NOTE: Answer five of the following six questions. Be sure to show your work and explain what you are doing. (Correct answers, for which we cannot follow the work, are worth no credit).

1. Assess whether the following infinite series and products converge or diverge, and give a justification for your answer (15 points: 5+5+5):
   (a) \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^5}{100+n^3 \ln n} \)
   (b) \( \sum_{n=1}^{\infty} \frac{\ln n}{n^{3/4}} \)
   (c) \( \prod_{n=1}^{\infty} \tanh \frac{n \pi}{x} \quad x \neq 0 \)

2. Derive the exact sum of the following infinite series, taking care to sketch the contour used and the singularities that are relevant, if you evaluate it using contour integration techniques (15 points).
\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{1+n^2}
\]

3. For the following function (15 points: 10+5)
\[
F(z) = \int_{-\infty}^{\infty} \frac{e^{-zt^4}}{\cosh t} \, dt
\]
(a) Using ideas adapted from the method of steepest descent, derive the first term of an asymptotic expansion of \( F(z) \) at \( z \to \infty \), assuming \( z \) is real. For what values of \( z \) this integral converges? (10 points)
(b) Find the next term in the asymptotic expansion (5 points)

If gamma functions or other standard special functions arise in your final answer, you do not need to evaluate them.
4. (15 points) The following questions relate to a system of linear equations of the form \( A\vec{x} = \vec{b} \), where \( \vec{b} \) is a given \( N \) vector, \( A \) is a square \( N \times N \) matrix, and \( \vec{x} \) is an unknown vector to be determined. State for each of the following examples whether a solution vector \( \vec{x} \) exists, and find a solution if one does exist; if more than one linearly independent solutions exists, give an orthogonal set of solution vectors.

(a) (15/2 points)

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

(b) (15/2 points)

\[
\begin{pmatrix}
2 & 4 & 6 \\
8 & 10 & 12 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

5. (15 points) The following definitions give the key properties of an orthogonal coordinate system, called prolate spheroidal coordinates. They are written in terms of \( r_1 \) and \( r_2 \), the distances from a given point in space to the two foci of a family of confocal ellipses and hyperbolas. Here \( R \) is the distance between the foci (later in the problem, the protons of a diatomic \( H_2^+ \) molecule will reside at these foci):

\[
\xi = \frac{r_1 + r_2}{R} \quad 1 < \xi < \infty
\]

\[
\eta = \frac{r_1 - r_2}{R} \quad -1 < \eta < 1 \quad \text{and} \quad \phi \quad 0 \leq \phi < 2\pi
\]

Here \( \phi \) is the usual azimuthal angle around the symmetry axis. The factors involved in the volume element and the Laplacian in these coordinates are:

\[
h_\xi = \frac{R}{2} \sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}} \quad h_\eta = \frac{R}{2} \sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}} \quad h_\phi = \frac{R}{2} \sqrt{(\xi^2 - 1)(1 - \eta^2)}
\]

The Hamiltonian for an electron in the field of two protons that are assumed to be clamped rigidly at a distance \( R \) apart, located at the foci of this coordinate system (see figure 1) is given (in atomic units \( m = 1 \), \( \hbar = 1 \), \( e = 1 \), etc...) by

\[
H = -\frac{1}{2} \nabla^2 + \frac{1}{R} - \frac{1}{r_1} - \frac{1}{r_2}
\]

Separate the partial differential equation and give the 3 separated ordinary differential equations that must be solved in order to obtain the energy eigenvalues for this problem.

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6. The Hydrogen-atom solution using SO(4) symmetry ideas: Classically the Runge-Lenz vector $\mathbf{A}$ is a constant of motion which characterizes the fixed axis of the orbit ellipse. There is an analog concept in the quantum problem where the operator

$$\mathbf{A} = \frac{1}{\sqrt{-2}} \left( \mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{F} - \mathbf{r} \right).$$

Here $H = \frac{1}{2} \mathbf{L}^2 - \frac{1}{2} \mathbf{r}^2$ (in atomic units). It satisfies $[\mathbf{A}, H] = 0$ and $\mathbf{A} \cdot \mathbf{A} = \mathbf{L} \cdot \mathbf{L} + 1 + \frac{1}{2\hbar}$. Let's define the operators $\mathbf{J} = \frac{1}{2}(\mathbf{L} + \mathbf{A})$ and $\mathbf{J}' = \frac{1}{2}(\mathbf{L} - \mathbf{A})$. It can be shown they satisfy $[\mathbf{J}_i, \mathbf{J}_j] = i\epsilon_{ijk} \mathbf{J}_k$, $[\mathbf{J}'_i, \mathbf{J}'_j] = i\epsilon_{ijk} \mathbf{J}'_k$ and $[\mathbf{J}_i, \mathbf{J}'_j] = 0$. Using this information (15 points: 5+5+5)

(a) Determine which group $\mathbf{J}$ and $\mathbf{J}'$ are generators of

(b) Show that $F = 1/2(\mathbf{L} \cdot \mathbf{L} + \mathbf{A} \cdot \mathbf{A})$ and $G = (\mathbf{L} \cdot \mathbf{A})$ are Casimir operators.

(c) Using $\mathbf{L} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{L} = 0$ use the $F$ Casimir operator to determine the H atom eigenenergies