Separating Matrix-valued Quantum Correlation via Embezzlement and Teleportation

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Joint work with Samuel J. Harris and Marius Junge

Quantum correlations

Quantum correlations are probabilistic correlations arose from quantum measurement. Suppose both Alice and Bob has d measurements (POVM) and each POVM has m values, i.e on some Hilbert space H

$$\sum_{a=1}^{m} E_a^x = \sum_{b=1}^{m} F_b^y = 1 \, , \qquad x, y = 1, \cdots, d \, .$$

We consider the correlation matrix $[p(a,b|x,y)]_{\substack{1 \le x,y \le d \\ 1 \le a,b \le m}}$ in $(\mathbb{R}_+)^{d^2 m^2}$ given

$$p(a,b|x,y) = \langle \psi | E_a^x F_b^y | \psi \rangle$$

where $|\psi\rangle \in H$ is a unit vector and $[E_a^x, F_b^y] = 0$.

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Correlation under different assumptions

A correlation matrix $\{p(a,b|x,y)\}$ is called

- *local* if $p(a,b|x,y) = \sum_n \lambda_n p_n(a|x) q_n(b|y)$ ($dimH = 1$, hidden variable)
- *quantum* if $p(a,b|x,y) = \langle\psi|E_a^x \otimes F_b^y|\psi\rangle$, where $|\psi\rangle \in H = H_A \otimes H_B$ with $dimH_A, dimH_B < \infty$, $\{E_a^x\}$ PVMs on H_A for $\{F_b^y\}$ PVMs on H_B.
  (finite dimensional separable quantum systems)
- *quantum spatial* if $p(a,b|x,y) = \langle\psi|E_a^x \otimes F_b^y|\psi\rangle$, where we allow $dimH_A = dimH_B = \infty$.
  (Infinite dimensional separable quantum systems)
- *quantum commuting* if $p(a,b|x,y) = \langle\psi|E_a^x F_b^y|\psi\rangle$, where E_a^x, F_b^y PVMs on H and $[E_a^x, F_b^y] = 0$.
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Hierarchy of correlation sets

Let $C_{loc}, C_q, C_{qs}, C_{qc}$ be the set of respectively the *local*, *quantum*, *quantum spatial*, *quantum commuting* correlations set. For each (d, m),

$$C_{loc}(d,m) \subset C_q(d,m) \subset C_{qs}(d,m) \subset \overline{C_{qs}(d,m)} \subset C_{qc}(d,m) \ .$$

- $d = 1$: collapse to classic correlations, $C_{loc}(1,m) = C_{qc}(1,m)$
- $C_{loc}(2,2) \neq C_q(2,2) = C_{qc}(2,2)$: Bell's inequality/CHSH game [Bell,'64] and experiments e.g. [Hanson et al,'15]

For $d, m \geq 2, (d,m) \neq (2,2)$, $C_{loc}(d,m) \neq C_q(d,m)$.

- Tsirelson's Problem '93: $\overline{C_{qs}}(d,m) \stackrel{?}{=} C_{qc}(d,m)$ for all $(d,m) > (2,2)$. Equivalent to Connes Embedding Problem! [Junge et al,'11],[Fritz '11],[Ozawa, '13].
- [Slofstra, '17]: for $d \approx 100$, $C_{qs}(d,8) \neq \overline{C_{qs}(d,8)}$. Explicit examples. [Dykma-Paulsen-Prakash, '17]: $C_{qs}(5,2) \neq \overline{C_{qs}(5,2)}$. Existence proof.
- [Coladangelo-Stark, '18]: $C_q(5,4) \neq C_{qs}(5,4)$. Explicit examples.

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Open Questions

C_q vs. C_{qs} vs. $\overline{C_{qs}}$ at smallest possible sizes?

$$C_q(3,2) \neq C_{qs}(3,2) \neq \overline{C_{qs}(3,2)}\ ?, \quad C_q(2,3) \neq C_{qs}(2,3) \neq \overline{C_{qs}(2,3)}\ ?$$

Matrix valued quantum correlations

We consider the following matrix-valued quantum correlations:

$$C^n_{qs}(d,m) = \left\{ \left[V^*(E^x_a \otimes F^y_b)V\right]_{a,b}^{x,y} \;\middle|\; \begin{array}{c} H_A, H_B \text{ Hilbert spaces,} \\ V : \mathbb{C}^n \to H_A \otimes H_B \text{ an isometry,} \\ (E^x_a)_{x=1}^m \text{ PVMs on } H_A \text{ for each } a \\ (F^y_b)_{y=1}^m \text{ PVMs on } H_B \text{ for each } b \end{array} \right\}$$

$C^n_{qc}(d,m) = [E^x_a, F^y_b] = 0$ on one Hilbert space H

$C^n_q(d,m) = H_A$ and H_B finite dimensional

Quantum correlations set $C_q(d,m) = C^1_q(d,m)$

$C^n_q(d,m)$: a vector $\psi \rightsquigarrow n$-dimensional subspace. Corresponds to games requires quantum answers to classical questions.

