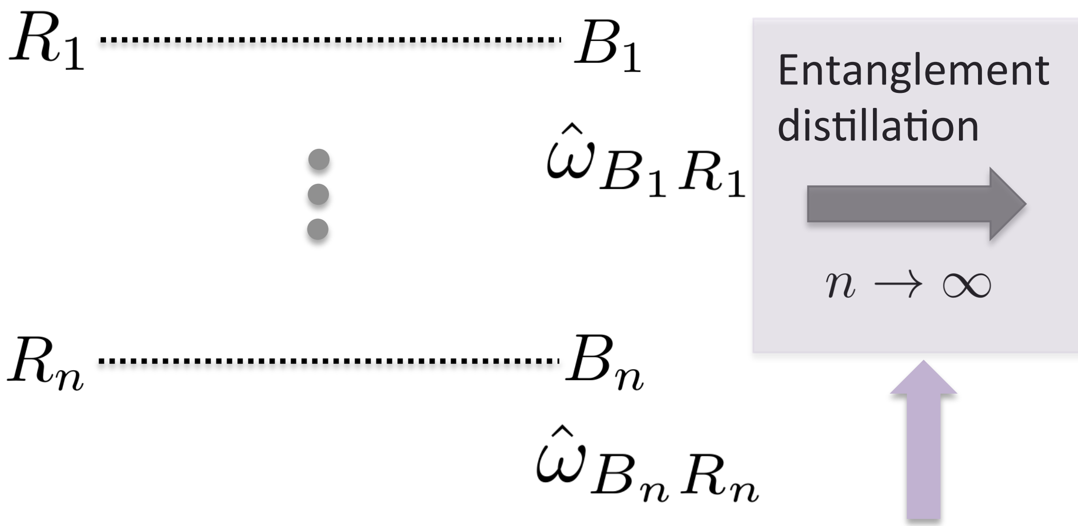
where

$$I_c(\hat{\omega}) = H(\hat{\omega}_B) - H(\hat{\omega}_{BR})$$

I. Devetak and A. Winter, Proc. R. Soc. A **461**, 207–235 (2005)

Entanglement distillation

I. Devetak and A. Winter, Proc. R. Soc. A **461**, 207–235 (2005)



$nI_c(\hat{\omega})$ perfect Bell pairs

where

$$I_c(\hat{\omega}) = H(\hat{\omega}_B) - H(\hat{\omega}_{BR})$$

1. Gaussian operations?

2. Explicit scheme?

1 : Entanglement distillation of Gaussian states with Gaussian operations only is impossible J. Eisert et al., Phys. Rev. Lett. **89**, 137903 (2002)

J. Niset et al., Phys. Rev. Lett. **102**, 120501 (2009)

2. Existence proven based on a random coding argument

(real-time fact-checking needed)

-Aim of this work

Find an **explicit** encoding / decoding strategy that achieves the quantum capacity of Gaussian thermal loss channels

-Earlier result

J. Harrington and J. Preskill, Phys. Rev. A **64**, 062301 (2001)

A family of **Gottesman-Kitaev-Preskill (GKP) codes** achieves the one-shot coherent information of **Gaussian random displacement channels** (or additive noise channels).

-Main result

KN, V. V. Albert, and L. Jiang, arXiv:1801.07271v2 (2018)

A family of **GKP codes** achieves an upper bound of the quantum capacity of **Gaussian thermal loss channels** up to at most a constant gap $\approx \log_2 e = 1.442\ldots$

Gottesman-Kitaev-Preskill (GKP) codes

-Working around the uncertainty principle

Heisenberg uncertainty principle :
position and momentum cannot be measured simultaneously

$$[\hat{q}, \hat{p}] = i \neq 0$$

BUT, can be measured simultaneously in modulo √π

$[\hat{S}_q, \hat{S}_p] = 0$ where

$$\hat{S}_q = e^{i2\sqrt{\pi}\hat{q}} = \hat{D}(i\sqrt{2\pi})$$

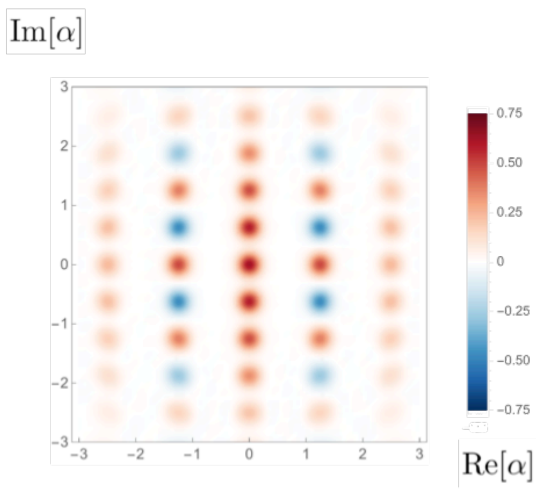
$$\hat{S}_p = e^{-i2\sqrt{\pi}\hat{q}} = \hat{D}(\sqrt{2\pi})$$

: stabilizers of the (square lattice) GKP code

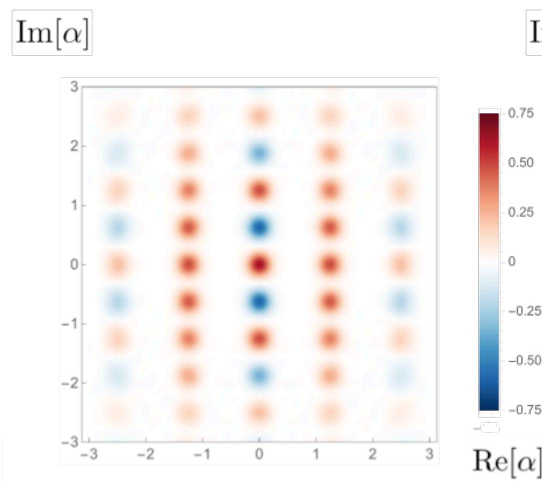
D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001)

