Universitat Autònoma de Barcelona

Grup d'Informació Quàntica

Bounds on Information Combining Revisited

Rocky Mountain Summit on Quantum Information, Boulder

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Física Teòrica: Informació i Fenòmens Quàntics

Information Combining

Quick recall!

Given random variables with side information (X_1, Y_1) and (X_2, Y_2):

- What do we know about $X_1 + X_2$ given $Y_1 Y_2$ and in particular $H(X_1 + X_2 | Y_1 Y_2)$?

Here simplest setting: binary random variables.

Without conditioning:

$$X_1 \sim \begin{bmatrix} p \\ 1-p \end{bmatrix}, \quad X_2 \sim \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

$$\Downarrow$$

$$X_1 + X_2 \sim \begin{bmatrix} pq + (1-p)(1-q) \\ p(1-q) + q(1-p) \end{bmatrix} \equiv \begin{bmatrix} p \star q \\ 1 - p \star q \end{bmatrix}$$

Therefore

$$H(X_1 + X_2) = h(p \star q) = h(h^{-1}(H(X_1)) \star h^{-1}(H(X_2)).$$

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With conditioning

Classical **bounds on information combining**.
Write $H(X_i|Y_i) = H_i$,

$$h(h^{-1}(H_1) \star h^{-1}(H_2)) \leq H(X_1 + X_2|Y_1 Y_2) \leq \log 2 - \frac{(\log 2 - H_1)(\log 2 - H_2)}{\log 2}$$

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Main ingredients:

$$g_c(H_1, H_2) := h(h^{-1}(H_1) * h^{-1}(H_2))$$

is convex in H_1 for fixed H_2, and vice versa, and

$$H(X|Y) = \sum_y p(y) H(X|Y = y).$$

With conditioning

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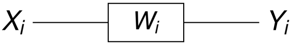
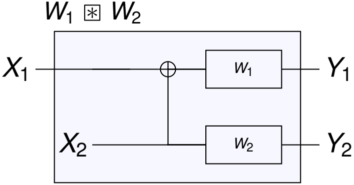
With $H_1 = H_2 = H$,

$$\begin{aligned} 0.799 \frac{H(\log 2 - H)}{\log 2} &\leq h(h^{-1}(H) \star h^{-1}(H)) - H \\ &\leq H(X_1 + X_2 | Y_1 Y_2) - H \\ &\leq \frac{H(\log 2 - H)}{\log 2} \end{aligned}$$

Channel picture



$$H(X_i|Y_i) = H(W_i)$$



$$H(X_1 + X_2|Y_1 Y_2) = H(W_1 \circledast W_2)$$

Lower Bound: Summary

Here for the simple $H_1 = H_2 = H$ case:

$$
\begin{aligned}
&H(X_1 + X_2|B_1 B_2) - H \\
&= I(A:C|B)_\tau && \textit{QCMI} \\
&\geq -2\log F(\tau_{ACB}, \mathcal{R}'_{B\to AB}(\tau_{CB})) && \textit{Fawzi} - \textit{Renner} \\
&\geq -2\log\cos\left[\frac{1}{2}\arccos[f^2] - \frac{1}{2}\arccos f\right] && \Delta - \textit{ineq.} \\
&\geq -2\log\cos\Big[\frac{1}{2}\arccos[(1-2h_2^{-1}(\log 2 - H))^2] && \textit{Concavity} \\
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&\Rightarrow \begin{cases} 0.083\cdot\frac{H}{1-\log H}, & H \leq \frac{1}{2}\log 2 \\ 0.083\cdot\frac{\log 2 - H}{1-\log(\log 2 - H)}, & H > \frac{1}{2}\log 2. \end{cases} && \textit{Duality/Simplify}
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with $f := F(\rho_0, \rho_1)$.

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Channel Duality

For every channel W we can define a dual channel $W^{\perp}$.

