Extendibility limits the performance of quantum processors

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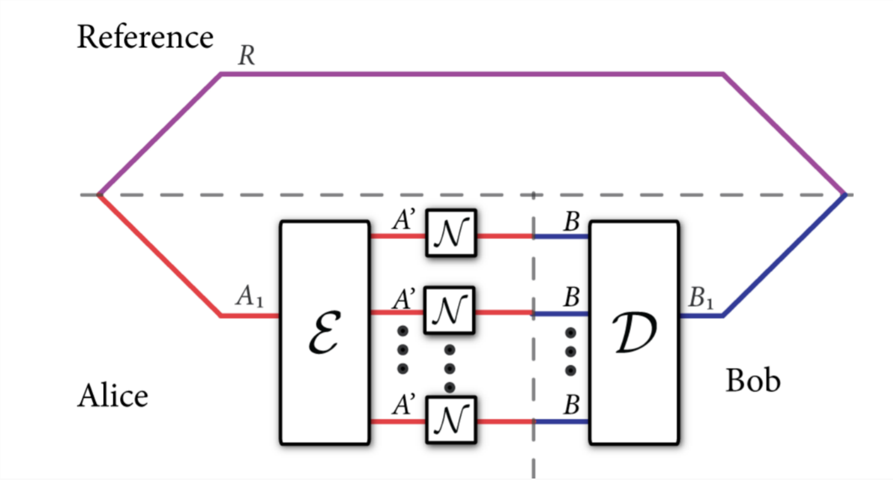
- In a quantum computer, as the qubits propagate through the circuit, they are subjected to noisy unitary gates, and are subjected to decoherence.
- In general, quantum processor consists noisy quantum gates, also called quantum channels.
- How is the ability of quantum processors to transmit entanglement limited?



Derive tighter bounds on the rate at which entanglement can be transmitted over finite number of uses of quantum channels in an unassisted quantum communication protocol.

Why resource theory of unextendibility?

- Resource theory of entanglement used to give bound on quantum capacity.

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- In prior work, Rains relaxed resource theory of entanglement by using PPT framework to obtain better bounds on quantum capacities.
- Here we relax resource theory of entanglement in a different way using framework of resource theory of unextendibility to get tighter bounds on non-asymptotic quantum capacity.

Resource theory of unextendibility

Resource theories

A resource theory consists of three main ingredients (e.g., a resource theory of entanglement):

- The resource states (e.g., entanglement)
- The free states (e.g., separable state)
- The restricted set of free operations (e.g., LOCC).

Resource theory of unextendibility

The resource theory of unextendibility constitutes:

Free states : k-extendible states.

k-extendible states: Definition

For a positive integer $k \geq 2$, ρ_{AB} is k-extendible w.r.t. system B if

- **State extension** $\exists$ a quantum state $\omega_{AB_1\cdots B_k}$ such that

$$\mathrm{Tr}_{B_2\cdots B_k}\{\omega_{AB_1\cdots B_k}\} = \rho_{AB},$$

  with $B_i \simeq B$ $\forall i \in [k]$.

k-extendible states: Definition

- **Permutation invariance**. The extension state $\omega_{AB_1\cdots B_k}$ is invariant with respect to permutations of the B systems

$$\omega_{AB_1\cdots B_k} = W^{\pi}_{B_1\cdots B_k}\omega_{AB_1\cdots B_k}W^{\pi\dagger}_{B_1\cdots B_k},$$

where $W^{\pi}_{B_1\cdots B_k}$ is a unitary representation of the permutation $\pi \in S_k$, with S_k denoting the symmetric group.

k-extendible states: Example

- Consider the state:

$$\frac{1}{2}\Phi_{AB_1} + \frac{1}{2}\pi_A \otimes \pi_{B_1}$$

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- A 2-extension of this state is

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- Only separable states are k-extendible for every k.
- For every finite k, there are k-extendible, yet entangled states.