Logical states of the (square-lattice) GKP code

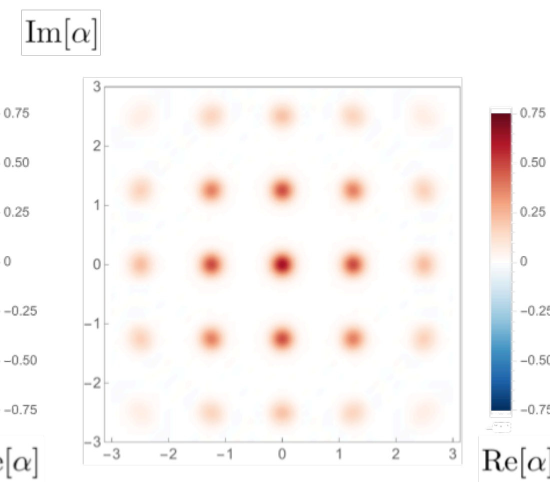
-Wigner function of the logical states



Logical 0



Logical 1



Maximally mixed

$$|0_L\rangle \propto \sum_{n\in\mathbb{Z}} |\hat{q} = (2n)\sqrt{\pi}\rangle \qquad |+_L\rangle \propto |0_L\rangle + |1_L\rangle \propto \sum_{n\in\mathbb{Z}} |\hat{p} = (2n)\sqrt{\pi}\rangle$$

$$|1_L\rangle \propto \sum_{n\in\mathbb{Z}} |\hat{q} = (2n+1)\sqrt{\pi}\rangle \qquad |-_L\rangle \propto |0_L\rangle - |1_L\rangle \propto \sum_{n\in\mathbb{Z}} |\hat{p} = (2n-1)\sqrt{\pi}\rangle$$

$$\Rightarrow \hat{q} = \hat{p} \equiv 0 \bmod \sqrt{\pi}$$

Gaussian random displacement channel

-Definition

$$\mathcal{N}_{B_2}[\sigma^2](\hat{\rho}) = \frac{1}{\pi\sigma^2}\int d^2\alpha e^{-\frac{|\alpha|^2}{\sigma^2}}\hat{D}(\alpha)\hat{\rho}\hat{D}^\dagger(\alpha)$$

σ^2 : variance of random displacement

$\Delta_q = \sqrt{2}\mathrm{Re}[\alpha]$: random displacement in the position quadrature

$\Delta_p = \sqrt{2}\mathrm{Im}[\alpha]$: random displacement in the momentum quadrature

-Decoding the square lattice GKP code

measure $\hat{q}$ and $\hat{p}$ in modulo $\sqrt{\pi}$

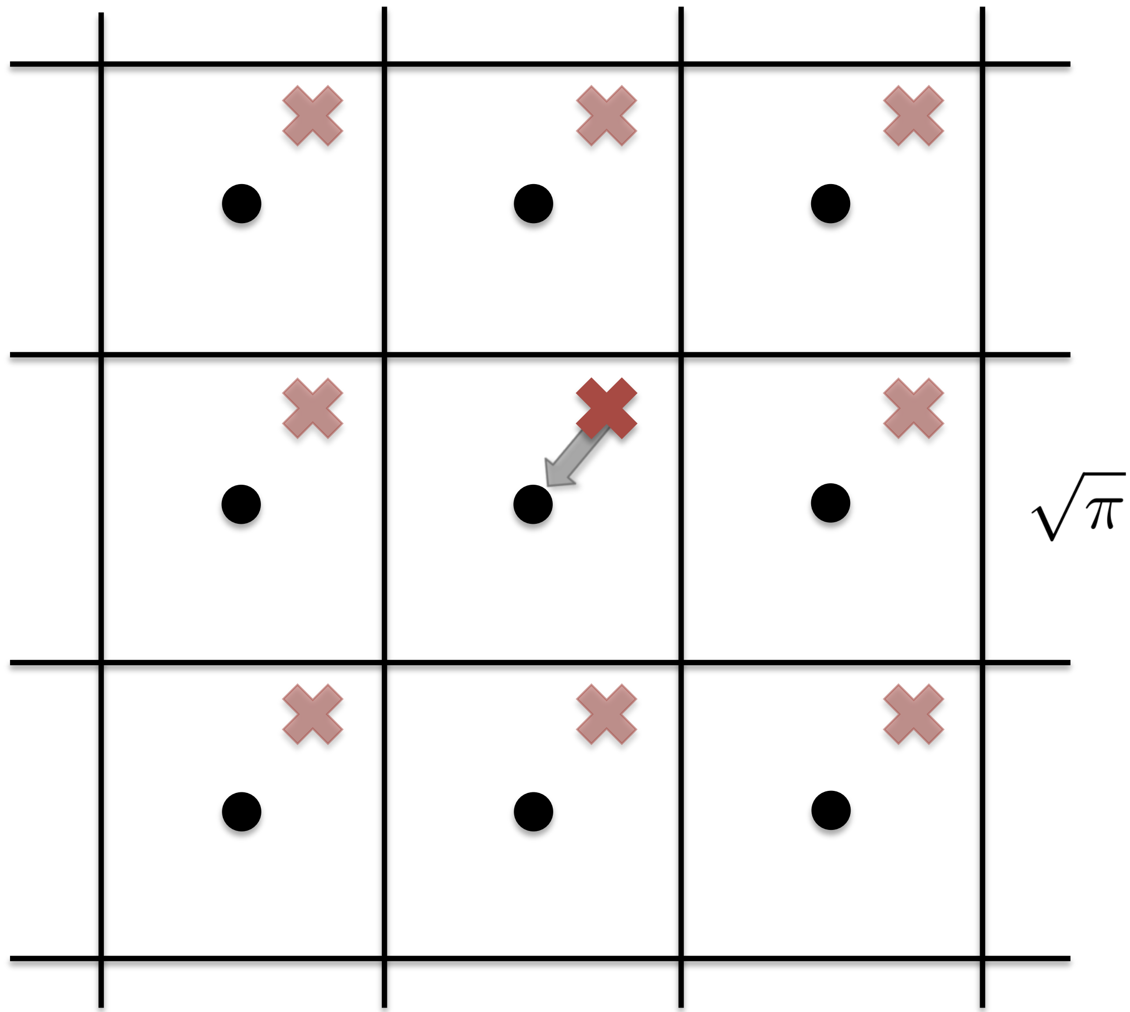
➡ $\Delta_q = \Delta_q^\star + n_q\sqrt{\pi}$ and $\Delta_p = \Delta_p^\star + n_p\sqrt{\pi}$

where $\Delta_q^\star, \Delta_p^\star \in \big(-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}\big]$

➡ infer $\Delta_q = \Delta_q^\star$ and $\Delta_p = \Delta_p^\star$, then implement the counter displacement

