Noisy quantum state redistribution with promise and the Alpha-bit

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Outline for section 1

The task

- Hayden, Pennington [2017]
- A channel $\mathcal{N}_{A\to B}$ between Alice and Bob.

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- Alice receives a quantum state $|\psi\rangle \in \mathcal{H}^d$. Bob receives the classical description of a subspace $S \subset \mathcal{H}^d$ of dimension $d^\alpha + 1$. Guaranteed that $|\psi\rangle \in S$.
- Alice needs to transmit $|\psi\rangle$ to Bob. For every subspace S, average error of decoding $|\psi\rangle \in S$ should be small.

The task

- Introduce a Reference.
- Fix an S. Alice and Reference share $|\Psi(S)\rangle_{RA}$, where Ψ_A is maximally mixed in S. Bob receives classical description of S.
- Transmit register A to Bob such that final state $\Psi'(S)_{RA}$ satisfies $\Psi'(S) \overset{\varepsilon}{\approx} |\Psi(S)\rangle\langle\Psi(S)|_{RA}$ for all S.

Capacities

- Think of $\alpha = 1$. Consider n uses of the channel and largest possible $d(n)$ such that decoding is successful with high fidelity.
- Capacity $= \frac{1}{n} \log d(n)$.

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- General α. Consider n uses of the channel and largest possible $d(n)$ such that decoding in the subspace S is successful with high fidelity.
- Capacity $= \frac{1}{n} \log d(n)$.
- Actual definition: allow many i.i.d. instances and take appropriate limit.
- Smaller α implies higher capacity: smaller α implies less quantum states to decode.

Capacities

- Entanglement assisted α-bit capacity:

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$$\frac{1}{1+\alpha}\max_{\psi_{AA'}} I(B:A')_{\rho_{BA'}}.$$

- Entanglement unassisted α-bit capacity:

$$\max_{\psi_{AA'}}\left[\min\left(\frac{1}{1+\alpha}\max_{\psi_{AA'}} I(B:A')_{\rho_{BA'}}, \frac{1}{\alpha}\max_{\psi_{AA'}} I(B>A')_{\rho_{BA'}}\right)\right]$$

- $\rho_{A'B} = \mathcal{N}_{A\to B}(\psi_{AA'})$.

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- Answer is $C = 1 + \alpha$. Reversible theory in asymptotic and i.i.d. setting (recall entanglement assisted capacity).
- Think of $\alpha = 1$. It is a qubit that can be transmitted perfectly. By superdense coding, 2 classical bits can be transmitted.

Features

- Key tool: subspace decoupling duality (a generalization of weak decoupling duality, Hayden, Winter [IEEE IT]) + decoupling via random unitary.
- Used for all above achievability results and converse.
- Several consequences for resource theory of quantum communication.

Our results

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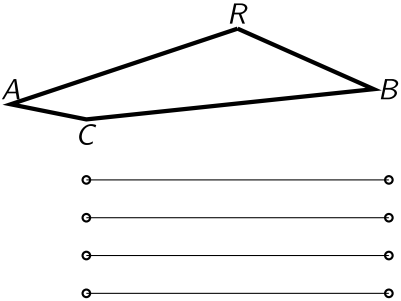
- Quantum state redistribution as a resource: asymptotically reversible theory.
- Reminiscent: Quantum Reverse Shannon Theorem (Bennett, Devetak, Harrow, Shor, Winter [arXiv 2009, IEEE IT 2014]; Berta, Christandl, Renner [CMP 2011]).

Outline for section 2

Quantum state redistribution (QSR)



Devetak and Yard [PRL 2008, IEEE IT 2009].
Communication rate in the asymptotic and i.i.d. setting:
$I(R : C|B)$.

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where $C_{ea}(\mathcal{N})$ is entanglement assisted classical capacity of $\mathcal{N}_{J\to K}$. Bennett, Shor, Smolin, Thapliyal [IEEE IT 2002].

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- Optimal in the asymptotic and i.i.d. setting. Quantum Reverse Shannon Theorem (Bennett, Devetak, Harrow, Shor, Winter [arXiv 2009, IEEE IT 2014]; Berta, Christandl, Renner [CMP 2011]).

A generalization of Alpha-bit task

- Quantum state redistribution with promise (Promise-QSR).
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- Alice, Bob and Reference share the state $|\Psi^y\rangle_{RABC}$. Reference and Bob know y.
- Alice needs to send a message to Bob, such that after Bob's decoding, the final state Φ^y_{RABC} satisfies $\Phi^y_{RABC} \overset{\varepsilon}{\approx} |\Psi^y\rangle\langle\Psi^y|_{RABC}$ for all y.