k-extendible states

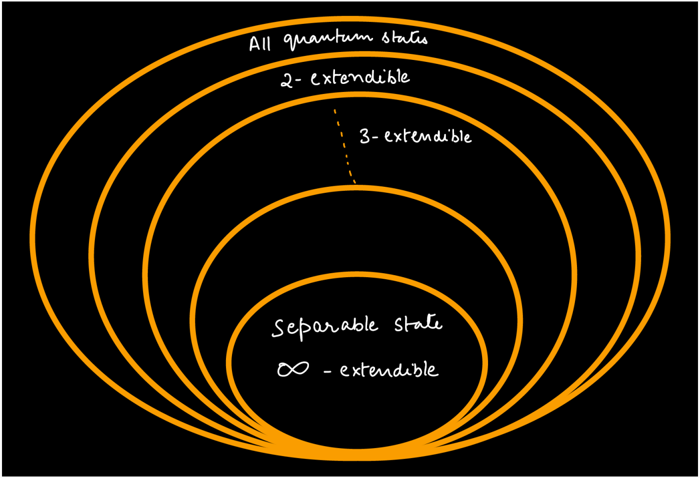


Figure 1: k-extendible states

Resource theory of unextendibility

The resource theory of k-unextendibility constitutes

Free states : k-extendible states.

Resource states : All k-unextendible states.

Free operations : k-extendible channels.

Resource theory of 3-unextendibility

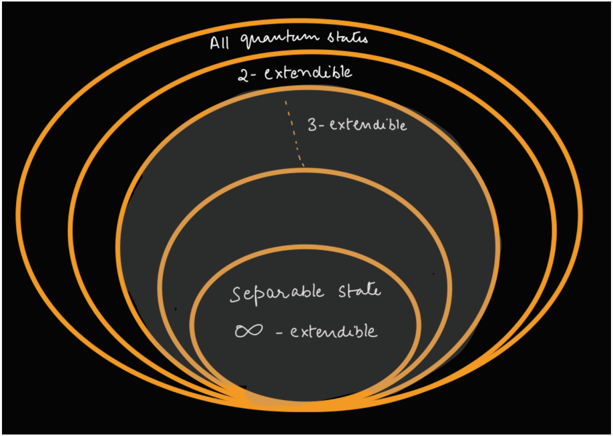
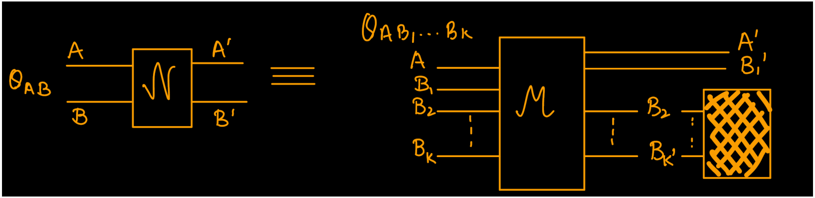


Figure 2: Resource states and free states in resource theory of 3-unextendibility.

k-extendible channels

A bipartite channel $\mathcal{N}_{AB\to A'B'}$ is called k-extendible if

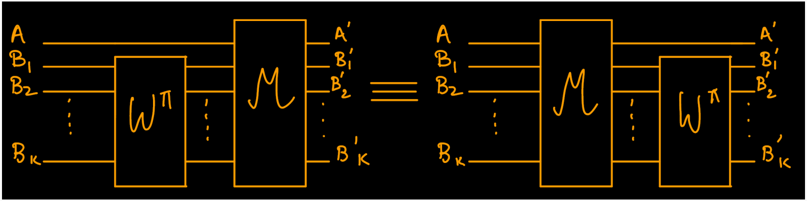


- **Channel extension**. $\exists$ a quantum channel $\mathcal{M}_{AB_1\cdots B_k\to A'B_1'\cdots B_k'}$ that extends $\mathcal{N}_{AB\to A'B'}$, such that for all quantum states $\theta_{AB_1\cdots B_k}$:

$$\mathrm{Tr}_{B_2'\cdots B_k'}\{\mathcal{M}_{AB_1\cdots B_k\to A'B_1'\cdots B_k'}(\theta_{AB_1\cdots B_k})\} = \mathcal{N}_{AB\to A'B'}(\theta_{AB_1}),$$

with $B_i \simeq B$ and $B_i' \simeq B'$ for all $i \in [k]$.