- [Junge et al,'11],[Fritz '11]: For any fixed $(d,m) > (2,2)$,

$$\overline{C^n_{qs}(d,m)} = C^n_{qc}(d,m) \;\forall\, n \geq 1 \;\Leftrightarrow\; \text{CEC}$$

- [Ozawa, '13]:

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Main result

Theorem (G.-Harris-Junge, 2017)

$C^3_{qs}(4,2) \neq \overline{C^3_{qs}(4,2)}$, $C^5_{qs}(3,2) \neq \overline{C^5_{qs}(3,2)}$, $C^{13}_{qs}(2,3) \neq \overline{C^{13}_{qs}(2,3)}$

An infinite dimensional correlation

Lemma

Let H be a Hilbert space. There exist unitaries u_0, u_1, v_0, v_1 on H and orthonormal vectors $|h_0\rangle, |h_1\rangle \in H \otimes H$ satisfying

$$u_0 \otimes v_0|h_0\rangle = u_1 \otimes v_1|h_0\rangle = (|h_0\rangle + |h_1\rangle)/\sqrt{2}\ ,$$
$$u_0 \otimes v_1|h_0\rangle = u_1 \otimes v_0|h_0\rangle = (|h_0\rangle - |h_1\rangle)/\sqrt{2},$$

if and only if H is infinite dimensional.

Proof. Denote $u := {u_1}^* u_0$, $v := {v_1}^* v_0$.

$$u \otimes 1|h_0\rangle = 1 \otimes v|h_0\rangle = |h_0\rangle\ , u \otimes 1|h_1\rangle = 1 \otimes v|h_1\rangle = -|h_0\rangle\ .$$

$$|h_0\rangle \in H_+ \otimes K_+\ , |h_1\rangle \in H_- \otimes K_-$$

where H_+ (resp. K_+) is the eigenspace of u (resp. v) of eigenvalue 1, H_- (resp. K_-) is the eigenspace of u (resp. v) of eigenvalue -1. $H_+ \perp H_-, K_+ \perp K_-$, hence for Schmidt Rank "Sr",

$$Sr(|h_0\rangle + |h_1\rangle) = Sr(|h_0\rangle) + Sr(|h_1\rangle)\ , Sr(|h_0\rangle + |h_1\rangle) = Sr(|h_0\rangle)\ .$$

$\Rightarrow Sr(|h_0\rangle) = \infty.$

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$$u \otimes 1(|h_0\rangle + |h_1\rangle) = 1 \otimes v(|h_0\rangle + |h_1\rangle) = (|h_0\rangle - |h_1\rangle) \ ,$$
$$u \otimes v(|h_0\rangle + |h_1\rangle) = (|h_0\rangle + |h_1\rangle)$$

$$u \otimes 1|h_0\rangle = 1 \otimes v|h_0\rangle = |h_0\rangle \ , u \otimes 1|h_1\rangle = 1 \otimes v|h_1\rangle = -|h_0\rangle \ .$$

$$|h_0\rangle \in H_+ \otimes K_+ \ , |h_1\rangle \in H_- \otimes K_-$$

where H_+ (resp. K_+) is the eigenspace of u (resp. v) of eigenvalue 1, H_- (resp. K_-) is the eigenspace of u (resp. v) of eigenvalue −1.

An infinite dimensional correlation

Lemma

Let H be a Hilbert space. There exist unitaries u_0, u_1, v_0, v_1 on H and orthonormal vectors $|h_0\rangle, |h_1\rangle \in H \otimes H$ satisfying

$$u_0 \otimes v_0 |h_0\rangle = u_1 \otimes v_1 |h_0\rangle = (|h_0\rangle + |h_1\rangle)/\sqrt{2}\ ,$$
$$u_0 \otimes v_1 |h_0\rangle = u_1 \otimes v_0 |h_0\rangle = (|h_0\rangle - |h_1\rangle)/\sqrt{2},$$

if and only if H is infinite dimensional.