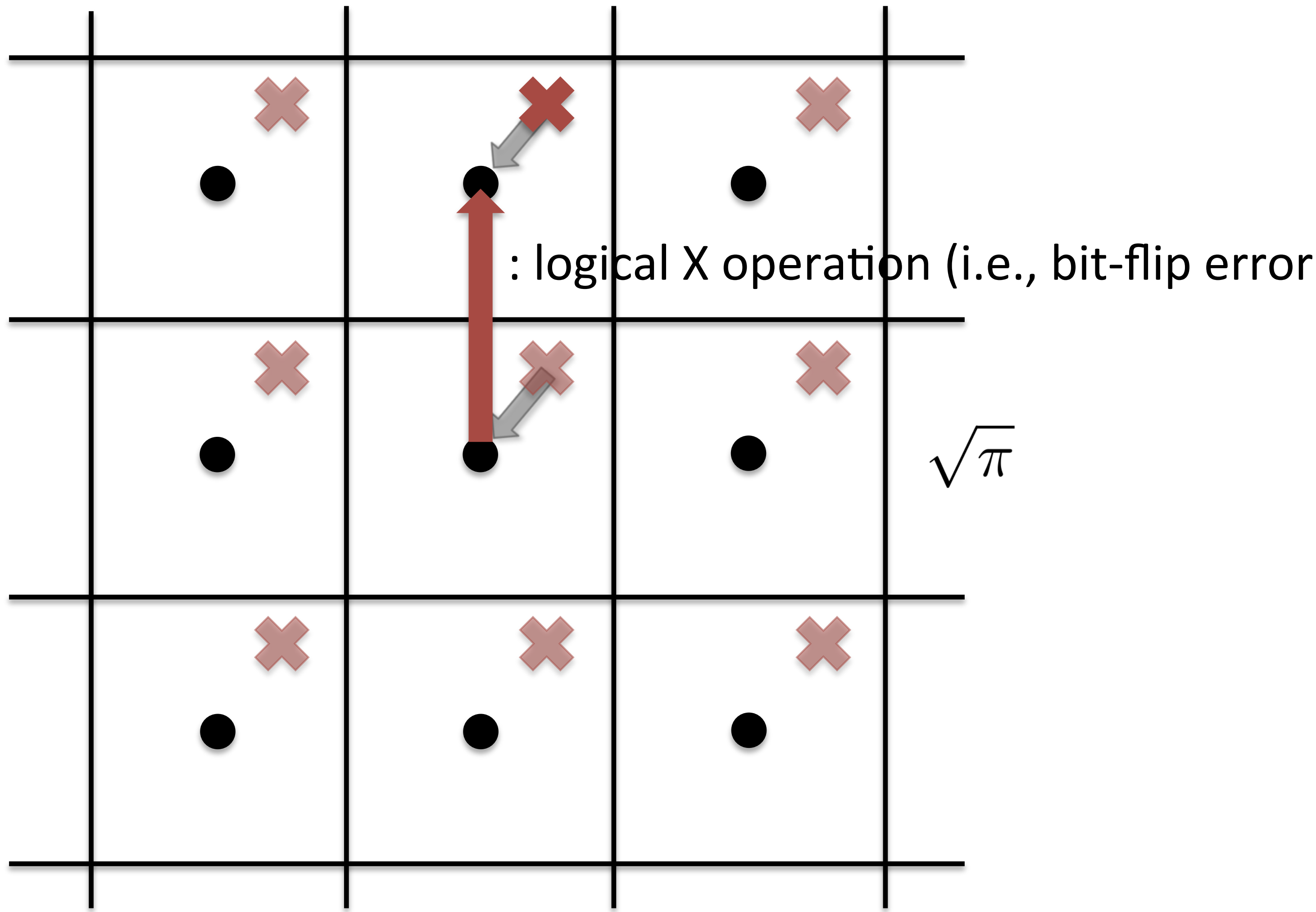
Conventional GKP decoding

-Successful decoding



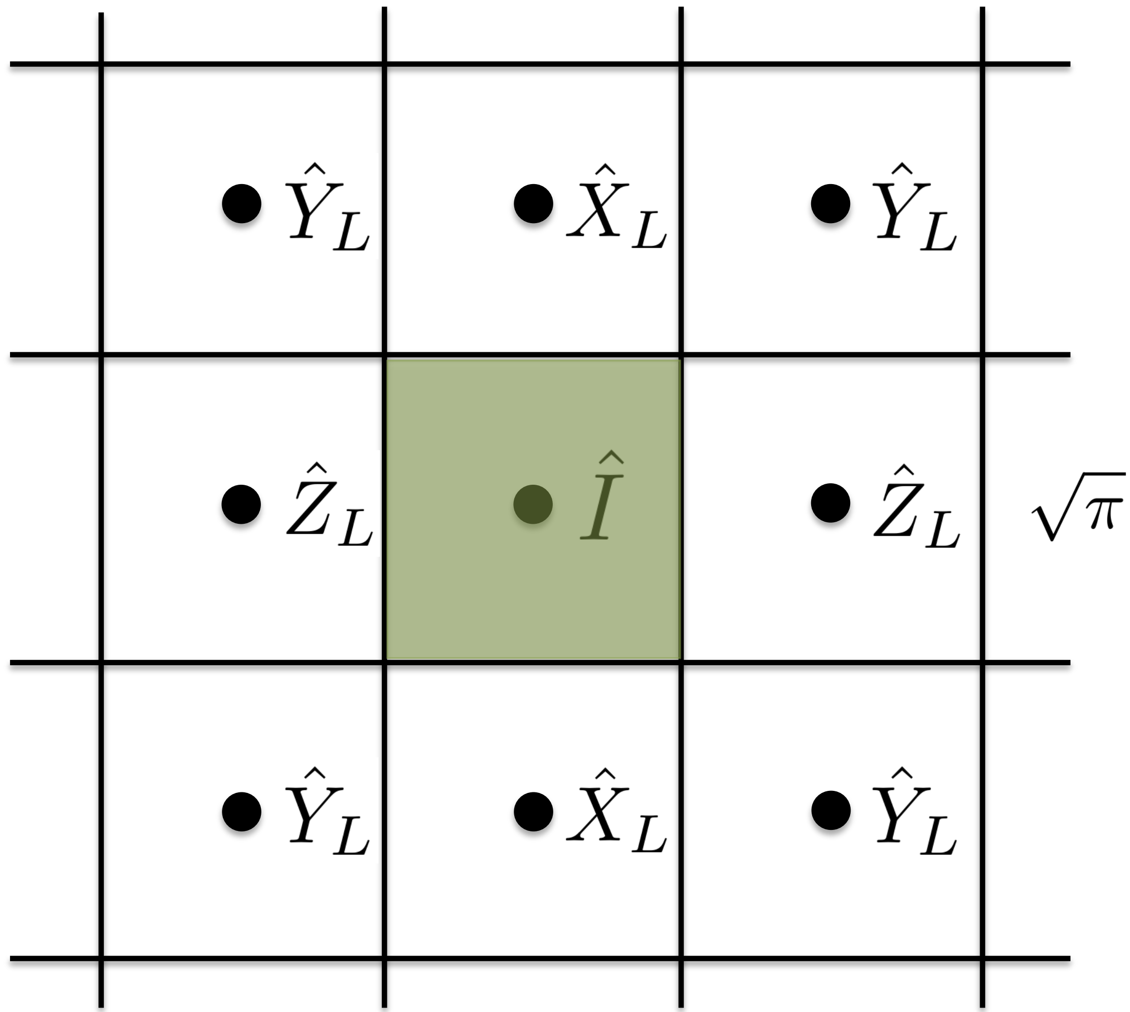
Conventional GKP decoding

-Failed decoding



Conventional GKP decoding

-Failed decoding



Success probability

$$P_{\text{succ}} = \frac{1}{2\pi\sigma^2}\int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} d\Delta_q \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} d\Delta_p e^{-\frac{\Delta_q^2+\Delta_p^2}{2\sigma^2}}$$

$$\geq \frac{1}{2\pi\sigma^2}\int_0^{\frac{\sqrt{\pi}}{2}} r dr \int_0^{2\pi} d\theta e^{-\frac{r^2}{2\sigma^2}}$$

$$= \int_0^{\frac{\pi}{8\sigma^2}} dx e^{-x}$$

$$= 1 - \exp\left[-\frac{\pi}{8\sigma^2}\right]$$

Generalization to symplectic lattices

-Stabilizers

$$\hat{S}_{\boldsymbol{S},q} \equiv \exp\left[i\sqrt{2\pi d}(S_{11}\hat{q} + S_{21}\hat{p})\right]$$

$$\hat{S}_{\boldsymbol{S},p} \equiv \exp\left[-i\sqrt{2\pi d}(S_{12}\hat{q} + S_{22}\hat{p})\right]$$

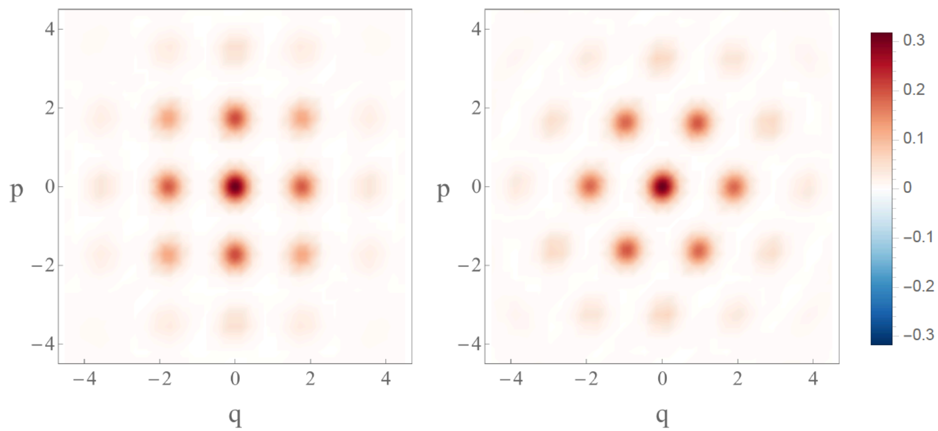
-Generator of the lattice

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$: symplectic matrix

$$\boldsymbol{S\Omega S^T} = \boldsymbol{\Omega}$$ where $$\boldsymbol{\Omega} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Efficient sphere packing

