The Fermi Pseudopotential

Importance of the scattering length in the long-wavelength limit

E. Fermi, Nuovo Cimento 11, 157 (1934)

He considered the spectrum of highly-excited (Rydberg) atoms (Na) embedded in a gas of ground state He atoms (perturbs)

\[ \text{He} - \text{Na}^+ - e \]

High-n Rydberg e⁻: very slow \((V \sim \frac{1}{n})\) with a long de Broglie wavelength

e⁻-He scattering: zero energy collision = s-wave only

The e⁻ Sch. equation looks like:

\[ \nabla^2 \psi + \frac{2m}{\hbar^2} (E - U - \xi V_i) \psi = 0 \]

\( U: e^-\text{Na}^+ \text{ potential} \)

\( \xi V_i: e^-\text{He interactions} \)

Idea: Find the average effect of all perturbers by utilizing the fact that the de Broglie wavelength is quite long i.e. call this average wavefunction \( \overline{\psi} \) and call the average of \( V \psi \to \overline{\psi} \).
Then
\[ \nabla^2 \Phi + \frac{2m}{\hbar^2} (E - u) \Phi - \frac{2m}{\hbar^2} \frac{\varepsilon}{\varepsilon_n} \Phi = 0 \]

Next consider \( \Phi \) for the \( e^- \), near a perturbing \( He \) atom in a coordinate system centered on the \( He \).

The \( e^-He \) scattering is at very low \( E \), so it should be dominated by the \( s \)-wave collision according to the Wigner threshold law: \( 8e \sim k^{2.01} \).

So let's examine the \( l=0 \) radial Schrödinger Eq.

Setting \( \Phi = \frac{u(r)}{r} \), \( \Psi_0 \theta, \phi \)

\[ u(r) \] obeys
\[ u''(r) = \frac{2m}{\hbar^2} \left( V(r)u(r) - E u(r) \right) \]
since \( \varepsilon=0 \)

So the solution for \( u(r) \) at \( r>r_0 \) (when \( u=0 \)) is in general just a straight line since

\[ u''(r) = 0 \] linear

The behavior of \( u(r) \) at \( E=0 \) looks like

\[ u(r) \text{ at } E=0 \]

\[ \Psi_0 \theta, \phi \]
Now, recall that $U(r) = \begin{cases} \text{complicated} & r < r_0 \\ r \sin (u r + 5) & r > r_0 \end{cases}$

We have seen that for $u \to 0$, $S_0 \to -k \text{asc}$.

Now to complete Fermi's derivation, we want to determine the average $\overline{\psi^4}$.

Observe that

$$\frac{2m}{\hbar^2} \int \psi^4 \, d^3r = 4\pi Y_0 \text{asc}$$

Proof:

$$\frac{2m}{\hbar^2} \int \psi^4 \, d^3r = \frac{2m}{\hbar^2} 4\pi Y_0 \int_0^{\infty} u(r) \frac{\psi(r)}{r} \, r^2 \, dr$$

$$= 4\pi Y_0 \int_0^{\infty} u''(r) \, dr \quad \text{replacing} \quad \frac{2m}{\hbar^2} \psi \, u = u''$$

Integrate by parts

$$= 4\pi Y_0 \left[ -\int_0^{\infty} u''(r) \, dr + r u'(r) \right]_0^{\infty}$$

$$= 4\pi Y_0 \left[ -u(\infty) \pm r_0 u'(r_0) \right]$$

$$= 4\pi Y_0 \left[ -C (r - \text{asc}) \pm r_0 \right]$$
\[ \bar{\psi} = \frac{\int d^3r \, \psi_{so}(r)}{\int d^3r} = \frac{\int d^3r \, \psi_{so}(C(r-R))}{\int d^3r} \]

or \( \bar{\psi} \approx C \psi_{so} \) provided that the averaging volume is large compared with \( a_{so}^3 \)

if \( (\text{density}) \, a_{so}^3 \ll 1 \)

\[ C = \frac{\bar{\psi}}{\psi_{so}} \Rightarrow 2 \pi \int \frac{V \psi}{k} \, d^3r = 4 \pi a_{so} \bar{\psi} \]

Which is often expressed in an equivalent form that can be used for \( V \) in the limit of low density and low energy.

\[ V(\bar{r}) = \frac{2\pi \, a_{so} \, k^2}{\hbar^2} \delta(\bar{r} - \bar{R}) \]

This is the zero-range Fermi–Pseudopotential

Definition: A pseudopotential is a simplified potential that can be used in place of the “true potential” if it has been designed to give certain properties correctly, such as energies, scattering phase shifts in the relevant range of energies.

Aside: The Fermi pseudopotential is too singular to be used in solving the exact 3D Schrödinger Eq. but it can be used in perturbative treatments.