NOTE: Be sure to show your work and explain what you are doing

1. Here we will explore some properties of rotation matrices. Consider a sequence of rotation represented by

\[ \hat{D}(\alpha, \beta, \gamma) = \exp \left( -i \hat{\sigma}_z \frac{\alpha}{2} \right) \exp \left( -i \hat{\sigma}_y \frac{\beta}{2} \right) \exp \left( -i \hat{\sigma}_z \frac{\gamma}{2} \right). \]  

(1)

Because of the group properties of the rotations, we expect this sequence is equivalent to a single rotation about some axis \( \hat{n} \) by an angle \( \phi \). (25 pts: 5+10+10):

(a) Write \( \hat{D}(\alpha, \beta, \gamma) \) in matrix form
(b) Find \( \phi \)
(c) Find \( \hat{n} \)

2. A neutron, magnetic moment \( \vec{\mu} = \mu_n \vec{S} \), travels with velocity \( \vec{v} = 2400 \text{ m/s} \) along the \( \hat{z} \) direction. Initially it travels in a region (region 1) of uniform magnetic field \( B_0 \hat{z} \) with spin up in the \( z \) direction. At time \( t_1 \) it enters another region (region 2) where the magnetic field is \( \vec{B} = B_0 (\cos \theta \hat{z} + \sin \theta \hat{x}) \). After traveling for 20 m in this region it enters a third region where the magnetic field is once again \( B_0 \hat{z} \). Assuming that \( \mu_n B_0 = \pi 60 \text{s}^{-1} \) answer the following questions (30 pts: 10+10+10):

(a) Calculate the state, \( |\psi(t_1)\rangle \), of the neutron at time \( t_1 \)
(b) Use \( |\psi(t_1)\rangle \) as an initial condition to evaluate the state of the neutron in region 2 as a function of time.
(c) Calculate the probability that the spin is down in the \( z \) direction after the neutron has traveled through region 2.
3. **Ramsey Spectroscopy**: Rabi spectroscopy was modified about 10 years later by Ramsey (his student and Physics Nobel prize laureate in 1989), and now serves as the basis for atomic clocks and the SI definition of the second. All precision atomic measurements, including modern atom-interferometers and quantum logic gates in atomic systems, have at their heart a Ramsey type geometry.

Consider an ensemble of two-level atoms with transition frequency \( \omega_0 \). In Ramsey spectroscopy atoms start in a state with spin down along \( \hat{z} \). The atoms are first illuminated with light with frequency \( \omega_L \) during a time \( T \) to generate a \( \theta \) pulse. Then the system is let to freely evolve during a time \( t \) and finally a second \( \pi/2 \) is applied for read out. This corresponds to the following sequence:

\[
|\psi(T)\rangle = \hat{R}(\pi/2, n_y) e^{-i(\delta T)/2\sigma_z} \hat{R}(\theta, n_y) |\downarrow\rangle_z \tag{2}
\]

where \( \delta = \omega_L - \omega_0 \). (40 pts=10+10+10+10)

(a) Represent the state of the system in a Bloch sphere: i) after the first pulse, ii) after the free evolution and iii) at the end of the sequence.

(b) Calculate \( \langle \psi(T)|\hat{S}|\psi(T)\rangle \) at the end of the experiment.

(c) An **echo sequence** adds an additional rotation \( \hat{R}(\pi, n_x) \) at the middle of the free evolution,

\[
|\dot{\psi}(T)\rangle = \hat{R}(\pi/2, n_y) e^{-i\delta T/2\sigma_z} \hat{R}(\pi, n_x) e^{-i\delta T/2\sigma_z} \hat{R}(\theta, n_y) |\downarrow\rangle_z \tag{3}
\]

Represent the state of the system in a Bloch sphere at the end of the sequence.

(d) Calculate \( \langle \dot{\psi}(T)|\hat{S}|\dot{\psi}(T)\rangle \) at the end of the echo sequence.

4. Consider a system composed of two spin 1/2 particles whose orbital variables are ignored, with Hamiltonian

\[
\hat{H} = \hbar \omega_1 \hat{\sigma}_1^z + \hbar \omega_2 \hat{\sigma}_2^z + \hbar J \hat{\sigma}_1 \cdot \hat{\sigma}_2. \tag{4}
\]

Suppose the initial state of the system at time \( t = 0 \) is \( |\psi(0)\rangle = (|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)/\sqrt{2} \). Compute

\[
\langle \psi(t)|\hat{S} \cdot \hat{S}|\psi(t)\rangle = \langle \psi(t)|(\hat{S}_1 + \hat{S}_2) \cdot (\hat{S}_1 + \hat{S}_2)|\psi(t)\rangle \tag{5}
\]

with \( \hat{S}_{1,2} = \hbar/2 \hat{\sigma}_{1,2} \), for the following cases:(40 pts=10+10+10+10)

(a) When \( \omega_1 = \omega_2 \). Interpret the result

(b) For the previous case draw the result in a Bloch sphere for the total vector \( \langle \hat{S} \rangle \)

(c) When \( \omega_1 \neq \omega_2 \).

(d) Take the limits \( J \gg \omega_1, \omega_2 \) and \( J \ll \omega_1, \omega_2 \). Interpret your results.