-One mode square vs. hexagonal lattice GKP codes



Wigner function of the maximally mixed code states (code fingerprint)

-Correctable radius of displacement

$r = \sqrt{|\Delta_q|^2 + |\Delta_p|^2}$: Radius of random displacement

Correctable if $r \le r_{\rm sq} = \sqrt{\frac{\pi}{4}}$: square $r \le r_{\rm hex} = \sqrt{\frac{\pi}{2\sqrt{3}}}$: hexagonal

➡ Lattices supporting more dense sphere packing provide better protection!

D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001)
J. Harrington and J. Preskill, Phys. Rev. A **64**, 062301 (2001)

Achievable rate of the GKP codes

-Gaussian random displacement channel

$$\mathcal{N}_{B_2}[\sigma^2](\hat{\rho}) = \frac{1}{\pi\sigma^2}\int d^2\alpha e^{-\frac{|\alpha|^2}{\sigma^2}}\hat{D}(\alpha)\hat{\rho}\hat{D}^\dagger(\alpha)$$

σ^2 : variance of the random displacement

-Achievable rate of the GKP code

J. Harrington and J. Preskill, Phys. Rev. A **64**, 062301 (2001)

$$R = \max\left(\log\left\lfloor\frac{1}{e\sigma^2}\right\rfloor, 0\right)$$

(assuming infinitely many modes and most efficient sphere packing)

