Example: The finite square well \( V(r) = \begin{cases} -V_0 & r < r_0 \\ 0 & r > r_0 \end{cases} \)

Since \( V(r) \) is constant at \( r < r_0 \), the solution is still a spherical Bessel function inside, namely

\[ U_{\text{ex}}(r) = N r j_x(qr) \quad \text{where} \quad q = \left( \frac{2m(E+V_0)}{h^2} \right)^{1/2} \]

The negative of the log derivative

\[ b_{cl} = -\frac{U_{\text{cl}}'(r_0)}{U_{\text{cl}}(r_0)} \quad \Rightarrow \quad S_{\text{cl}}(E) = \arctan \left[ \frac{F_{\text{ex}}(r_0) b_{cl} + F_{\text{ex}}'(r_0)}{C_{\text{ex}}(r_0) b_{cl} + C_{\text{ex}}'(r_0)} \right] \]

\[ b_{cl} = -\left| \frac{\frac{d}{dr} \left[ r j_x(qr) \right]}{r j_x(qr)} \right|_{r = r_0} \]

We know \( b_{cl} \), \( F_{\text{ex}}(r) = \left( \frac{2m}{\pi^{1/2} h^2} \right)^{1/2} \frac{1}{r} r j_x(qr) \), and \( C_{\text{ex}} = \left( \frac{2m}{\pi^{1/2} h^2} \right)^{1/2} \frac{1}{r} r j_x(qr) \)

So we know \( S_{\text{cl}}(E) \) and with that we can compute
So we know $S_e (E)$ and with that we can compute

$$
\frac{d\sigma}{d\Omega} = \left| \frac{4\pi}{2\sin \theta} \right|^2 \sum_{m} \left( e^{i S_e} - 1 \right) \langle \ell m | \ell m \rangle
$$

and $S_e = \frac{4\pi}{\hbar^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \theta_e$

**Limiting Behavior at Low Energies**

Various names associated: 
- Wigner threshold law
- Effective range theory

Idea: All low energy the behavior of $(F, G)$ is

$$
F_{ee} (r) \xrightarrow{\nu \ll \epsilon_e} \left( \frac{2m}{\pi \hbar^2 \nu} \right)^{1/2} \frac{(\nu r)_{\ell+1}}{(2\ell+1)!} \sim \nu^{\ell+1/2} r^{\ell+1} \delta_e
$$

and

$$
G_{ee} (r) \xrightarrow{\nu \ll \epsilon_e} -\left( \frac{2m}{\pi \hbar^2 \nu} \right)^{1/2} \frac{(2\ell-1)!}{(\nu r)^{\ell}} \sim \nu^{-\ell-1/2} r^{-\ell} \delta_e
$$

Here $\delta_e$ and $\delta_e$ are constants independent of $r$ and $\nu$.

So at low energies where $\nu \to 0$, we have

$$
\tan \left( S_e (E) \right) \xrightarrow{\nu \to 0} \left[ \frac{b_{ee} r_0^{\ell+1} + (c_{ee} r_0^\ell)}{C_{ee} r_0^\ell - 2 r_0^{\ell-1}} \right] \delta_e \nu^{\ell+1/2} \nu^{-\ell-1/2}
$$

Since $b_{ee} \sim$ constant $+ O(\epsilon)$ at $E \to 0$

$$
\tan \left( S_e (E) \right) \xrightarrow{\nu \to 0} -\nu^{\ell+1} a_e \left( 1 + c_{ee} E + \cdots \right) = -\nu^{\ell+1} a_e
$$
This leading order behavior is called the Wigner threshold law for low energy scattering phase shifts by a finite range potential.

Note: It does not apply to potentials that become Coulombic at larger r.

**Important Special Case**

It is normally the s-wave (\( l=0 \)) partial wave that dominates at low energy, so the constant \( a = a_{s=0} \) with dimension of length is particularly important, and \( a \) is called the

**Scattering Length**

\[
q = \lim_{\epsilon \to 0} \left( \frac{\tan \delta_{0}}{\epsilon} \right)
\]

The low energy expansion is sometimes written with the next term too as

\[
h \cot \delta_{s=0} = -\frac{1}{a} + \frac{1}{2} \frac{n^2 \text{Re} \rho}{m}
\]

Called effective range

Let's consider the square well potential for \( r > r_0 \)

\[
\frac{d^2u}{dr^2} = 0 \quad u = \text{constant} (r - a)
\]
\[
\frac{d^2 u}{dr^2} = 0 \quad u = \text{constant} (r-a)
\]

But before we wrote

\[ U_{z=0} = A_0 \left[ Fe_0 \cos \phi_0 - Ge_0 \sin \phi_0 \right] \]

Note \( Fe_0 = \sin (\pi r) \quad Ge_0 = -\cos (\pi r) \)

Then \( U_{z=0} \sim \sin (\pi r + \phi_0) \)

\[
\frac{u'_0}{u} = \frac{1}{1 + \cot [\pi (r+\phi_0)]} \xrightarrow{r \to a} \frac{1}{r-a}
\]

At \( r = 0 \) \( \lim_{u \to 0} u \cot (\pi a) = \frac{1}{-a} \)

\[ f_0 = \frac{a}{1 + \cot a} \]

\[ \text{Sel} \left|_{z=0} = \frac{4 \pi}{h^2} \sin^2 \phi_0 \sim \alpha \pi a^2 \right. \]

Even though \( a \) has the dimension of the range of the potential, \( a \) and \( r_0 \) can differ by orders of magnitude. In particular for an attractive potential, it is possible that \( |a| \gg r_0 \).

To understand the physical meaning of \( a \) note that \( a \) is the intercept of the outside wave function. For a repulsive potential \( a \geq 0 \) and it is roughly of the order of \( r_0 \).
The sign change resulting from the increased attraction is related to the development of a bound state.

For a very large and positive $r$, the wave function is essentially flat for $r > R$. But with a very large $r$, it is not too different from $e^{-ur}$ with $u$ essentially zero. Now $e^{-ur}$ with $u = 0$ is just a bound state function for $r > R$ with energy $E$ infinitesimally negative. The inside wave function ($r < R$) for the $E = 0^+$ (scattering with zero kinetic energy) and the $E = 0^-$ case (bound state with infinitesimally small binding energy) are essentially the same because in both cases $u$ is determined by

$$\frac{\hbar^2 n^2}{2m} = E - V_0 \approx |V_0| \quad \text{with } E \text{ infinitesimal}$$

Because the inside wave function are the same for both physical situations ($E = 0^+$ and $E = 0^-$) we can equate the logarithmic derivative of the bound state with
that of the solution involving zero-metric energy scattering

\[
- \frac{\nu e^{-i\nu r}}{e^{i\nu r}} = \left( \frac{1}{(r-a)} \right) \bigg|_{r=r_0}
\]

If \( R \ll a \quad \frac{\nu}{a} \quad \frac{1}{a}

The binding energy satisfies

\[
E_{\text{ SCE}} = E_{\text{core state}} = \frac{\hbar^2 n^2}{2m} = \frac{\hbar^2}{2am^2}
\]

Nice result: If there is a loosely bound state we can infer its binding energy by performing a scattering experiment near kinetic energy, provided \( a \) is large compared to the range of the potential \( r_0 \)