A simplification: min-max duality

- Consider the task: for a distribution $\mu(y)$, perform the QSR of

$$\sum_y \sqrt{\mu(y)}\,|y\rangle_Y\,|y\rangle_{R_Y}\,|\Psi^y\rangle_{RABC}\,.$$

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- Let $\mathcal{P}_C$ be the set of all protocols with communication at most C (finite amount of entanglement used).

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- Let $\mathcal{P}_C$ be the set of all protocols with communication at most C (finite amount of entanglement used).
- Min-max principle [Yao, FOCS 1979]:

$$\max_{\mu} \min_{P \in \mathcal{P}_C} \mathrm{error}(\mu, P) = \min_{P \in \mathcal{P}_C} \max_{\mu} \mathrm{error}(\mu, P).$$

- Promise-QSR $<=$ QSR for all distributions μ.

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- Similar statements (carefully done) for the one-shot setting.
- But is it optimal?

Decoupling Alice's entanglement

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- QSR protocols have the property that Alice's final entanglement is independent of y.
- A protocol for Promise-QSR may not have this property.
- But subspace decoupling duality implies that this is the case for Alpha-bit.
- Alpha-bit = Alpha-bit for all μ.

Recovering Alpha-bit capacities

- Consider $\sum_S \sqrt{\mu(S)} \left|S\right\rangle_Y \left|S\right\rangle_{R_Y} \left|\Psi(S)\right\rangle_{RC}$.

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- Capacity is largest d for one channel use.
- $\log d = \frac{C_{ea}(\mathcal{N}_{J\to K})}{(1+\alpha)}$.
- Similar one-shot results available as well.

Recovering Alpha-bit capacities

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- Entanglement always gained (for above instance).
- Using protocol for either entanglement assisted or unassisted (Lloyd, Shor, Devetak [1997-2005]) capacities of $\mathcal{N}_{J\to K}$ leads to an optimal protocol for unassisted capacity of Alpha-bit in asymptotic and i.i.d. setting.

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- Weakness: capacity for one instance of Alpha-dit not recovered (feature implicit in Hayden, Pennington [2017] owing to subspace decouping duality).

A classical Alpha-bit

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- Bob receives a subset $S \in [1, 2, \ldots d]$ and Alice receives an $i \in S$.
- Communicate i to Bob.
- Entanglement assisted classical capacity of classical Alpha-bit:

$$\frac{1}{\alpha} \max_{\psi_{JJ'}} I(K : J')_{\mathcal{N}_{J \to K}(\psi_{JJ'})}.$$

Outline for section 3

Definition of simulation

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- Given a m, Alice singles out some registers A_m, C_m, which are subsets of E_A.
- She runs P on A_m, C_m. Communication to Bob is implicit in P.
- Bob decodes on his final state, to recover m.

Our result

- Any protocol P for QSR of Ψ_{RABC} can be used to communicate a message in $[1, 2, \ldots 2^N]$ with error $\varepsilon + 2\delta$, where N is equal to

$$I_H^{\varepsilon}(R : BC)_{\Psi} - I_{\max}^{\delta}(R : B)_{\Psi} + \log \varepsilon\delta.$$

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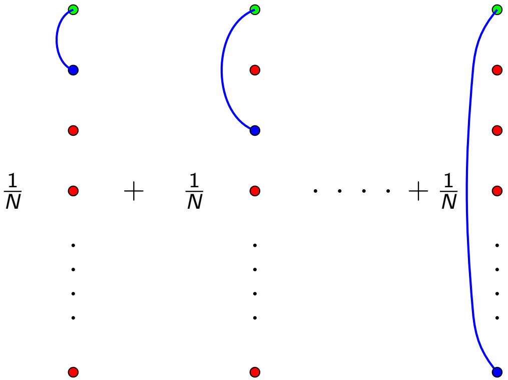
Recovering entanglement assisted classical capacity of Alpha-bit

- Appeal to subspace-decoupling duality: any protocol for Alpha-bit is also a protocol for state redistribution under distribution μ.
- Alice's operations ensure that state is transmitted to Bob with remaining registers independent of S.
- Use result in previous slide.

Position-based decoding



If $N < I_H(R : B)$, correlation can be located (A., Jain, Warsi [IEEE IT, 2018]).

Outline for section 4

Take-home message

Any QSR task can be used as a resource, with optimal simulation cost in asymptotic and i.i.d. setting.

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- Extending above results to multi-party setting.

Thank you!