NOTE: Be sure to show your work and explain what you are doing

1. Shifted harmonic oscillator (revisited again)

Suppose a one-dimensional harmonic oscillator (mass $m$, frequency $\omega$) is subjected to a driving force of the form $F(t) = m\omega^2 f(t)$, where $f(t)$ is some specified function. Assume that the force was first turned on at time $t = 0$; $f(t) = 0$ for $t \leq 0$. This system can be solved exactly, both in classical mechanics and in quantum mechanics [32pts:=8+8+8+8]

(a) Determine the classical position of the oscillator, $x_c(t)$, assuming it started from rest at the origin $[x_c(0) = \dot{x}_c(0) = 0]$.

(b) Show that the solution to the (time-dependent) Schrodinger equation for this oscillator, assuming it started out in the $n$th state of the undriven oscillator $\Psi(x, 0) = \psi_n(x)$ can be written as

$$\Psi(x,t) = \psi_n(x) \exp\left\{ -\frac{(n+1/2)\hbar\omega t + \frac{m\omega^2}{2} \int_0^t f(t') x_c(t') dt'}{2} \right\}$$  \hspace{1cm} (1)

(c) Show that the eigenfunctions and eigenvalues of the instantaneous Hamiltonian are

$$\psi_n(x,t) = \psi_n(x-f) \hspace{1cm} E_n(t) = (n + \frac{1}{2})h\omega - \frac{1}{2}m\omega^2 f^2$$ \hspace{1cm} (2)

(d) Show that in the adiabatic approximation the classical position reduces to $x_c \sim f(t)$. State the precise criterion for adiabaticity, in this context, as a constraint on the time derivative of $f(t)$. Hint: Write $\sin[w(t-t')]$ as $(1/\omega)(d/dt')\cos[w(t-t')]$ and use integration by parts

2. The adiabatic approximation can be regarded as the first term in an adiabatic series for the coefficients $c_m(t)$. Suppose the system starts out in the $n$th state; in the adiabatic approximation, it remains in the $n$th state, picking up only a time-dependent geometric phase factor $c_m(t) = \delta_{mn} \exp(i\gamma_m(t))$ [30pts:=15+15]

(a) Substitute this into the right side of the Equation for the amplitude coefficients we derived in class (see Eq. 5.6.8 of Sakurai) to obtain the "first correction" to adiabaticity. This enables us to calculate transition probabilities in the nearly adiabatic regime.

(b) As an example, apply the "first correction" formula you derived in a) to the driven oscillator (Problem 1). Compute in the near-adiabatic approximation the transition amplitudes and show they are possible only to the two immediately adjacent levels.

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3. Consider the case of low energy scattering from a spherical delta function potential

\[ V(r) = A\delta(r - R) \]  

with A and R constants. Assuming that \( kR \ll 1 \) so only the \( l = 0 \) term significantly contributes calculate [32pts:=8+8+8+8]

(a) The phase shift \( \delta_0 \)
(b) The scattering amplitude \( f(\theta) \)
(c) The differential cross section \( d\sigma/d\Omega \)
(d) The total cross section

4. For the same delta function potential use the Born approximation to calculate [32pts:=8+8+8+8]

(a) The scattering amplitude \( f(\theta) \)
(b) The differential cross section \( d\sigma/d\Omega \)
(c) The total cross section
(d) Show that your results are consistent to the ones obtained in problem 3 for the appropriated regime.