Additional uncertainty relation

$$H(W) = \log 2 - H(W^{\perp})$$

and symmetry relation

$$\begin{aligned} & H(W_1 \boxast W_2) - \left(H(W_1) + H(W_2)\right)/2 \\ =& H(W_1^{\perp} \boxast W_2^{\perp}) - \left(H(W_1^{\perp}) + H(W_2^{\perp})\right)/2. \end{aligned}$$

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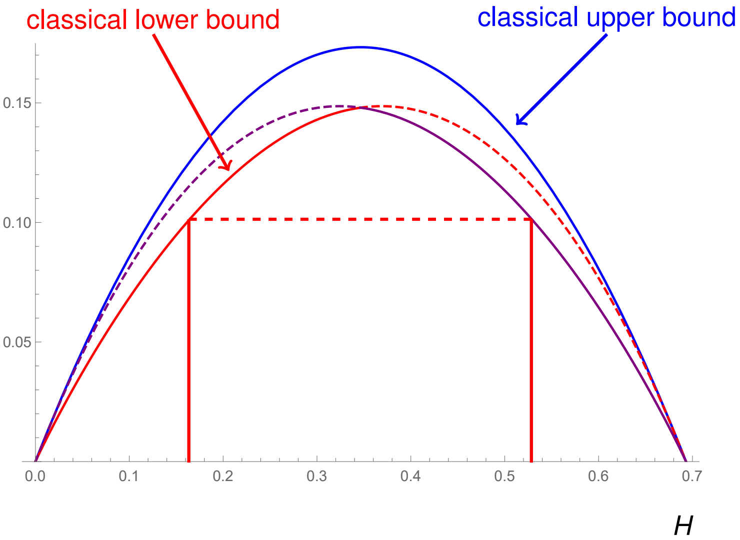
$$\begin{aligned} & H(W_1 \circledast W_2) - (H(W_1) + H(W_2))/2 \\ = & H(W_1^{\perp} \circledast W_2^{\perp}) - \left(H(W_1^{\perp}) + H(W_2^{\perp})\right)/2. \end{aligned}$$

Channel Duality



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Proposition 2 in *Renes, 2017*

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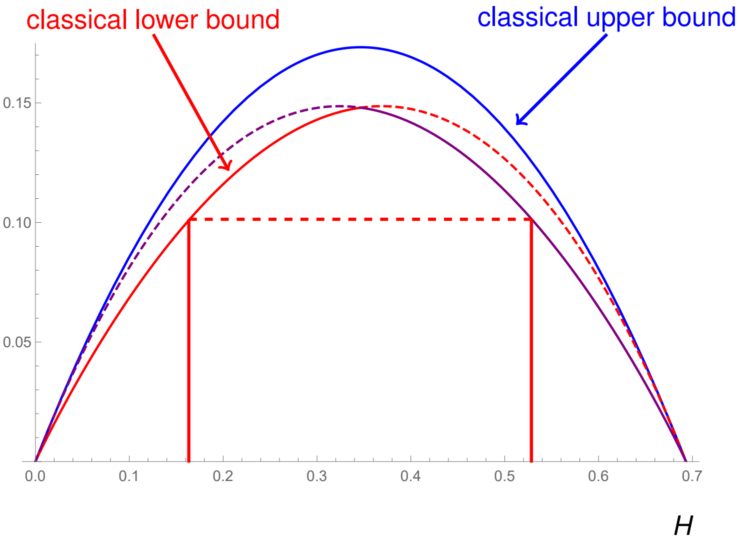
Theorem

Any bound on $H(X_1 + X_2|Y_1 Y_2)$ proven for symmetric channels, also holds for asymmetric channels.

Channel Duality



CQMI

Conditional Quantum Mutual Information

$$H(X_1 + X_2 | B_1 B_2) - H_1 = I(X_1 + X_2 : X_2 | B_1 B_2)$$

Lower bounds on CQMI

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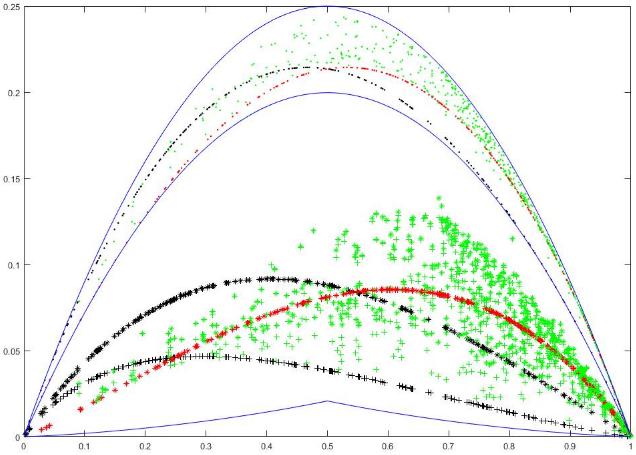
Wait!