-Loss + amplification = displacement

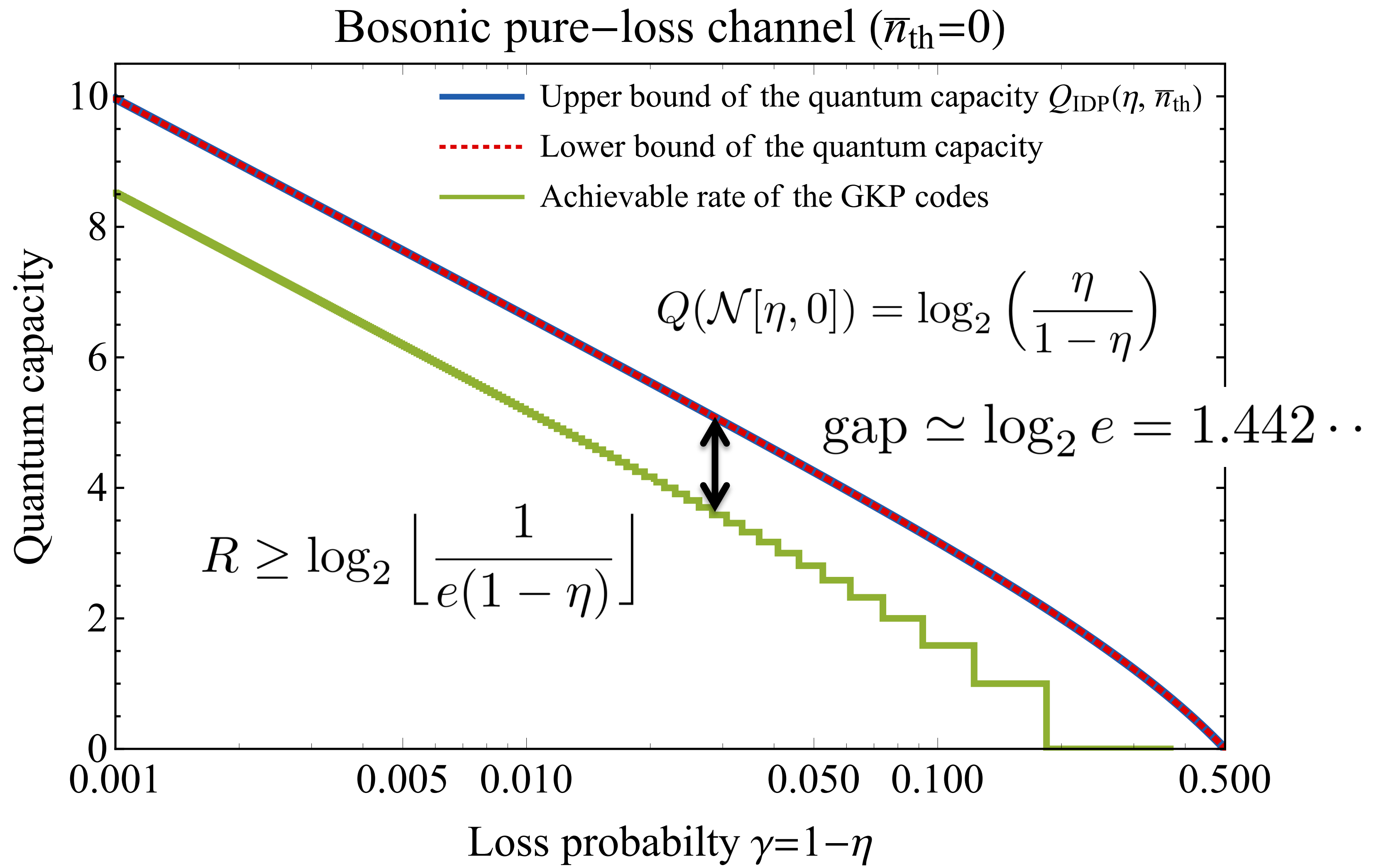
$$\mathcal{N}[\eta, \bar{n}_{th}] \cdot \mathcal{A}[1/\eta] = \mathcal{N}_{B_2}[\tilde{\sigma}^2_{\eta,\bar{n}_{th}}]$$

where $\tilde{\sigma}^2_{\eta,\bar{n}_{th}} \equiv (1-\eta)(\bar{n}_{th}+1)$

V. V. Albert, **KN**, … , L. Jiang, PRA **97**, 032346 (2018)
KN, V. V. Albert, and L. Jiang, arXiv:1801.07271v2 (2018)

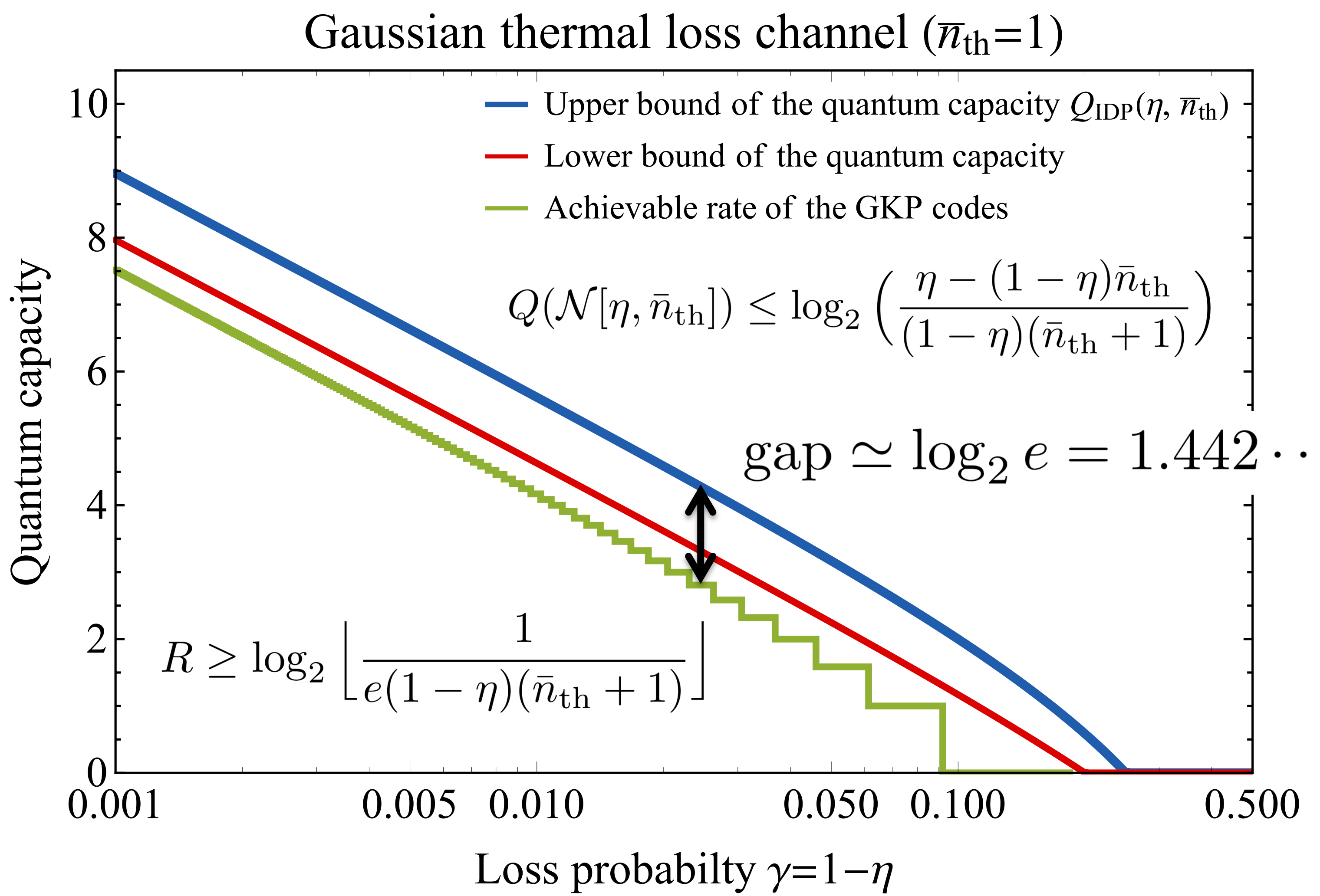
➡ Can use amplification + conventional GKP decoding for the displacement channel to correct loss errors

