Summation of infinite series
(Morse + Feshback, Vol I pag. 413)

\[ S = \sum_{n=-\infty}^{\infty} f(n) \]

We want to show that this is a sum of residues which we can convert back to an integral.

Idea: replace the sum by a contour integral

Observation

1. \( \text{Res} \left( f(z) \pi \cot(\pi z), z = n \right) = f(n) \quad n = 0, \pm 1, \pm 2, \ldots \)

2. \( \text{Res} \left( f(z) \pi \csc(\pi z), z = n \right) = (-1)^n f(n) \quad n = 0, \pm 1, \pm 2, \ldots \)

Let use this for alternating series.

Why:

Note \( \pi \cot(\pi z) \) has simple poles of residue 1 at \( z = n \)

\( \pi \csc(\pi z) \) has simple poles of residue \((-1)^n\) at \( z = n \)

Then:

\[ \frac{1}{2\pi i} \oint_C f(z) \pi \cot(\pi z) \, dz = \sum_{n=-\infty}^{\infty} f(n) + \sum_{n=-\infty}^{\infty} \text{Res} \left( f(z) \pi \cot(\pi z), z = n \right) \]

Finding \( C \):

\( \infty \): poles of \( f(z) \)
Take the limit of a box that goes to infinity through points on the real axis exactly halfway the points at \( n \)-integers.

If \( f(z) \) has no essential singularities anywhere on 
If \( |z f(z)| \to 0 \) as \( |z| \to \infty \), the infinite contour integral will be zero.
Since \( \pi \cot(n \pi) \) is bounded at \( |z| \to \infty \).

\[
\left| \oint_C \cot(z) \frac{dz}{z} \right| \leq A \implies |f(n)| \to 0 \quad \text{as} \quad |n| \to \infty
\]

So

\[
\sum_{n=-\infty}^{2k} f(n) = \sum_{n=-\infty}^{2k} \text{Res} \left( \cot(z) f(z), \frac{z}{2k} \right)
\]

All poles at \( f(z) \).
Alternating series

\[ \sum_{n=0}^{\infty} (-1)^n f(n) = -\sum_{n=0}^{\infty} \text{Res} \left( \frac{\pi \csc(n\pi z)}{z^n} f(z) \right) \]

Example

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{(a+n)^2} \]

\[ f(z) = \frac{1}{(z+a)^2} \quad \text{a real integer} \]

Note we do not want the poles of \( f(z) \) to coincide with the poles of \( \csc(nz) \).

\( f(z) \) has 2\text{nd} order pole at \( z = -a \)

\[ S = -\text{Res} \left[ \frac{n \csc(n^2)}{(a+n)^2}, z = -a \right] \]

Note \( \csc(n^2) = \csc(-n) + \frac{d}{dz} \csc(nz) \)

\[ \left. \left( z+a \right) + \frac{d^2 \csc(nz)}{dz^2} \right|_{z=-a} \]

Then

\[ S = -\pi \left( \csc(n^2) \right) \left. \left( \cot(nz) \csc(nz) \right) \right|_{z=-a} = \pi^2 \cot(na) \csc(na) \]

So

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{(a+n)^2} = \pi^2 \cot(na) \csc(na) \]
Infinite Products

Consider a succession of positive factors

\[ f_1 \cdot f_2 \cdot f_3 \cdots f_n \quad (f_i > 0) \]

\[ f_1 \cdot f_2 \cdots f_n = \prod_{i=1}^{n} f_i \]

We define \( P_n \) as a partial product (in analogy to \( S_n \), the partial sum).

\[ P_n = \prod_{i=1}^{n} f_i \]

And look at the limit \( \lim_{n \to \infty} P_n = P \)

If \( P \) is finite (but not zero) we say that the infinite product is convergent. If \( P \) is infinite or zero the infinite product is labeled as divergent.

Since the product will diverge to infinity if

\[ \lim_{n \to \infty} f_n > 1 \]

or to zero if \( \lim_{n \to \infty} f_n < 1 \) (and \( < 0 \)).

It is then convenient to write our product as

\[ P = \prod_{n=1}^{\infty} (1 + a_n) \]

Notes:

1. \( P \) diverges to \( \infty \) if \( \lim_{i \to \infty} a_i > 0 \)

2. \( P \) diverges to 0 if \( -1 < \lim_{i \to \infty} a_i < 0 \)
The necessary condition for convergence of this product is $\lim_{n \to \infty} a_n = 0$.