Proof. Denote $u := {u_1}^* u_0$, $v := {v_1}^* v_0$.

$$u \otimes 1 |h_0\rangle = 1 \otimes v |h_0\rangle = |h_0\rangle\ , u \otimes 1 |h_1\rangle = 1 \otimes v |h_1\rangle = -|h_0\rangle\ .$$
$$|h_0\rangle \in H_+ \otimes K_+\ , |h_1\rangle \in H_- \otimes K_-$$

where H_+ (resp. K_+) is the eigenspace of u (resp. v) of eigenvalue 1, H_- (resp. K_-) is the eigenspace of u (resp. v) of eigenvalue -1.

$H_+ \perp H_-, K_+ \perp K_-$, hence for Schmidt Rank "Sr",

$$Sr(|h_0\rangle + |h_1\rangle) = Sr(|h_0\rangle) + Sr(|h_1\rangle)\ , Sr(|h_0\rangle + |h_1\rangle) = Sr(|h_0\rangle)\ .$$

$\Rightarrow Sr(|h_0\rangle) = \infty.$

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$\Rightarrow Sr(|h_0\rangle) = \infty$.

A non-spatial correlation

Lemma

There are no unitary $u_0, u_1, u_2, v_0, v_1, v_2$ *on* H *and two orthonormal vector* $|h_0\rangle, |h_1\rangle \in H \otimes H$ *satisfying the equations*

$$u_0 \otimes v_0|h_0\rangle = u_1 \otimes v_1|h_0\rangle = (|h_0\rangle + |h_1\rangle)\sqrt{2}\ , \tag{1}$$

$$u_0 \otimes v_1|h_0\rangle = u_1 \otimes v_0|h_0\rangle = (|h_0\rangle - |h_1\rangle)\sqrt{2}\ , \tag{2}$$

$$u_2 \otimes v_2|h_0\rangle = |h_1\rangle\ . \tag{3}$$

Proof. (1), (2) implies

$$\text{Schmidt coefficients}\left(\frac{|h_0\rangle + |h_1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\text{Schmidt coefficients}(|h_0\rangle) \cup \frac{1}{\sqrt{2}}\text{Schmidt coefficients}(|h_1\rangle)\ .$$

(3) implies $|h_0\rangle$, $|h_1\rangle$ and $\frac{1}{\sqrt{2}}(|h_0\rangle + |h_1\rangle)$ have the same Schmidt coefficients, which is a contradiction.

Theorem (van Dam-Hayden,'02)

For any $\epsilon > 0$, there exists a finite dimensional H, a unit vector $|R\rangle \in H \otimes H$ and unitaries U, V on $\mathbb{C}^2 \otimes H$ such that

$$\| U \otimes V |0\rangle|R\rangle|0\rangle - \frac{1}{\sqrt{2}}(|0\rangle|R\rangle|0\rangle + |1\rangle|R\rangle|1\rangle) \| \leq \epsilon \, .$$

Let $X = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Choose

$$u_0 = U, v_0 = V, u_1 = (X \otimes 1)U, v_1 = (X \otimes 1)V,$$
$$u_2 = v_2 = Z \otimes 1,$$
$$|h_0\rangle = |0\rangle|\mathcal{R}\rangle|0\rangle, |h_1\rangle = |1\rangle|R\rangle|1\rangle.$$

When $\epsilon \to 0$, $u_0 \otimes v_0 |h_0\rangle \sim \frac{|h_0\rangle + |h_1\rangle}{\sqrt{2}} \sim u_1 \otimes v_1 |h_0\rangle$

$u_0 \otimes v_1 |h_0\rangle \sim \frac{|h_0\rangle - |h_1\rangle}{\sqrt{2}} \sim u_1 \otimes v_0 |h_0\rangle$

Approximation by Embezzlement of Entanglement

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Matrix-valued correlation by group embeddings

- n-valued PVM $\sum_{j=0}^{n-1} E_j = 1 \Leftrightarrow$ order n unitary $u = \sum_{j=0}^{n-1} e^{\frac{2\pi j i}{n}} E_j$.
- In previous lemma, we have two order 2 unitary X, Z and an abstract unitary U in the limit $\epsilon \to 0$ has infinity order, hence corresponds to a vector state of the unitary representation of

$$\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z} \times \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$$

- Four 2-valued quantum correlations $\rightsquigarrow$ states of unitary representation of

$$\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \times \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$

- Using group embedding, $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z} \hookrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

$$X \mapsto \sigma_0 \ , \ Z \mapsto \sigma_1 \ , \ U \mapsto \sigma_2\sigma_3 \ .$$