Achievable rate of the GKP codes



KN, V. V. Albert, and L. Jiang, arXiv:1801.07271v2 (2018)

Achievable rate of the GKP codes

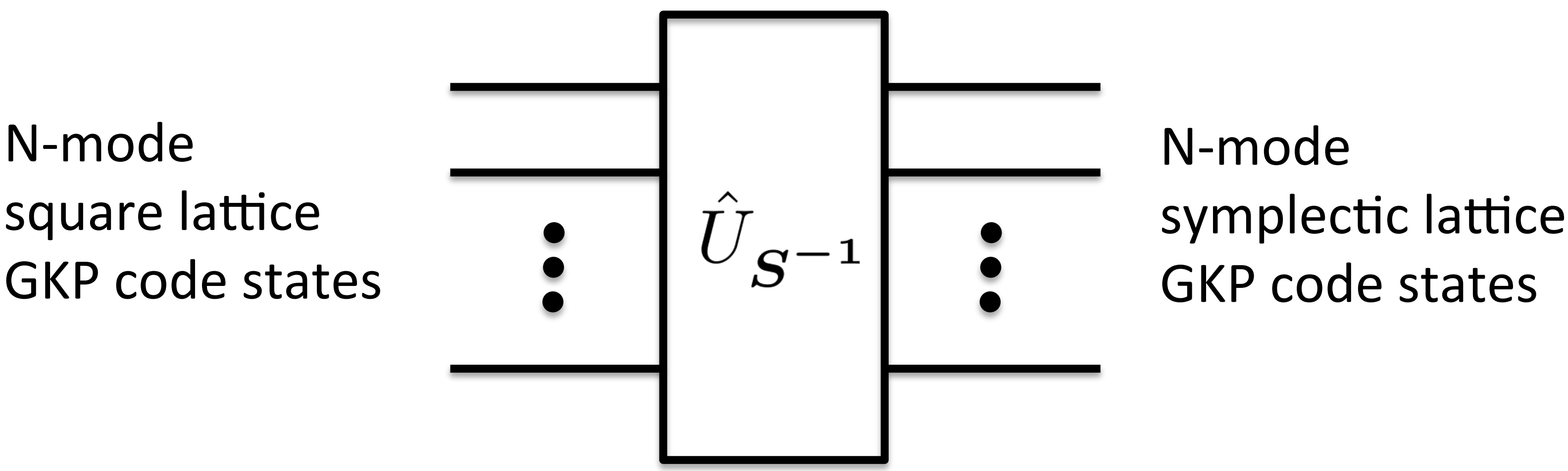


KN, V. V. Albert, and L. Jiang, arXiv:1801.07271v2 (2018)

Several known lattices

E_8 lattice and the Leach lattice Λ_{24} are symplectic, and thus can be used to define 4-mode and 12-mode GKP code, respectively.

-Implementation



Gaussian unitary associated with a symplectic lattice (e.g., E_8 and Λ_{24})

Constant gap from the capacity

- Achievable rate of GKP codes deviates from the quantum capacity by $\approx \log_2 e = 1.442\ldots$
- The decoding we assumed (amplification followed by the conventional GKP decoding) to established the rate is not optimal.
  - Petz recovery may yield a better rate.
- It could also be that GKP codes are not optimal for Gaussian thermal loss channels.
  - Numerical search for better codes (next slide)

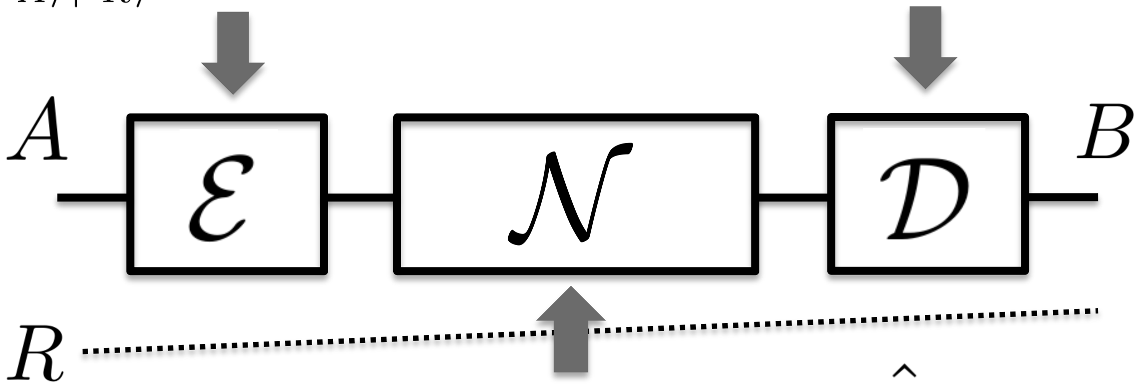
Entanglement fidelity

$|\Phi^+_{AR}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i_A\rangle |i_R\rangle$

: Maximally entangled state

Qudit-into-an-oscillator Encoding (Alice)

Decoding (Bob)



Gaussian thermal loss channel

$\hat{\omega}_{BR}$: approximately maximally entangled state

$$\hat{\omega}_{RB} = (\mathcal{D} \cdot \mathcal{N} \cdot \mathcal{E} \otimes \mathrm{id}_R)(|\Phi^+_{AR}\rangle\langle\Phi^+_{AR}|)$$

$\Rightarrow$ $F_e = \langle \Phi^+_{BR} | \hat{\omega}_{BR} | \Phi^+_{BR} \rangle$: Entanglement fidelity