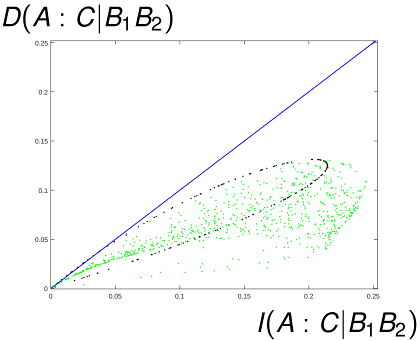
What about other recoverability bounds?

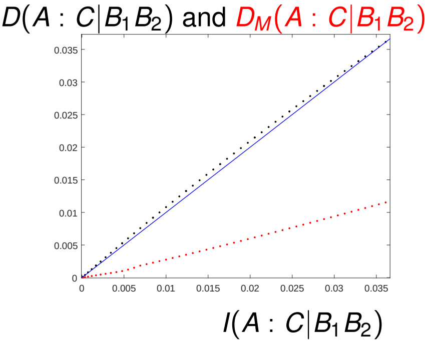
$$I(A : C|B)_\tau \not\geq D(A : C|B)$$
$$I(A : C|B)_\tau \geq D_M(A : C|B)$$

Recovery bounds



Isn't the $D(A:C|B)$ bound wrong?





It still is! (Also for A and C classical)

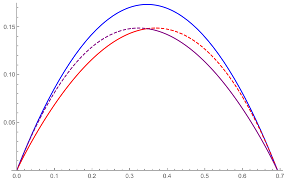
Conjectured Bounds

Let $\rho^{X_1 B_1}$ and $\rho^{X_2 B_2}$ be cq.-states with H_1 and H_2. Then:

$$H(X_1 + X_2|B_1 B_2) - (H_1 + H_2)$$
$$\geq \begin{cases} h(h^{-1}(H_1) * h^{-1}(H_2)) - (H_1 + H_2) & H_1 + H_2 \leq \log 2 \\ h(h^{-1}(\log 2 - H_1) * h^{-1}(\log 2 - H_2)) - \log 2 & H_1 + H_2 \geq \log 2 \end{cases}$$

and

$$H(X_1 + X_2|B_1 B_2) \leq \log 2 - \frac{(\log 2 - H_1)(\log 2 - H_2)}{\log 2}.$$



Renyi entropies

Four conditional quantum Renyi entropies.

$$\bar{H}_\alpha^{\downarrow}(A|B) = -\bar{D}_\alpha(\rho_{AB} \| \mathbb{1} \otimes \rho_B)$$
$$\bar{H}_\alpha^{\uparrow}(A|B) = \sup_{\sigma_B} -\bar{D}_\alpha(\rho_{AB} \| \mathbb{1} \otimes \sigma_B)$$
$$\tilde{H}_\alpha^{\downarrow}(A|B) = -\tilde{D}_\alpha(\rho_{AB} \| \mathbb{1} \otimes \rho_B)$$
$$\tilde{H}_\alpha^{\uparrow}(A|B) = \sup_{\sigma_B} -\tilde{D}_\alpha(\rho_{AB} \| \mathbb{1} \otimes \sigma_B)$$

Reduce classically to:

$$H_\alpha^{\uparrow}(X|Y) = \frac{\alpha}{1-\alpha} \log \left(\sum_y p(y) \left(\sum_x p(x|y)^\alpha \right)^{\frac{1}{\alpha}} \right)$$

$$H_\alpha^{\downarrow}(X|Y) = \frac{1}{1-\alpha} \log \left(\sum_y \sum_x p(y) p(x|y)^\alpha \right),$$

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$$H^{\downarrow}_{\alpha}(X|Y) = \frac{1}{1-\alpha}\log\left(\sum_y\sum_x p(y)p(x|y)^{\alpha}\right),$$