- We have a non-spatial state witnessed on the set

$$\{\sigma_0, \sigma_1, \sigma_2\sigma_3, \sigma_1\sigma_2\sigma_3\} \otimes \{\sigma_0, \sigma_1, \sigma_2\sigma_3, \sigma_1\sigma_2\sigma_3\}$$

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A concrete example in $\overline{C_{qs}^3(4,2)}$

Let $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ be self adjoint unitaries. $\sigma_x = E_x^+ - E_x^-$

$$\Psi(\sigma_0 \otimes \sigma_0) = \begin{bmatrix} 0 & 1 & * \\ 1 & 0 & * \\ * & * & * \end{bmatrix}, \Psi(\sigma_2 \otimes \sigma_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} * & * & 1 \\ * & * & 1 \\ 1 & 1 & * \end{bmatrix},$$

$$\Psi(\sigma_3 \otimes \sigma_3) = \begin{bmatrix} * & * & 1 \\ * & * & * \\ 1 & * & * \end{bmatrix}, \Psi(\sigma_1 \otimes \sigma_1) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ * & * & * \end{bmatrix},$$

$$\Psi(\sigma_1 \otimes 1) = \begin{bmatrix} 1 & 0 & * \\ 0 & -1 & * \\ * & * & * \end{bmatrix}, \Psi(1 \otimes \sigma_1) = \begin{bmatrix} 1 & 0 & * \\ 0 & -1 & * \\ * & * & * \end{bmatrix},$$

Separation between C_q^n vs C_{qs}^n.

Theorem (G.-Harris-Junge, 2017)

$C_{qs}^3(4,2) \neq \overline{C_{qs}^3(4,2)}$, $C_{qs}^5(3,2) \neq \overline{C_{qs}^5(3,2)}$, $C_{qs}^{13}(2,3) \neq \overline{C_{qs}^{13}(2,3)}$

Theorem (Harris, 2018)

$C_q^2(3,2) \neq C_{qs}^2(3,2)$, $C_q^4(2,3) \neq C_{qs}^4(3,2)$.

Observation:

C_{qs} $vs.$ $\overline{C_{qs}}$: infinite "amount" of entanglement (entropy)

C_q $vs.$ C_{qs}: infinite "rank" of entanglement (Schmidt rank)

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Prefect embezzlement of entanglement

Prefect embezzlement of entanglement: a unit vector $|R\rangle \in H \otimes H$ and unitaries U, V on $\mathbb{C}^2 \otimes H$ such that

$$U \otimes V|0\rangle|R\rangle|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|R\rangle|0\rangle + |1\rangle|R\rangle|1\rangle).$$

Let $U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$ and $V = \begin{bmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{bmatrix}$. This means

$$\langle R|u_{00} \otimes v_{00}|R\rangle = \langle R|u_{10} \otimes v_{10}|R\rangle = \frac{1}{\sqrt{2}} ,$$
$$\langle R|u_{10} \otimes v_{00}|R\rangle = \langle R|u_{00} \otimes v_{10}|R\rangle = 0 .$$

Teleportation and dense coding for unitary

$$2 \times 2 \text{ unitary } U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} \xrightarrow{\mathcal{T}} \text{4 unitaries}$$

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}, \ (1 \otimes Z)U(1 \otimes Z) = \begin{bmatrix} u_{00} & -u_{01} \\ -u_{10} & u_{11} \end{bmatrix},$$

$$(1 \otimes X)U(1 \otimes X) = \begin{bmatrix} u_{11} & u_{10} \\ u_{01} & u_{00} \end{bmatrix}, \ (1 \otimes Y)U(1 \otimes Y) = \begin{bmatrix} u_{11} & -u_{10} \\ -u_{01} & u_{00} \end{bmatrix}$$

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$$4 \text{ unitaries } g_1, g_2, g_3, g_4 \xrightarrow{\mathcal{S}} 2 \times 2 \text{ unitary entries :}$$

$$u_{00} = \frac{1}{2}\begin{bmatrix} g_1 + g_2 & \\ & g_3 + g_4 \end{bmatrix}, u_{01} = \frac{1}{2}\begin{bmatrix} & g_1 - g_2 \\ g_3 - g_4 & \end{bmatrix},$$

$$u_{10} = \frac{1}{2}\begin{bmatrix} & g_3 - g_4 \\ g_1 - g_2 & \end{bmatrix}, u_{11} = \frac{1}{2}\begin{bmatrix} g_3 + g_4 & \\ & g_1 + g_2 \end{bmatrix},$$

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4 unitaries $g_1, g_2, g_3, g_4 \xrightarrow{\mathcal{S}} 2 \times 2$ unitary entries:

$$U = \frac{1}{2} \begin{bmatrix} g_1 + g_2 & & & g_1 - g_2 \\ & g_3 + g_4 & g_3 - g_4 & \\ & g_3 - g_4 & g_3 + g_4 & \\ g_1 - g_2 & & & g_1 + g_2 \end{bmatrix}$$

C^*-Teleportation and Dense Coding

$C^*(\mathbb{F}_4)$: the "universal" C^*-algebra of four unitaries $g_{00}, g_{10}, g_{01}, g_{11}$.

