1. An atom in an external magnetic field: The energy of an atom in an external magnetic field pointing along the $\hat{z}$ directions is $E = -g\mu_0 J_z B$. Here $\hbar J_z$ is the angular momentum along the $z$ direction, $\mu_0$ is the standard unit of magnetic moment (usually the Bohr magneton) and $g$ the so called $g$-factor of the atom. A number of order of unity. In a quantum mechanical description the eigenvalues of $J_z$ are discrete and given my $m$, where $m$ can take values between $-J$ and $J$, i.e. $m \in \{-J, -J+1, \ldots, J-1, J\}$. Thus there are $2J+1$ possible values.[20pts:10+10]

(a) Compute the partition function $Z$. Use the fact that the probability to me in a state with energy $\epsilon$ is $P_\epsilon = e^{-\epsilon \beta} / Z$ with $\beta = 1/(k_B T)$ with $T$ the temperature and $k_B$ the Boltzmann constant.

(b) If the $z$ component of the magnetic moment of an atom with eigenvalue $m$ is $\mu_z = g\mu_0 m$ compute $\bar{\mu}_z$, the mean $z$ component of the magnetic moment of an atom. Using the definition of the ”Brillouin function” $B_J(\chi) = \bar{\mu}_z/(g\mu_0 J)$ with $\chi = \beta g\mu_0 B$. Determine the high temperature $\chi \ll 1$ and low temperature $\chi \gg 1$ limits of $B_J(\chi)$. Determine how $B$ needs to scale with temperature to make $B_J(\chi)$ converge to the same value in both limits.

2. In this problem you will explore Rabi oscillations in the presence of radiative damping (spontaneous emission). To add the damping let $H = H_0 + V$ with (in matrix form) [40pts:10+10+10+10]

$H_0 = \frac{1}{2} \begin{pmatrix} \hbar \omega_1 & 0 \\ 0 & \hbar \omega_2 \end{pmatrix}$ \quad $V = \frac{1}{2} \begin{pmatrix} 0 & \hbar \Omega \exp(-i\omega t) \\ \hbar \Omega \exp(i\omega t) & -i\gamma_2 \end{pmatrix}$

(1)

(a) Find the probability of measuring the atom in spin up $P_\uparrow = \langle \psi(t) | \uparrow \rangle \langle \uparrow | \psi(t) \rangle$ and spin down $P_\downarrow = \langle \psi(t) | \downarrow \rangle \langle \downarrow | \psi(t) \rangle$, if at time $t = 0$ the system is in state $| \psi(t) \rangle = | \downarrow \rangle$.

(b) Using Mathematica plot the excited and ground state probabilities as a function of time. In particular make the following plots
- Resonant Rabi oscillations with no damping
- Resonant Rabi oscillations with moderate damping
- Resonant drive with strong damping.

(c) What is wrong with this calculation in terms of modeling the effect of spontaneous emission in a two level atom? Answer in a complete sentence.

(d) In the limit of weak drive (excited probability $\ll 1$) and for strong damping, what is the lineshape vs detuning? What is its full width at half maximum (FWHM)?
3. Inhomogeneous broadening: Consider a macroscopic ensemble of spins in static magnetic field in the z direction, $B$, but with an inhomogeneous magnitude. If a given spin starts in $|+\rangle_x$ (Bloch vector points along +x) and sees a magnetic field $B$ it will Larmor precess. This signal however will decay due to the averaging over different local precession frequencies. Suppose the distribution of $B$ is Gaussian, with mean $B_0$ and rms $\Delta B \ll B_0$. [30pts:15+15]

(a) Calculate and sketch, $\langle \hat{\sigma}_x \rangle$, averaged over the ensemble, as a function of time.
(b) What the characteristic decay time, known as $T_2^*$ due to inhomogeneity.

4. Solve Sakurai 3.26 [30pts:10+10+10]