Renyi entropies

We have the following equalities:

$$H_\alpha^\uparrow(X|Y) = \frac{\alpha}{1-\alpha}\log\left(\sum_y p(y)e^{\frac{1-\alpha}{\alpha}H_\alpha^\uparrow(X|Y=y)}\right)$$

$$H_\alpha^\downarrow(X|Y) = \frac{1}{1-\alpha}\log\left(\sum_y p(y)e^{(1-\alpha)H_\alpha^\downarrow(X|Y=y)}\right).$$

This motivates us to define the following quantities:

$$K_\alpha^\uparrow(X|Y) = e^{\frac{1-\alpha}{\alpha}H_\alpha^\uparrow(X|Y)}$$

$$K_\alpha^\downarrow(X|Y) = e^{(1-\alpha)H_\alpha^\downarrow(X|Y)}.$$

Renyi entropies

In analogy to the Shannon entropy case, we get

$$H_\alpha(X_1 + X_2) = h_\alpha(h_\alpha^{-1}(H_\alpha(X_1) \star h_\alpha^{-1}(H_\alpha(X_2))).$$

We will see that the crucial quantity in the Renyi setting is the following

$$\mathbb{k}_\alpha^\star(x, y) = k_\alpha^\star\left(k_\alpha^{\star\,-1}(x) \star k_\alpha^{\star\,-1}(y)\right)$$

for $\star \in \{\uparrow, \downarrow\}$.

Renyi entropies

Theorem (BSC-bound)

If, for a given α and $\star \in \{\uparrow, \downarrow\}$, the function $\mathbb{k}_\alpha^\star(x, y)$ is convex in x for fixed y and vice versa, then one of the two following equations holds:
If $\alpha > 1$, then

$$H_\alpha^\star(X_1 + X_2 | Y_1 Y_2) \leq h_\alpha(h_\alpha^{-1}(H_\alpha^\star(X_1|Y_1)) \star h_\alpha^{-1}(H_\alpha^\star(X_2|Y_2))).$$

If $\alpha < 1$, then

$$H_\alpha^\star(X_1 + X_2 | Y_1 Y_2) \geq h_\alpha(h_\alpha^{-1}(H_\alpha^\star(X_1|Y_1)) \star h_\alpha^{-1}(H_\alpha^\star(X_2|Y_2))).$$

If $\mathbb{k}_\alpha^\star(x, y)$ is concave instead, the inequalities hold with $\leq$ and $\geq$ exchanged. These bounds are optimal, in the sense that equality is achieved by binary symmetric channels.

Renyi entropies

Theorem (BEC-bound)

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$$H_\alpha^\downarrow(X_1 + X_2|Y_1Y_2) \geq \frac{1}{1-\alpha} \log \frac{(\delta_\alpha^\downarrow - K_\alpha^\downarrow(X_1|Y_1))(\delta_\alpha^\downarrow - K_\alpha^\downarrow(X_2|Y_2))}{1-\delta_\alpha^\downarrow} + \delta_\alpha^\downarrow,$$

with $\delta_\alpha^\uparrow = 2^{\frac{1-\alpha}{\alpha}}$ and $\delta_\alpha^\downarrow = 2^{1-\alpha}$. If $\alpha < 1$, the inequalities hold with $\leq$ and $\geq$ exchanged.
If $\mathbb{k}_\alpha^\star(x, y)$ is concave instead, the inequalities hold with $\leq$ and $\geq$ exchanged.
These bounds are optimal, since equality is achieved by binary erasure channels.

Renyi entropies

Lemma (Convexity result from HASC18)

For $\alpha \geq 2$, the function

$$k_\alpha^\uparrow ({k_\alpha^\uparrow}^{-1}(x) \star {k_\alpha^\uparrow}^{-1}(y))$$

is convex in x for fixed y and vice versa.

Renyi entropies

Conjecture

There exists a value $\hat{\alpha}$, such that

$$k_\alpha^\uparrow({k_\alpha^\uparrow}^{-1}(x) \star {k_\alpha^\uparrow}^{-1}(y))$$

is convex for $0 < \alpha < 1$ and $\alpha \geq \hat{\alpha}$ and concave for $1 < \alpha \leq \hat{\alpha}$.
Numerics suggests that $1.6 < \hat{\alpha} < 1.7$.