k-extendible channels



- **Permutation covariance**. For all quantum states $\theta_{AB_1\cdots B_k}$:

$$\mathcal{M}_{AB_1\cdots B_k\to A'B_1'\cdots B_k'}(W^{\pi}_{B_1\cdots B_k}\theta_{AB_1\cdots B_k}W^{\pi\dagger}_{B_1\cdots B_k}) = W^{\pi}_{B_1'\cdots B_k'}\mathcal{M}_{AB_1\cdots B_k\to A'B_1'\cdots B_k'}(\theta_{AB_1\cdots B_k})W^{\pi\dagger}_{B_1'\cdots B_k'},$$

where $W^{\pi}_{B_1\cdots B_k}$ and $W^{\pi}_{B_1'\cdots B_k'}$ are unitary representations of the permutation $\pi \in S_k$.

k-extendible channels: Property

- k-extendible channels preserve k-extendible states.

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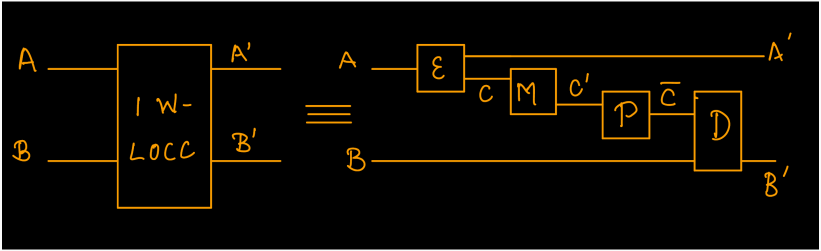
- This definition of k-extendible channel is consistent with the resource theory of unextendibility as k-extendible channels are free operations.

k-extendible channels: Example

- Any 1W-LOCC channel is a k-extendible channel.



- Any 1W-LOCC can be expressed as

$$\mathcal{D}_{\bar{C}B\to B'} \circ \mathcal{P}_{C'\to\bar{C}} \circ \mathcal{M}_{C\to C'} \circ \mathcal{E}_{A\to A'C}$$

where $\mathcal{E}_{A\to A'C}$, $\mathcal{M}_{C\to C'}$ is a measurement channel, $\mathcal{P}_{C'\to\bar{C}}$ is a preparation channel and $\mathcal{D}_{\bar{C}B\to B'}$ is the quantum channel, and C' is a classical register.

- Any 1W-LOCC channel is a k-extendible channel.

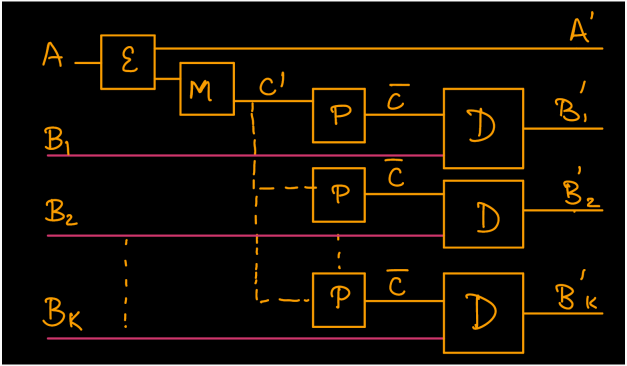


Figure 3: Extension of a 1W-LOCC channel.

Why resource theory of unextendibility?

- Set of free states in the resource theory of unextendibility is larger than the set of free states in the resource theory of entanglement.

Why resource theory of unextendibility?

- Set of free states in the resource theory of unextendibility is larger than the set of free states in the resource theory of entanglement.

- By relaxing the resource theory of entanglement, we obtain tighter, non-asymptotic upper bounds on quantum communication rates of a quantum channel.

Quantum communication protocol

Quantum communication protocol

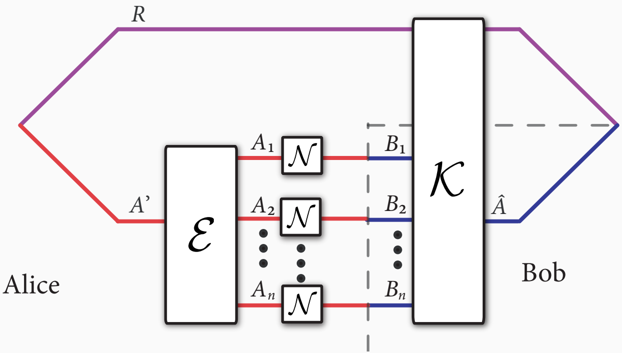
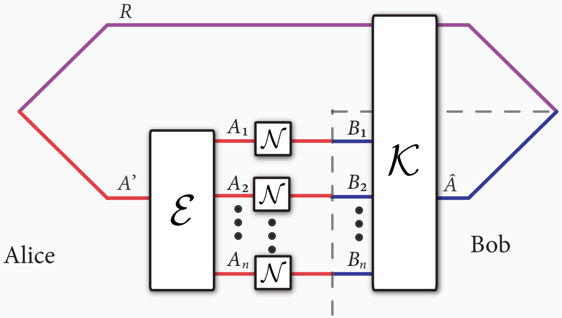


