\[ \frac{m!}{n! (m-n)!} \]

Binomial coefficient

**Binomial distribution**

A random variable \( Y \) follows the binomial distribution with parameters \( n \in \mathbb{N} \) and \( p \in [0,1] \).

The probability of getting exactly \( k \) successes in \( n \) trials is given by

\[ \Pr(Y = k) = \binom{n}{k} p^k (1-p)^{n-k} \]

Compute the expected value of \( X \)

\[ \mathbb{E}(Y) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \]

\text{Trick:} \quad (p+q)^n = q^n \left(1+\frac{p}{q}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{p}{q}\right)^j q^{n-j}

\[ q = 1-p \]

\[ = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \]

\[ (np+q)^n = \sum_{j=0}^{n} \binom{n}{j} (np)^j q^{n-j} \]

\[ \frac{\partial}{\partial a} (np+q)^n = \sum_{j=0}^{n} \binom{n}{j} j (np)^{j-1} p q^{n-j} \]
\[
\frac{\partial}{\partial \psi} \left[ \psi \right]^n = \sum_{j=0}^{n} \binom{n}{j} j \cdot \psi^j q^{n-j} = \langle \psi \rangle
\]

So \( \langle \psi \rangle = n (\varphi \psi + q) p \bigg|_{\varphi = 1}^n = n (p+q) p = np \)

How about \( \langle \psi^2 \rangle \)

\[
\langle \psi^2 \rangle = \sum_{k=0}^{n} u^2 \binom{n}{k} p^k (1-p)^{n-k} = \sum_{j=0}^{n} \binom{n}{j} j (j-1) \psi^j q^{n-j} \sum_{k=2}^{n} \binom{n}{k-2} \psi^k q^{n-k} = \sum_{j=0}^{n} \binom{n}{j} (j^2 - j) \psi^j q^{n-j} \]

So \( \langle \psi^2 \rangle = \frac{\partial^2}{\partial \psi^2} \left[ \varphi \psi + q \right]^n \bigg|_{\varphi = 1} + np = n(n-1)p^2 + np \)
The binomial formula generalizes to an infinite Taylor series if \( m \neq 0, 1, 2, 3 \ldots \) i.e. if \( m \neq \text{integer} \) or \( m = \text{integer} < 0 \). However, we must interpret

\[ m! = \Gamma(m+1) = \text{Gamma function} \]

The function is defined by the relations given in the book. Here, just note that \( \Gamma(z) \) obeys

\[ (i) \quad \Gamma(z+1) = z \Gamma(z) \]
\[ (ii) \quad \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(z\pi)} \]

When the binomial series converges,

Look at \[ \frac{a_{n+1}}{a_n} = \frac{x^{m+1}}{(n+1)!} \frac{n!}{(m-n)!} \]

\[ = x \frac{m-n}{n+1} \]

And \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \) so the binomial series converges absolutely if \( |x| < 1 \).

Aside: Polynomial expansion for integer \( M, N \)

\[ (a_1 + a_2 + \ldots + a_M)^N = \sum \frac{N!}{n_1! \ldots n_M!} a_1^{n_1} a_2^{n_2} \ldots a_M^{n_M} \]

where \( P = (n_1, n_2, \ldots, n_M) \) all permutations of nonnegative integers \( n_i \) such that \( \sum n_i = N \)
\[(a+b+c)^5 = a^5 + 5(a^4b + a^4c + b^4a + b^4c + c^4a + c^4b) + 10(a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2) + 120(a^2b^2c + b^2ac^2 + c^2ab^2) + 30(a^2b^2c + ab^2c^2 + ba^2c^2) + b^5 + c^5\]

How many terms

\[\frac{N! + M! - 1}{M-1! N!}\]

Check \(N=5\) \(M=3\) \(\frac{7!}{2!5!} = \frac{6!}{1!2!} = 21\)

How? Think how many ways I can put \(M-1\) walls to separate \(N\) particles.

This relation is useful to count the dimension of the Hilbert space of \(N\) bosons in \(M\) sites.
Power Series

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n \]

\( a_n \): constants independent of \( x \)

- Convergence

\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R} \]

Then the series converges for \(-R < x < R\)

\( a_n = \frac{1}{n} \) then \( R = 1 \) and \( a_n = n! \) the \( R = 0 \): diverges for all \( x \neq 0 \)

- Uniform and Absolute Convergence

Suppose a power series has been found convergent for \(-R < x < R\), then it will be uniformly and absolutely convergent in any interior interval \(-S < x < S\) where \( 0 < S < R \)

May be proved by the Weierstrass M test as we did for \( \sum_{n}^{} \)

- Continuity

Since each term \( \frac{a_n x^n}{n} \) is a continuous function of \( x \) and \( f(x) = \sum_{n}^{} a_n x^n \) converges uniformly for \(-S \leq x \leq S\), \( f(x) \) must be continuous in the interval of uniform convergence

- Differentiation and Integration

With \( \sum_{n=0} a_n x^n \) continuous and \( \sum_{n=0}^{} a_n x^n \) uniformly convergent, then the differentiated series...
is a power series with continuous function and the same radius of convergences than the original series. Note new factors introduced by differentiation (integration) do not affect the ratio test. Therefore:

- A power series might be integrated and differentiated as often as desired within the interval of convergence.

- Uniqueness

\[ \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \]

\[-R_a < x < R_a \quad -R_b < x < R_b\]

with overlapping intervals of convergence including the origin then \( a_n = b_n + n \)

Inversion of Power Series

Sometimes we know \( y(x) = a_0 + a_1 x + a_2 x^2 + \ldots \)

but instead we want to know \( x \) as a function of \( y \)

\( x(y) = b_0 + b_1 (y - a_0) + b_2 (y - a_0)^2 + \ldots \)

Then \( b_0 = 0 \), \( b_1 = 1/a_1 \), \( b_2 = -a_2/a_1^3 \)

\[ b_3 = \frac{2a_3 - a_1 a_3}{a_1^5} \]

why?
\[ x - b_0 = b_1 (a_1 x + a_1 x^2 + \ldots) + b_2 (a_1 x + a_1 x^2 + \ldots)^2 + \ldots \]

Then \( b_0 = 0 \)
\begin{align*}
\Rightarrow x - b_0 &= (b_1 a_1)x^2 + (b_1 a_2 + b_2 a_1^2) x^2 \\
&\quad + (b_1 a_3 + 2b_2 a_1 a_2 + b_3 a_1^3) x^3 + \ldots
\end{align*}

Then \( b_0 = 0 \) \quad \Rightarrow b_1 a_1 = x \quad \Rightarrow b_1 = \frac{y}{a_1}

\( b_1 a_2 + b_2 a_1^2 = 0 \) \quad \Rightarrow b_2 = \frac{-b_1 a_2}{a_1^2} = -\frac{a_2}{a_1^3}

\( b_3 = -\left(\frac{b_1 a_3 + 2b_2 a_1 a_2}{a_1^3}\right) \)

\[ = \frac{2a_2^2 - a_1 a_3}{a_1^5} \ldots \]