Renyi entropies

Lemma

The function

$$k_\alpha^\downarrow({k_\alpha^\downarrow}^{-1}(x) \star {k_\alpha^\downarrow}^{-1}(y))$$

is convex for $0 < \alpha < 1$ *and* $2 < \alpha \leq 3$ *and concave for* $1 < \alpha \leq 2$ *and* $\alpha \geq 3$.

Renyi entropies

Lemma

The function

$$k_\alpha^{\downarrow}({k_\alpha^{\downarrow}}^{-1}(x) \star {k_\alpha^{\downarrow}}^{-1}(y))$$

is linear in x and y for $\alpha = 2$ and $\alpha = 3$.

This lemma is interesting as it tells us that the BSC-bound and the BEC-bound both hold with equality. We have

$$H_2^{\downarrow}(X_1 + X_2 | Y_1 Y_2) = h_2(h_2^{-1}(H_2^{\downarrow}(X_1|Y_1)) \star h_2^{-1}(H_2^{\downarrow}(X_2|Y_2)))$$
$$H_3^{\downarrow}(X_1 + X_2 | Y_1 Y_2) = h_3(h_3^{-1}(H_3^{\downarrow}(X_1|Y_1)) \star h_3^{-1}(H_3^{\downarrow}(X_2|Y_2))).$$

These equations are remarkable as they give an equality in the conditional case, something we usually only get for unconditioned entropies.

Renyi entropies

And quantum?
Difficult to make a conjecture! For symmetry from duality we needed two main tools:

- Uncertainty:
$$H(W) = \log 2 - H(W^{\perp})$$
- Chain rule for mutual information leading to:

$$H(X_1 + X_2|Y_1 Y_2) + H(X_2|X_1 + X_2, Y_1 Y_2) = H(X_1|Y_1) + H(X_2|Y_2),$$

Entropy optimization

Alternative way of investigating convexity?

Witsenhausen and Wyner (1975) investigated the following optimization problem:

$$F(x) = \min_{\substack{p(w|x) \\ H(X|W)\geq x}} H(Y|W) \, .$$

It was furthermore shown that $F(x)$ is always convex and that when $p(y|x)$ is given by a binary symmetric channel with channel parameter δ, then the following holds

$$F_{BSC}(x) = h(h^{-1}(x) \star \delta).$$

Due to the convexity of $F(x)$, this substitutes an important step in the information combining proofs.

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It was furthermore shown that $F(x)$ is always convex and that when $p(y|x)$ is given by a binary symmetric channel with channel parameter δ, then the following holds

$$F_{BSC}(x) = h(h^{-1}(x) \star \delta).$$

Due to the convexity of $F(x)$, this substitutes an important step in the information combining proofs.

Entropy optimization

Alternative way of investigating convexity?
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Information Bottleneck

The task can be expressed by the following rate function:

$$R(x) = \max_{\substack{p(w|x) \\ I(W;X) \leq x}} I(W;Y).$$

First note, that an equivalent rate function is given by (in the sense, that is describes the same curve)

$$\hat{R}(x) = \min_{\substack{p(w|x) \\ I(W;Y) \geq x}} I(W;X),$$

which is a very common alternative formulation.
It can easily be seen that

$$R(x) = H(Y) - F(H(X) - x). \tag{1}$$

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And quantum information?

Quantum versions of the information bottleneck have recently been investigated by Salek, Cadamuro, Kammerlander and Wiesner (2017). In particular an information theoretic task was given with optimal rate given by a quantum generalization of the information bottleneck. The given generalization is as follows:

$$R_Q(x) = \min_{\substack{\mathcal{N}_{X\to W} \\ I(W;Y)_{\rho_{WY}} \geq x}} I(X';W)_{\tau_{X'W}} ,$$

with $\tau_{X'X}$ a purification of ρ_X.

Unfortunately the result was not fully proven, but relies on the conjecture that $R_Q(x)$ is convex in x. Note that in the classical case this convexity leads to the convexity which is a crucial step in proving information combining inequalities.

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Wrap-up

Results:

- Sufficiency of symmetric channels.
- Counterexamples for recovery conjecture in ccq-case.
- Information combining for Renyi entropies.

...but even more open problems!

- Prove conjectures!
- Extensions to other (quantum) entropies.
- Conditioning, information bottleneck, partial orders on channels...

Ask me for notes!