$\mathcal{B}_2$: the "universal" C^*-algebra of $u_{00}, u_{01}, u_{10}, u_{11}$ such that $\begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$ is a unitary.

Theorem (G.-Harris-Junge, 2017)

The teleportation and dense coding scheme gives the following C^*-algebras embedding:

$$B_2 \overset{\mathcal{T}}{\hookrightarrow} M_2(C^*(\mathbb{F}_4)) \longrightarrow B_2 \ , \ C^*(\mathbb{F}_4) \overset{\mathcal{S}}{\hookrightarrow} M_2(B_2) \longrightarrow C^*(\mathbb{F}_4)$$

$$\begin{array}{ccccc}
\mathcal{B}_2 \otimes \mathcal{B}_2 & \xrightarrow{\mathcal{T}\otimes\mathcal{T}} & M_2(C^*(\mathbb{F}_4)) \otimes M_2(C^*(\mathbb{F}_4)) & \xrightarrow{\mathcal{S}^{(2)}\otimes\mathcal{S}^{(2)}} & M_4(\mathcal{B}_2) \otimes M_4(\mathcal{B}_2) \\
\downarrow \psi & \swarrow \phi & & & \downarrow \simeq \\
\mathbb{C} & & \xleftarrow{\rho\otimes\psi} & & M_4 \otimes M_4 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_2)
\end{array}$$

ψ is the state of perfect embezzlement. ρ is an entangled bit.

ϕ is our M_2-valued non-spatial correlation for unitaries.

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All Teleportation and Super Dense Coding Schemes

Let ϕ be a maximally entangled state.

Theorem (Werner, 2001)

There are one to one correspondence between the following objects:

1. *Super dense coding Schemes* $|j\rangle\langle j| \to |\phi_j\rangle = U_j \otimes 1|\phi\rangle \to |j\rangle\langle j|$
2. *Orthonomal bases of Maximally entangled states.* $|\phi_j\rangle = U_j \otimes 1|\phi\rangle$
3. *Unitary orthonomal bases in Hilbert-Schmidt norm i.e.* $tr(U_k^* U_j) = \delta_{jk}$
4. *Teleportation schemes* $\rho \to \frac{1}{d^2}\sum_{j=1} U_j \rho U_j^* \otimes |j\rangle\langle j| \to \rho$

Theorem (G.-Werner, 2017)

Teleportation & dense coding schemes are one to one corresponds

i) *completely positive* 1*-complemented embeddings to*

$$S_1^d \overset{\mathcal{T}}{\hookrightarrow} M_d(l_1^{d^2}) \longrightarrow S_1^d \ , \ l_1^{d^2} \overset{\mathcal{S}}{\hookrightarrow} M_d(S_1^d) \longrightarrow l_1^{d^2} \ .$$

ii) *injective* $*$*-homomorphism which sends* $\{u_{jk}\}$ *to* $M_d(span\{g_{jk}\})$

$$B_d \overset{\mathcal{T}}{\hookrightarrow} M_d(C^*(\mathbb{F}_{d^2})) \longrightarrow B_d \ , \ C^*(\mathbb{F}_{d^2}) \overset{\mathcal{S}}{\hookrightarrow} M_d(B_d) \longrightarrow C^*(\mathbb{F}_{d^2})$$

Summary

1. We have separation of Matrix valued quantum correlation for C_q^n $vs.$ C_{qs}^n $vs.$ $\overline{C_{qs}^n}$ at smallest measurement size $(d, m) = (3, 2), (2, 3)$.
2. Our correlations are explicit.
3. An interesting key lemma relates to embezzlement of entanglement. A quantum game scenario?

Remains Open

$C_q(3, 2) \neq C_{qs}(3, 2) \neq \overline{C_{qs}(3, 2)}$?, $C_q(2, 3) \neq C_{qs}(2, 3) \neq \overline{C_{qs}(2, 3)}$?

Thank you for your attention!