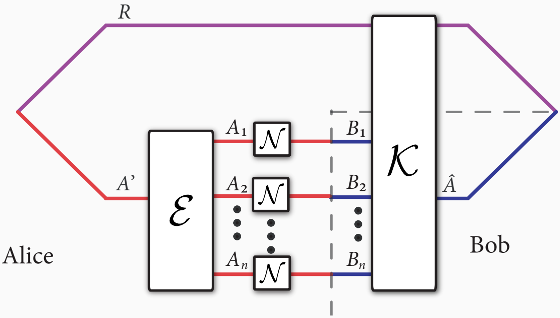
Figure 4: Quantum communication protocol with post-processing by k-extendible channels.

Quantum communication protocol



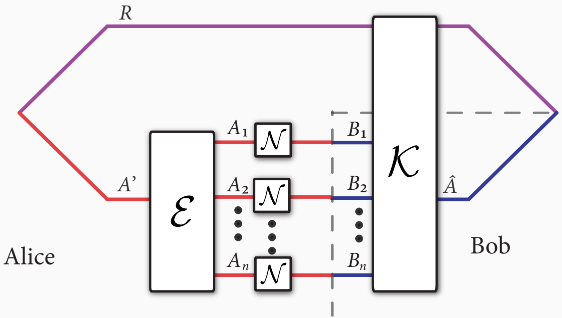
- (n, M, ε) quantum communication protocol assisted by a k-extendible post-processing begins with Alice preparing a state ρ_{RA^n}.

Quantum communication protocol



- She transmits the systems $A^n \equiv A_1 \cdots A_n$ using the channel $\mathcal{N}_{A\to B}^{\otimes n}$.

Quantum communication protocol



- Alice and Bob then perform a k-extendible channel $\mathcal{K}_{RB^n \to M_A M_B}$, and the resulting state satisfies the following performance condition:

$$\mathcal{F}(\mathcal{K}_{RB^n \to M_A M_B}(\mathcal{N}^{\otimes n}_{A \to B}(\rho_{RA^n})), \Phi_{M_A M_B}) \geq 1 - \varepsilon,$$

where $\Phi_{M_A M_B}$ is the maximally entangled state of Schmidt rank M.

Quantum communication protocols: Converse bound

For all integer $k \geq 2$ and for any $(1, M, \varepsilon)$ quantum communication protocol that uses a quantum channel $\mathcal{N}$ assisted by a k-extendible post-processing:

$$-\log_2 \left[\frac{1}{M} + \frac{M-1}{Mk} \right] \leq \sup_{\psi_{RA}} E_k^{\varepsilon}(R;B)_{\tau},$$

where

$$E_k^{\varepsilon}(R;B)_{\tau} \coloneqq \inf_{\sigma_{RB} \in \mathrm{EXT_k}(R;B)} D_h^{\varepsilon}(\tau_{RB} \| \sigma_{RB}),$$

$\tau_{RB} \coloneqq \mathcal{N}_{A \to B}(\psi_{RA})$, and the optimization is with respect to pure states ψ_{RA} such that $|R| = |A|$, and for $\varepsilon \in [0,1]$

$$D_h^{\varepsilon}(\tau \| \sigma) := -\log_2 \inf_{\Lambda} \{ \mathrm{Tr}\{\Lambda \sigma\} : \ 0 \leq \Lambda \leq I \wedge \mathrm{Tr}\{(I - \Lambda)\tau\} \leq \varepsilon \}.$$

Converse bound: Proof outline

- An isotropic state $\rho_{AB}^{(t,d)}$ is $U \otimes U^*$-invariant for an arbitrary unitary U, where $|A| = d = |B|$. For $t \in [0,1]$, it can be expressed as

$$\rho_{AB}^{(t,d)} = t\Phi_{AB}^{d} + (1-t)\frac{I_{AB} - \Phi_{AB}^{d}}{d^2 - 1}.$$

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- An isotropic state $\rho_{AB}^{(t,d)}$ is k-extendible if and only if $t \in \left[0, \frac{1}{d}\left(1 + \frac{d-1}{k}\right)\right]$ [JV13].