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Lower bound

Main ingredient

$$g_c(H_1, H_2) := h(h^{-1}(H_1) * h^{-1}(H_2))$$

is convex in H_1 for fixed H_2, and vice versa.

$$\begin{aligned}
&H(X_1 + X_2 | Y_1 Y_2) \\
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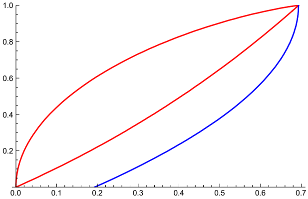
Concavity of von Neumann Entropy

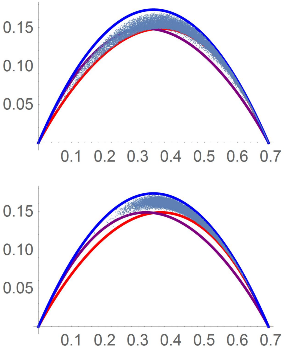
Let $\rho_i \in \mathcal{B}(\mathbb{C}^d)$ and $\{p_i\}_{i=1}^n$ be a probability distribution.

$$H\left(\sum_{i=1}^n p_i\rho_i\right) - \sum_{i=1}^n p_i H(\rho_i)$$
$$\geq H(\{p_i\}) - \log\left(1 + 2\sum_{1\leq i<j\leq n} \sqrt{p_i p_j} F(\rho_i, \rho_j)\right).$$

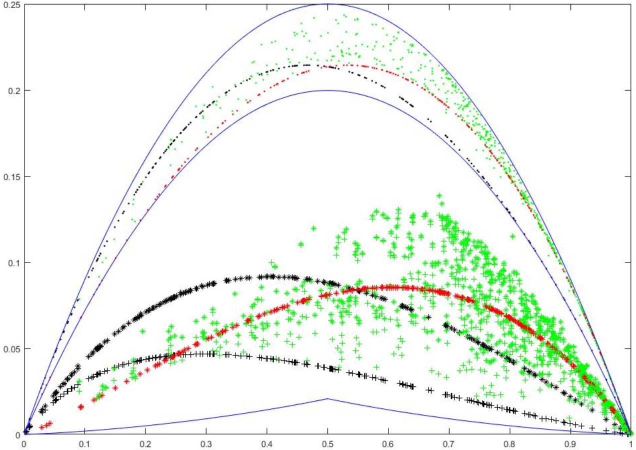


Evidence

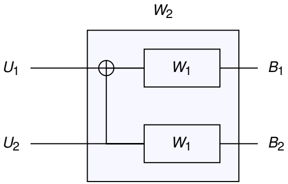
- States with Equality.
- Numerics:



Recovery bounds



What are polar codes?



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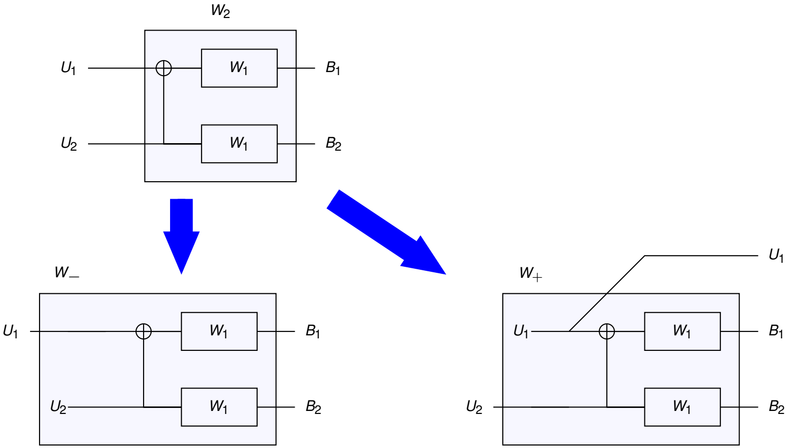