Maximization of entanglement fidelity

M. Reimpell and R. F. Werner, Phys. Rev. Lett. **94**, 080501 (2005)
A. S. Fletcher, P. W. Shor, and M. Z. Win, Phys. Rev. A **75**, 012338 (2007)

$$\max_{\hat{X}_{\mathcal{E}},\hat{X}_{\mathcal{D}}} \mathrm{Tr}[\hat{X}_{\mathcal{D}}^{\dagger} f_{\mathcal{N}}(\hat{X}_{\mathcal{E}})],$$

$$\text{s.t.} \quad \hat{X}_{\mathcal{D}} = \hat{X}_{\mathcal{D}}^{\dagger} \succeq 0, \; \mathrm{Tr}_{\mathcal{H}_0} \hat{X}_{\mathcal{D}} = \hat{I}_{\mathcal{H}_n},$$

$$\hat{X}_{\mathcal{E}} = \hat{X}_{\mathcal{E}}^{\dagger} \succeq 0, \; \mathrm{Tr}_{\mathcal{H}_n} \hat{X}_{\mathcal{E}} = \hat{I}_{\mathcal{H}_0},$$

$$\text{and } \mathrm{Tr}[(\hat{E} \otimes \hat{I}_{\mathcal{H}_0})\hat{X}_{\mathcal{E}}] \leq \bar{E} d,$$

where $\left(f_{\mathcal{N}}(\hat{X})\right)_{[l'i'],[li]} \equiv \sum_{k,k'=0}^{n-1} (\hat{X}_{\mathcal{N}})_{[kl],[k'l']} (\hat{X})_{[ik],[i'k']}$

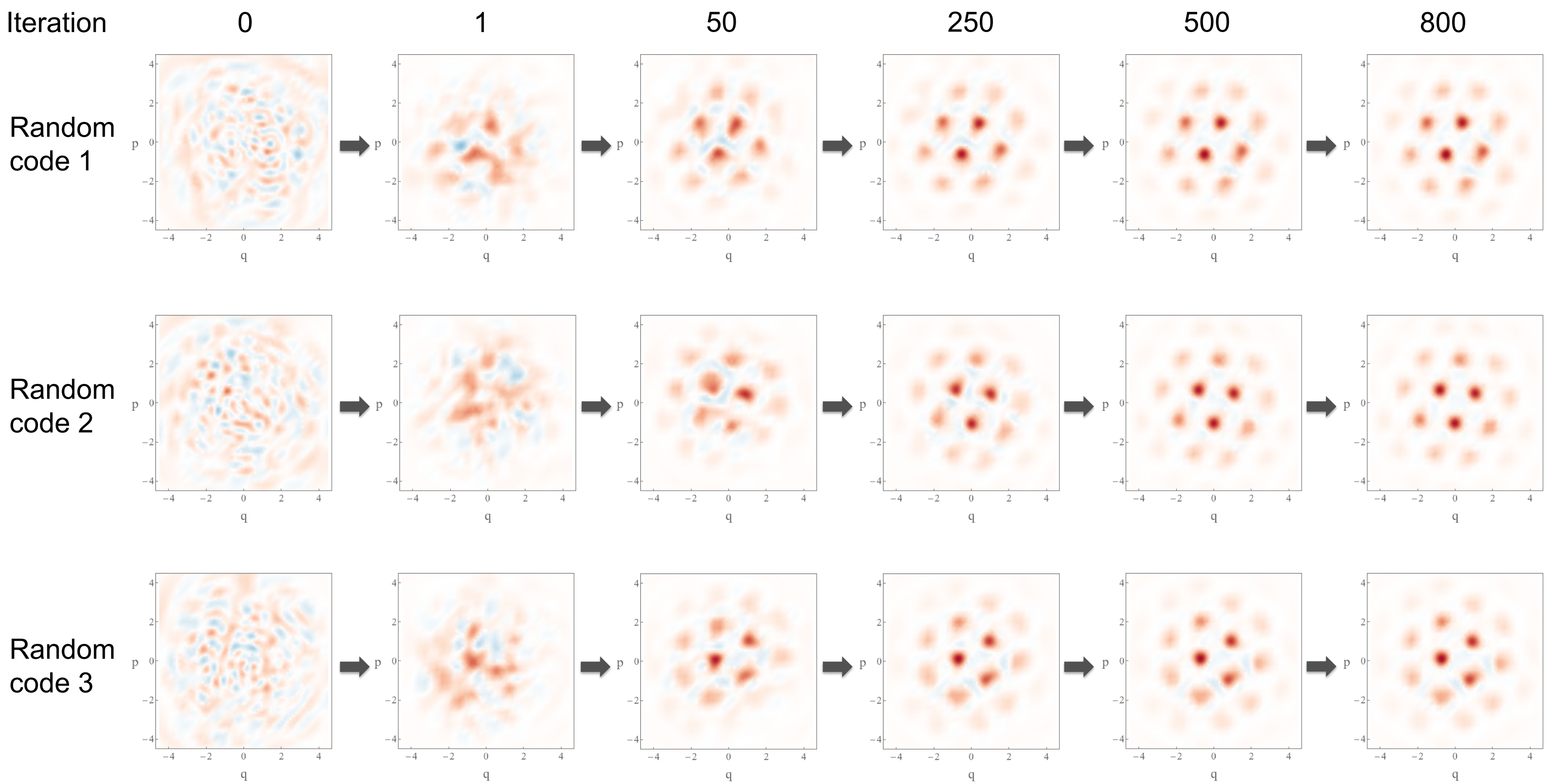
is a linear map

➡ Biconvex optimization : Can be tackled by alternating semidefinite programming (SDP)

R. L. Kosut and D. A. Lidar, Quantum Information Processing **8**, 443–459 (2009)
KN, V. V. Albert, and L. Jiang, arXiv:1801.07271v2 (2018)

Optimization results : GKP supremacy

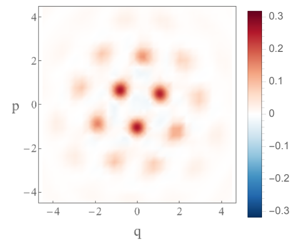
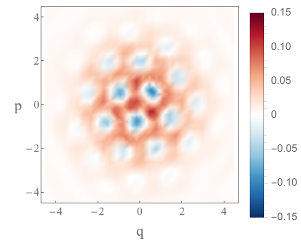
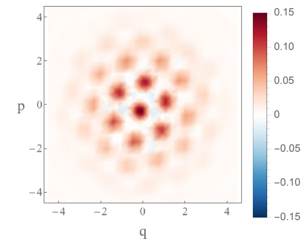
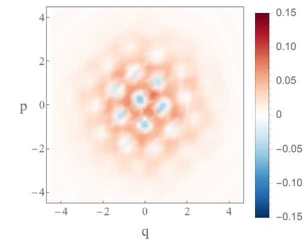
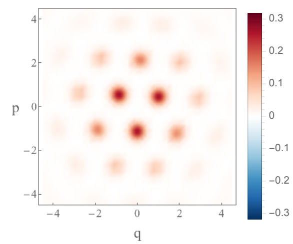
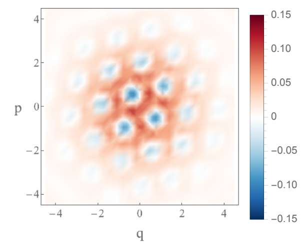
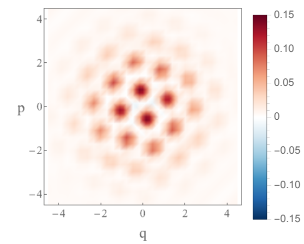
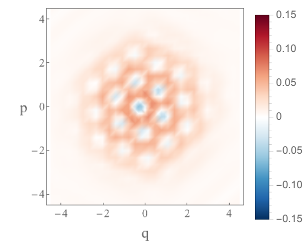
-Qubit-into-an-oscillator code for a pure-loss channel (η=0.9)