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- The extendibility of a k-extendible state does not change under the action of $U \otimes U^*$ for a unitary U.
- E_k^{ε} does not increase under the action of a k-extendible channel.

Examples: Depolarizing and Erasure channels

Depolarizing channels

- A qubit depolarizing channel acts on any input density operator ρ as

$$\mathcal{D}^{p}_{A\to B}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z),$$

where $p \in [0, 1]$ is the parameter determining the deviation from a unitary evolution, and X, Y, and Z are the Pauli operators.

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- We want to upper bound the following.

$$\log M \leq \log\left(\frac{k-1}{k}\right) - \log\left(2^{-E^{\varepsilon}_{k}(R^n;B^n)} - \frac{1}{k}\right),$$

$$E^{\varepsilon}_{k}(R^n;B^n)_{\rho} \coloneqq \min_{\sigma_{R^nB^n}\in \mathrm{EXT_k}(R^n;B^n)} D^{\varepsilon}_{h}(\tau_{R^nB^n}\|\sigma_{R^nB^n}).$$

Depolarizing channel

$$E_k^{\varepsilon}(R^n;B^n)_{\rho} := \min_{\sigma_{R^nB^n}\in \mathrm{EXT_k}(R^n;B^n)} D_h^{\varepsilon}(\tau_{R^nB^n}\|\sigma_{R^nB^n}),$$

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- Choose $\sigma_{R^n B^n}$ to be a tensor power of qubit isotropic states $\sigma_{RB}^{\otimes n}$, where $\sigma_{RB} = (1-t)\Phi_{RB} + \frac{t}{3}(I - \Phi_{RB})$.

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- $\tau_{RB} = (1-p)\Phi_{RB} + \frac{p}{3}(I-\Phi_{RB})$
- Choose $\sigma_{R^nB^n}$ to be a tensor power of qubit isotropic states $\sigma_{RB}^{\otimes n}$, where $\sigma_{RB} = (1-t)\Phi_{RB} + \frac{t}{3}(I-\Phi_{RB})$.
- $E_k^{\varepsilon}(R^n;B^n)_\rho \le \min_t D_h^{\varepsilon}\left(\{1-p,p\}^{\otimes n}\|\{t,1-t\}^{\otimes n}\right)$

Depolarizing channels

- We then obtain the following bound:

$$\log M \leq \log\left(1-\frac{1}{k}\right)-\log\left(2^{-D_h^{\varepsilon}\left(\{1-p,p\}^{\otimes n}\|\{t,1-t\}^{\otimes n}\right)}-\frac{1}{k}\right).$$

Depolarizing channels

- We then obtain the following bound:

$$\log M \leq \log\left(1 - \frac{1}{k}\right) - \log\left(2^{-D_h^{\varepsilon}\left(\{1-p,p\}^{\otimes n} \| \{t,1-t\}^{\otimes n}\right)} - \frac{1}{k}\right).$$

- This reduces to calculating the hypothesis testing relative entropy between Bernoulli distributions.

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- This reduces to calculating the hypothesis testing relative entropy between Bernoulli distributions.
- The optimal measurement (Neyman-Pearson test) for the resulting classical hypothesis testing relative entropy between Bernoulli distributions is then well known, giving an explicit upper bound on $\log M$.