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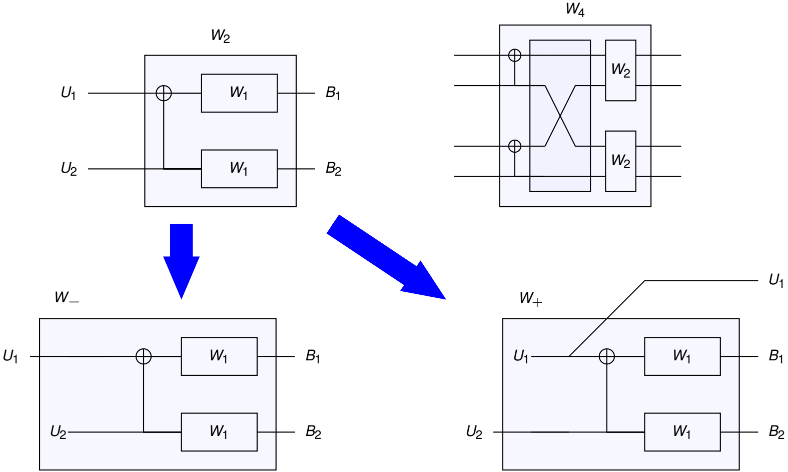

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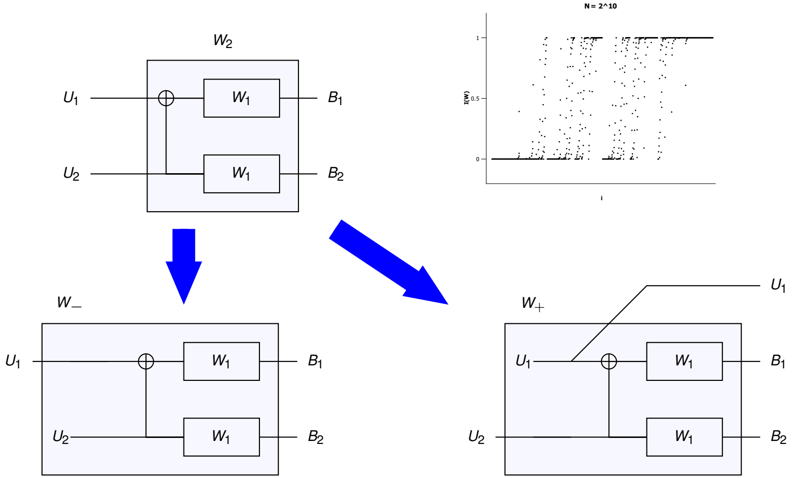
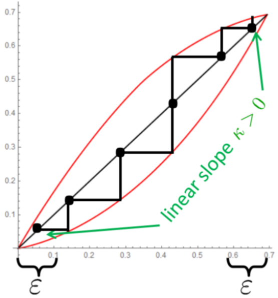
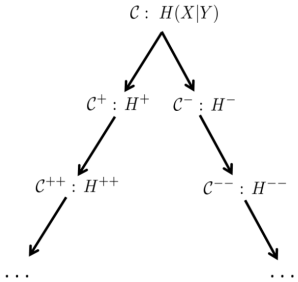


Speed of Polarization





$\#$ steps to reach $[0, \epsilon] \cup [\log 2 - \epsilon, \log 2]$: $n \approx \frac{1}{\kappa} \log \frac{1}{\epsilon}$
$\Rightarrow$ Rate $R = I(W) - \epsilon$ with polynomial blocklength $\approx poly(1/\epsilon)$.

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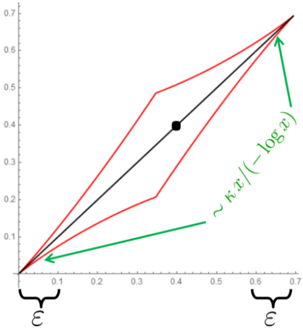

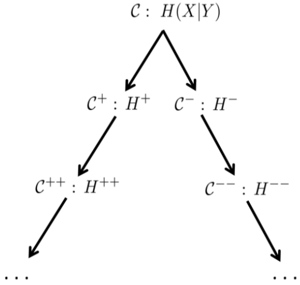

$\#$ steps to reach $[0, \epsilon] \cup [\log 2 - \epsilon, \log 2]$: $n \approx \frac{1}{\kappa}(\log \frac{1}{\epsilon})^2$
$\Rightarrow$ Rate $R = I(W) - \epsilon$ with subexponential blocklength $\approx (1/\epsilon)^{\log(1/\epsilon)}$.

Non-stationary channels

Also:
Bounds for $H_1 \neq H_2$ give

- a conceptually simple proof of polarization (without martingales),
- that also works for non-stationary channels.