A Hexagonal GKP code emerges as an optimal encoding from a random Haar initial code! (n=20 and nbar ≤ 3)

KN, V. V. Albert, L. Jiang, arXiv:1801.07271v2 (2018)

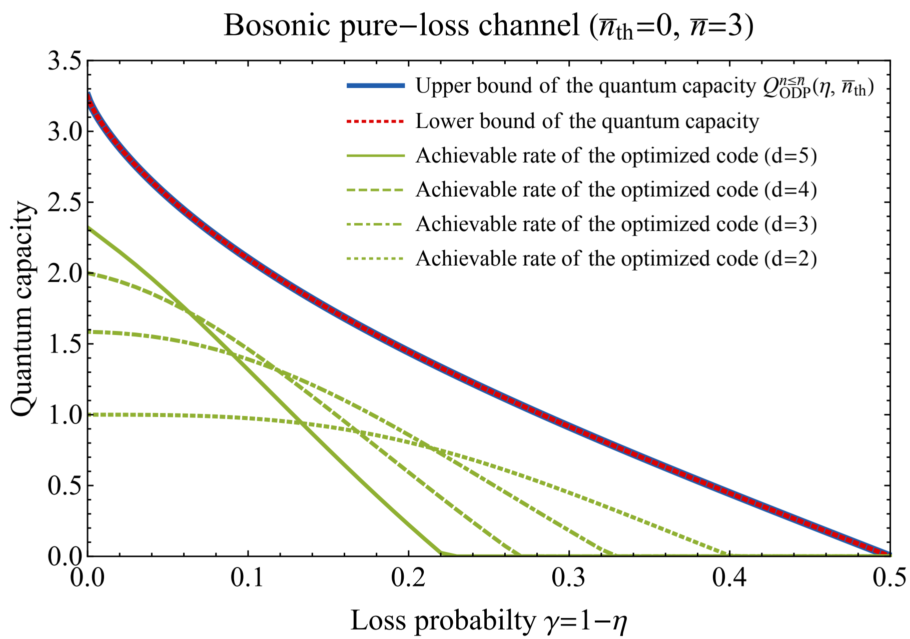
Optimal qudit-into-an-oscillator codes

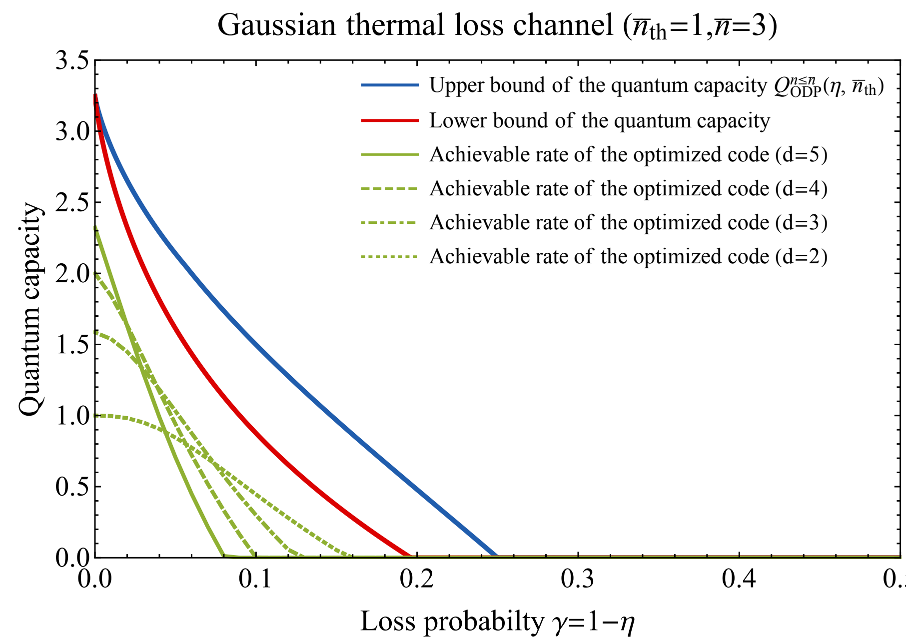
	$d=2$	$d=3$	$d=4$	$d=5$
$\mathcal{N}[\eta,\bar{n}_{\text{th}}]=\mathcal{N}[0.9,0]$				
$\mathcal{N}[\eta,\bar{n}_{\text{th}}]=\mathcal{N}[0.9,1]$				

Again, a Hexagonal GKP code emerges as an optimal encoding from a random Haar initial code both for a bosonic pure-loss channel and a Gaussian thermal loss channel!
(n=30 and nbar ≤ 3)

KN, V. V. Albert, L. Jiang, arXiv:1801.07271v2 (2018)

Achievable rate of the optimized codes





$$\hat{W}(F_e, d) \equiv F_e|\Phi^+\rangle\langle\Phi^+| + \frac{(1-F_e)}{d^2-1}(\hat{I} - |\Phi^+\rangle\langle\Phi^+|)$$

$$\Rightarrow \quad R(F_e, d) = \log d + F_e \log F_e + (1-F_e)\log\left(\frac{1-F_e}{d^2-1}\right)$$

Outlook

- A family of Gottesman-Kitaev-Preskill (GKP) codes achieves the quantum capacity of Gaussian thermal loss channels up to at most a constant ($\approx \log_2 e = 1.442\ldots$) number of qubits per channel use.
- Numerical optimization suggests that the GKP code defined over an optimal lattice (supporting the most efficient sphere packing) may be the optimal encoding for Gaussian thermal loss channels.
- **Open problem : Find the optimal achievable rate of GKP codes using Petz recovery (or a variant of Petz recovery).**

For more details please see

V. V. Albert, **KN**, … , L. Jiang, PRA **97**, 032346 (2018)

KN, V. V. Albert, L. Jiang, arXiv:1801.07271v2 (2018)