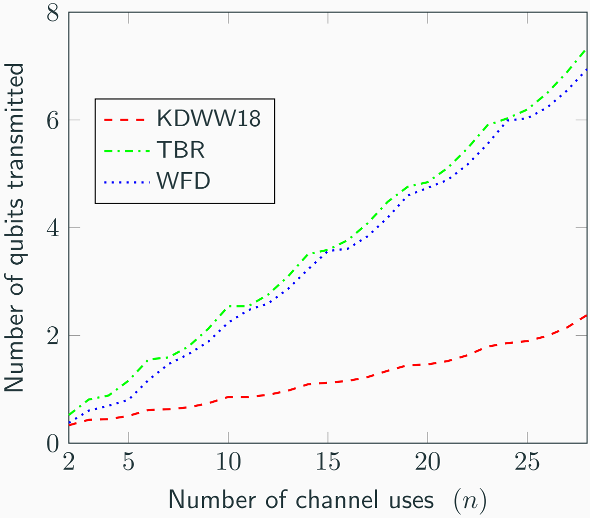

Figure 5: Non-asymptotic upper bounds on the number of qubits reliably transferred over a depolarizing channel with $p = 0.15$, and $\varepsilon = 0.05$.

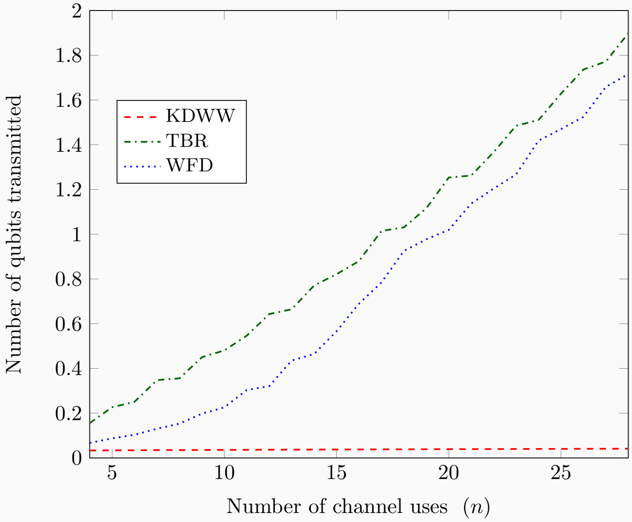

Figure 6: Non-asymptotic upper bounds on quantum capacity of a depolarizing channel with $p = 0.24$, and $\varepsilon = 0.05$.

Erasure channel

- An erasure channel acts on a qubit as

$$\mathcal{E}^{p}_{A\to B}(\rho_A) = (1-p)\rho_B + p\,|e\rangle\langle e|_B\,.$$

  where $p \in [0,1]$ is the parameter of the erasure channel and $|e\rangle\langle e|$ is a pure state, orthonormal to the input state.

- We want to upper bound the following.

$$\log M \leq \log\left(\frac{k-1}{k}\right) - \log\left(2^{-E_k^{\varepsilon}(R^n;B^n)} - \frac{1}{k}\right),$$

$$E_k^{\varepsilon}(R^n;B^n)_\rho := \min_{\sigma_{R^nB^n}\in\mathrm{EXT_k}(R^n;B^n)} D_h^{\varepsilon}(\tau_{R^nB^n}\|\sigma_{R^nB^n}).$$

How to choose $\sigma_{R^n B^n}$?

- For two channel uses, the output state can be written as

$$\mathcal{E}^p_{A\to B}(\Phi_{RA}) = (1-p)^2 \left(\Phi_{R_1B_1} \otimes \Phi_{R_2B_2}\right) + p(1-p)\left(\Phi_{R_1B_1} \otimes \pi_{R_2} \otimes |e\rangle\langle e|_{B_2}\right)$$
$$+p(1-p)\left(\pi_{R_1} \otimes |e\rangle\langle e|_{B_1} \otimes \phi_{R_2B_2}\right) + p^2\left(\pi_{R_1} \otimes |e\rangle\langle e|_1 \otimes \pi_{R_2} \otimes |e\rangle\langle e|_{B_2}\right).$$

- The minimizing state is of the form:

$$\sigma_{A_1B_1A_2B_2} := c_0\Phi_{A_1B_1} \otimes \Phi_{A_2B_2} + c_2\pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2}$$
$$c_1\left(\Phi_{A_1B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2} + \Phi_{A_2B_2} \otimes \pi_{A_1} \otimes |e\rangle\langle e|_{B_1}\right),$$

  where $c_0 + 2c_1 + c_2 = 1$.

- Want to find constraints on c_0, c_1 and c_2 such that the state is k-extendible.

