Feshbach Resonances:

The physical origin of a Feshbach resonance can be understood by considering two molecular potentials, $V_{bg}(r)$ and $V_{cc}(r)$.

For large internuclear distances $r$, the background potential connects to free particles. For a collision process having low energy $E$, this potential represents the open channel, referred to as entrance channel. The other potential, $V_{cc}(r)$, representing the closed channel is important as it can support bound molecular states closed to the open channel.

A. Feshbach resonance occurs when the bound molecular state in the closed channel approaches the scattering state in the open channel. Then a weak coupling can lead to a strong mixing between the two channels. The corresponding energy difference can be controlled via a magnetic field when the corresponding magnetic moments are different. This leads to a magnetically tuned feshbach resonance. The magnetic tuning
method is the common way to achieve resonance coupling and it has found numerous applications. An alternative tuning method can be achieved by optical methods, leading to optical Feshbach resonances. These ones have many conceptual similarities to the magnetic resonances.

More explicitly:

Both channels obey a Schrödinger Eq. and if there is some interaction between the two channels then their Schrödinger equations are coupled

\[(\hat{S} + V_{bg}(r) - E) \Psi_{bg}(r) + g \Psi_c(r) = 0 \quad (1)\]

\[(\hat{S} + V_{cc}(r) - E) \Psi_c(r) + g \Psi_{bg}(r) = 0 \quad (2)\]

Here \(\hat{S}\) is the kinetic energy operator. Let \(\Psi_c^m\) a complete set of orthonormal wave functions

\[(\hat{S} + V_{cc}(r) - E_m) \Psi_c^m = 0\]

And let expand \(\Psi_c\) in terms of them

\[\Psi_c(r) = \sum_{m=1}^{\infty} d_m \Psi_c^m(r) \quad (3)\]

Substitution of Eq. (3) into Eq. (1) and taking the inner product with \(\Psi_c^m\) yields:

\[(E_n - E) d_n = -\langle \Psi_c^m | g(r) | \Psi_{bg}(r) \rangle \]

\[d_n = -\frac{\langle \Psi_c^m | g(r) | \Psi_{bg}(r) \rangle}{E_n - E}\]

Close to resonance only one of these dres will have an appreciable physical effect, so the closed
channel wave function is approximately given by

\[ \Psi_c(r) \approx -\frac{\langle \Psi_{c,\text{res}}^* | g(r) | \Psi_{bg}(r) \rangle}{E_{\text{res}} - E} \Psi_{c,\text{res}}(r) \]

Now we substitute this expression into (1) to find the effect of the nearly resonance closed channel on the Sc. equation for the open channel

\[ (\hat{F} + V_{bg}(r) - E)\Psi_{bg}(r) = g \frac{\langle \Psi_{c,\text{res}}^* | g | \Psi_{bg} \rangle}{E_{\text{res}} - E} \Psi_{c,\text{res}}(r) \]

To solve this Eq. we can move use the homogeneous solution

\[ (\hat{F} + V_{bg}(r) - E)X = 0 \Rightarrow X \propto (1 - \frac{a_{bg}}{r}) \]

Here \( a_{bg} \) is the scattering length of the open channel in the absence of the effects of the closed channel.

There are many possible solutions of the inhomogeneous equation

\[ (\hat{F} + V_{bg}(r) - E)\eta = g \frac{\langle \Psi_{c,\text{res}}^* | g | \Psi_{bg} \rangle}{E_{\text{res}} - E} \Psi_{c,\text{res}}(r) \]

We will choose one which behaves as

\[ \eta = A \left(1 - \frac{b}{r}\right) \]

where \( A \) and \( b \) are determined by \( V_{bg}(r) \) and \( g \).

The complete solution is a superposition of both homogeneous and inhomogeneous

\[ \Psi_c(r) = dX(r) + B\eta(r) = (d + BA) - \frac{d a_{bg} + BAb}{r} \]

which corresponds to an scattering length.
\[ a = a_b g + \frac{A (b-a_b g)}{A + \frac{1}{\beta}} \]

where all the parameters are new with the exception of \( \frac{1}{\beta} \). This can be written such that \( (1+\frac{1}{\beta a_b})d(E_0 - \varepsilon) \)

\[ a = a_b g \left( 1 + \frac{\Delta}{E_{E_0} - \varepsilon} \right) \]

For low energy collisions

\[ a = a_b g \left( 1 + \frac{\Delta}{E_{E_0}} \right) \]

The scattering length then diverges near the magnetic field \( B_0 \) at which the 'cloud channel ground state is resonant with the threshold energy of the open channel.

One should note that the binding energy of the weakly bound level is not equal to the energy of the resonant level.

\[ |E_0| \approx \frac{\hbar^2}{2m a_b^2} \approx \frac{\hbar^2}{2m a_b^2} \left( \frac{E_{E_0}}{a_b} \right)^2 \approx E_{E_0} \]