Connection with infinite series

$$P = \exp \left[ \ln \prod_{n=1}^{\infty} (1 + a_n) \right] = \exp \left[ \sum_{n=1}^{\infty} \ln (1 + a_n) \right]$$

Thus the infinite product converges if the infinite series $\sum_{n=1}^{\infty} \ln (1 + a_n)$ converges.

Moreover, for small $|a_n|$, $\ln (1 + a_n) = a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \ldots$

Therefore, $\sum_{n=1}^{\infty} \ln (1 + a_n)$ converges or diverges exactly as the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

Convergence See A&W

If $|a_n| < 1$

$$\prod_{n=1}^{\infty} (1 + a_n) = \begin{cases} \text{converges} & \text{if } \sum_{n=1}^{\infty} a_n \text{ converges} \\ \text{diverges} & \text{if } \sum_{n=1}^{\infty} a_n \text{ diverges} \end{cases}$$
Some important useful infinite product representations.

Recall that a polynomial can be written as a product of roots, i.e.

\[ p_n = c (x-x_1)(x-x_2) \cdots (x-x_n) \]

all roots in the complex plane

This generalizes for many functions having an infinite number of zeros.

e.g. \[ \sin(x) = x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2 \pi^2} \right) \]

\[ \prod_{n=1}^{\infty} \left( 1 - \frac{x}{n\pi} \right) \left( 1 + \frac{x}{n\pi} \right) \]

Likewise \[ \cos(x) = \prod_{i=1}^{\infty} \left( 1 - \frac{x^2}{(2n-1)^2 \pi^2} \right) \]

An equivalent form can be seen to be

\[ \sin(x) = x \left( 1 - \frac{x}{\pi} \right) \left( 1 + \frac{x}{\pi} \right) \left( 1 - \frac{x}{2\pi} \right) \left( 1 + \frac{x}{2\pi} \right) \cdots \]

or \[ \frac{\sin x}{x} = \prod_{n=1}^{\infty} \left( 1 + \frac{x}{n\pi} \right) \]

However, \[ \sum_{n=1}^{\infty} \frac{|x|}{n\pi} \] diverges!
We have gone from an absolutely convergent to a conditionally convergent infinite product

\[
\frac{\sin x}{x} = \left[ \frac{(1 - \frac{x}{\pi})^\frac{x}{\pi}}{\pi} \right] \left[ \frac{(1 + \frac{x}{\pi})^\frac{-x}{\pi}}{\pi} \right] \left[ \frac{(1 - \frac{x}{2\pi})^\frac{x}{2\pi}}{\pi} \right] \left[ \frac{(1 + \frac{x}{2\pi})^\frac{-x}{2\pi}}{\pi} \right] \ldots
\]

\[
\frac{\sin x}{x} = \prod_{n=0}^{\infty} \left[ \frac{(1 + \frac{x}{n\pi})^\frac{-x}{n\pi}}{n\pi} \right] \text{ which is absolutely convergent in this form}
\]

\[
= \prod_{n=1}^{\infty} \left( \frac{1 + \frac{x}{n\pi}}{n\pi} \right)^{-\frac{x}{n\pi}} \prod_{n=1}^{\infty} \left( \frac{1 - \frac{x}{n\pi}}{n\pi} \right)^{\frac{x}{n\pi}}
\]

Another fundamental infinite product is

\[
\frac{1}{\Gamma(x)} = x^{-x} \prod_{n=1}^{\infty} \left( 1 + \frac{x}{n} \right)^{-\frac{x}{n}}
\]

with \( \gamma \) = Euler-Mascheroni constant

\[
\lim_{m \to \infty} \left( \sum_{k=1}^{m} \frac{1}{k} - \ln m \right) = 0.577216...
\]
So choose that

\[
\frac{1}{\Gamma(x)\Gamma(1-x)} = -x^2 \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2} \right)
\]

\[
= -x \left( \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2} \right) \right)
\]

\[
= -\frac{x}{\pi} \sin(\pi x)
\]

\[
\left( \frac{\Gamma(x)}{\Gamma(-x)} \right) = -\frac{x}{\pi \sin(\pi x)}
\]

or using \( \Gamma(z+1) = z \Gamma(z) \) with \( z = -x \)

\[
\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}
\]

Aside: For numerical calculation, infinite products like this are less efficient than series and thus they are rarely used.