How to choose $\sigma_{R^n B^n}$?

$$\sigma_{A_1 B_1 A_2 B_2} := c_0 \Phi_{A_1 B_1} \otimes \Phi_{A_2 B_2} + c_2 \pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2}$$
$$c_1 \left(\Phi_{A_1 B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2} + \Phi_{A_2 B_2} \otimes \pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \right)$$

- Make each individual state in the sum a k-extendible state.

How to choose $\sigma_{R^n B^n}$?

$$\sigma_{A_1B_1A_2B_2} := c_0\Phi_{A_1B_1} \otimes \Phi_{A_2B_2} + c_2\pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2}$$
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- Make each individual state in the sum a k-extendible state.
- Replace all the terms $\Phi_{A_iB_i}$ in the above state with the k-extendible state $\frac{1}{k}\Phi_{A_iB_i} + \left(1 - \frac{1}{k}\right)\pi_{A_i} \otimes |e\rangle\langle e|_{B_i}$.

How to choose $\sigma_{R^n B^n}$?

$$\sigma_{A_1B_1A_2B_2} := c_0\Phi_{A_1B_1} \otimes \Phi_{A_2B_2} + c_2\pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2}$$
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- The two-extendible state obtained from $\sigma_{A_1B_1A_2B_2}$ is

$$\frac{c_0}{4}\Phi_{A_1B_1}\otimes\Phi_{A_1B_1}+\left(\frac{c_0}{4} + c_1 + c_2\right)\left(\pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2}\right)$$
$$\left(\frac{c_0}{4} + \frac{c_1}{2}\right)\left(\Phi_{A_1B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2} + \pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \Phi_{A_2B_2}\right).$$

How to choose $\sigma_{R^n B^n}$?

$\sigma_{A_1 B_1 A_2 B_2}$ is two-extendible:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 2 \cdot \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

The test state then is:

$$\sigma_{A_1 B_1 A_2 B_2} := b_0 \Phi_{A_1 B_1} \otimes \Phi_{A_2 B_2} + b_2 \pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2}$$
$$b_1 \left(\Phi_{A_1 B_1} \otimes \pi_{A_2} \otimes |e\rangle\langle e|_{B_2} + \Phi_{A_2 B_2} \otimes \pi_{A_1} \otimes |e\rangle\langle e|_{B_1} \right)$$

How to choose $\sigma_{R^n B^n}$

- We then have

$$\min_{\sigma'_{A_1 B_1 \cdots A_n B_n} \in \mathrm{EXT}_k} D_h^\varepsilon \left(\rho_{A_1 B_2 \cdots A_n B_n} \| \sigma'_{A_1 B_1 \cdots A_n B_n}\right)$$
$$\leq \min_{b_0, b_1, \ldots, b_n} D_h^\varepsilon \left(\{a_0, a_1, \ldots, a_n\} \,\|\, \{b_0, b_1, \ldots, b_n\}\right),$$

  where the distribution $\{a_0, a_1, \ldots, a_n\}$ is induced by measuring the number of erasures in $\rho_{A_1 B_2 \cdots A_n B_n}$.
- The RHS above is a linear program, so we can find the minimizing b_i's.

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- The RHS above is a linear program, so we can find the minimizing b_i's.
- We obtain the following upper bound.

$$\log M \leq \log\left(\frac{k-1}{k}\right) - \log\left(2^{-D_h^{\varepsilon}(\{a_0, a_1, \ldots, a_n\} \| \{b_0, b_1, \ldots, b_n\})} - \frac{1}{k}\right),$$

Erasure channels

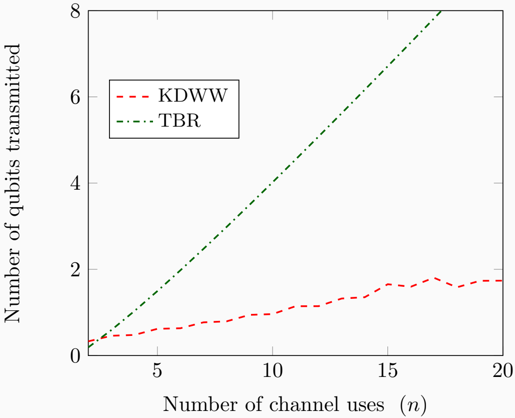

Figure 7: Non-asymptotic upper bounds on quantum capacity of an erasure channel with $p = 0.35$, and $\varepsilon = 0.05$.

Erasure channels

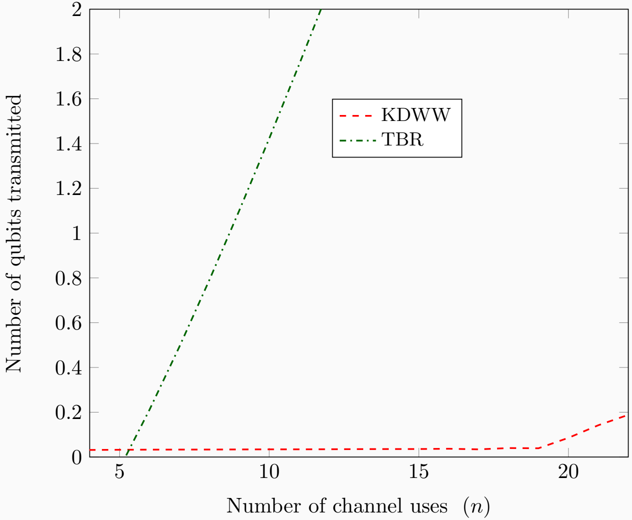

Figure 8: Non-asymptotic upper bounds on quantum capacity of an erasure channel with $p = 0.49$, and $\varepsilon = 0.05$.

Adaptive protocols and Pretty strong converse

Adaptive protocols

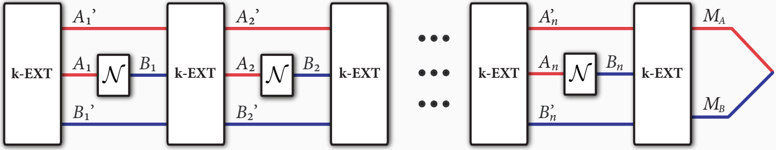

Figure 9: Each channel use is interleaved with a k-extendible channels.

- If $\mathcal{N}$ is a depolarizing channel or an erasure channel (or any k-simulable channel) then the bounds given above still hold.

Adaptive protocols: converse bound

For all $k \geq 2$ and for any (n, M, ε) quantum communication protocol employing n uses of a channel $\mathcal{N}$ interleaved by k-extendible channels:

$$-\log_2\left[\frac{1}{M} + \frac{M-1}{Mk}\right] \leq n \sup_{\psi_{RA}} E_k^{\max}(R;B)_\tau + \log_2\left(\frac{1}{1-\varepsilon}\right),$$

where

$$E_k^{\max}(R;B)_\rho := \min_{\sigma_{RB} \in \mathsf{EXT}_k(R;B)} D_{\max}(\rho_{RB} \| \sigma_{RB}),$$

$\tau_{RB} := \mathcal{N}_{A\to B}(\psi_{RA})$, and the optimization is with respect to pure states ψ_{RA} with $|R| = |A|$.

Antidegradable channels: Pretty strong converse

- A channel $\mathcal{N}_{A\to B}$ is antidegradable if the output state $\mathcal{N}_{A\to B}(\rho_{RA})$ is two-extendible for any input state ρ_{RA}. Due to this property, antidegradable channels have zero asymptotic quantum capacity.

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- Fix $\varepsilon \in [0, 1/2)$. The following bound holds for any (n, M, ε) quantum communication protocol employing n uses of an antidegradable channel $\mathcal{N}$ interleaved by two-extendible channels:

$$\frac{1}{n}\log_2 M \leq \frac{1}{n}\log_2\left(\frac{1}{1-2\varepsilon}\right).$$

Conclusion

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  - analyzing these bounds for other noise models such as thermal channels and pure loss channels
  - understanding the structure of the k-extendible channels.

Conclusion

- We introduced the resource theory of unextendibility.
- Improved significantly on the prior non-asymptotic upper bounds for quantum communication over depolarizing and erasure channels.
- Other directions include
  - analyzing these bounds for other noise models such as thermal channels and pure loss channels
  - understanding the structure of the k-extendible channels.
  - to link the bounds developed here with the open problem of finding a strong converse for the quantum capacity of degradable channels.

For more details please refer to: arXiv:1803.10710.

Further reading

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