SO(2) and SO(3) rotation groups

For SO(2) there is just one independent generator and it can be found by differentiating the finite rotation matrix and evaluating it at the identity element

\[ -\frac{i}{\hbar} \frac{d}{d\phi} R(\phi) \bigg|_{\phi=0} = -\frac{i}{\hbar} \left( \begin{array}{cc} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{array} \right) \bigg|_{\phi=0} = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) = i\sigma_z \]

For SO(3) the finite rotation matrix about the z axis is

\[ R_z(\phi) = \left[ \begin{array}{ccc} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{array} \right] \]

and the corresponding generator is

\[ -\frac{i}{\hbar} \frac{d}{d\phi} R_z = S_z = \left( \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \]

\[ R_z(\hbar \phi) = 1 + i\hbar \phi S_z \]

A finite rotation can be composed of successive infinitesimal rotations

\[ R_z(\hbar \phi_1, + \hbar \phi_2) = (1 + i\hbar \phi_1 S_z) (1 + i\hbar \phi_2 S_z) \]
or \( R_x(z) = \lim_{N \to \infty} \left[ 1 + \frac{i \theta}{N} S_z \right]^N = e^{i \theta S_z} \)

\( R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta + i \sin \theta & 0 \\ 0 & -i \sin \theta & \cos \theta \end{pmatrix} \) 

\( R_y(\theta) = \begin{pmatrix} \cos \theta & 0 & -i \sin \theta \\ 0 & 1 & 0 \\ i \sin \theta & 0 & \cos \theta \end{pmatrix} \) 

Convention is not arbitrary. Signs are chosen to make the operators cyclic.

\( R_y R_x = i \sin \theta \)

\( R_y R_x = + \sin \theta \)

\( R_z x \quad \text{whereby} \)

\( S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \)

\( S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \)

Note that \( S_z \) is the generator of the group \( R_z \), which is an Abelian subgroup of \( SO(3) \).

**Connection with orbital angular momentum**

Let us act the rotation operator \( R \) on a state \( \psi(x, y, z) \) and remember that if we use \( R_z(\theta) \) which rotates the axis \( (x, y, z) \) CCW by \( \theta \) about \( \hat{z} \), this is equivalent to rotating the lobes of \( \psi \) by \( -\theta \).

i.e. let \( \vec{r}' = \vec{r} \) and define

\( R \psi(x, y, z) = \psi'( \vec{r} ) = \psi( \vec{r}' ) \)

Going to the infinitesimal limit, the

\( R_z(\delta \theta) \psi(x, y, z) = \psi( x+y \delta \theta, y-x \delta \theta, z ) \)
An expansion into a Taylor series gives

\[ R_z(\phi) \psi(x, y, z) = \psi(x, y, z) - i \hbar \frac{\partial}{\partial y} \psi + \frac{\partial^2}{\partial x^2} \psi + O(\phi^2) \]

\[ = \left(1 - i \phi \Omega \right) \psi(x, y, z) \]

\[ \Omega \rightarrow R_z(\phi) = 1 - i \phi \Omega \zeta \rightarrow R_z(\phi) = e^{-i \phi \Omega} \]

This connects with the definition of the orbital angular momentum operator

\[ L_z = xy - yx = -i \left( \frac{x}{\partial y} - \frac{y}{\partial x} \right) \]

We can also write

\[ L_z = \left( x, y, z \right) S_z \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \]

\[ L_x = \left( x, y, z \right) S_x \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \end{pmatrix} \]

\[ L_y = \left( x, y, z \right) S_y \begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \]

\[ [L_i, L_j] = i \left( \hbar \right) \epsilon_{ijk} L_k \]
The following results are valid for \( SU(n) \) and \( SO(n) \).

To begin, choose a set of linearly independent and mutually commuting operators \( \mathbf{t}_i = \{ \cdots, I \} \), where \( l \) is the maximum number obeying

\[
[t_i, t_j] = 0 \quad \text{think } t_i \text{ as generalizations of } S^2
\]

\( l \): is the rank of the Lie group or its Lie algebra.

\( H \): raising and lowering operators

\( \mathbf{E} \): the other generators besides the \( t_i \)

\( [H, \mathbf{E}] = \alpha \mathbf{E} \)

where we call \( N \): the order of the Lie group

\( \# \) of linearly independent generators.

\begin{align*}
\text{Lie Algebra} & \quad A_\ell \quad B_\ell \quad D_\ell \\
\text{Lie group} & \quad SU(2\ell + 1) \quad SO(2\ell + 1) \quad SO(2\ell - 1)
\end{align*}

\begin{align*}
\text{Rank} & \quad \ell \quad \ell \quad \ell \\
\text{Order} & \quad \ell (2\ell + 2) \quad \ell (2\ell + 1) \quad \ell (2\ell - 1)
\end{align*}

eg \( SU(2) \): spin \( \frac{1}{2} \) angular momentum

\( \text{Order } 3, \text{ Rank } 1 \) (one commuting generator)

\( SO(3) \): \( \text{Order } 3, \text{ Rank } 1 \) (one commuting generator)

It is not surprising that \( SU(2) \) and \( SO(3) \) are homomorphic.

The \( SU(2) \) rotations for spin \( \frac{1}{2} \) involve half angles.
\[ \text{e.g. } U_3(d) = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} \]

"parameter" \(\alpha/2\) ranges from 0 to \(\pi\).

where as for \(SO(3)\) rotations for vectors involve

\[ R_3(d) = \begin{pmatrix} \cos d & \sin d & 0 \\ -\sin d & \cos d & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

"parameter" \(d\) ranges from 0 to \(2\pi\).

The correspondence is 2-to-1 not 1-to-1 and therefore it is a homomorphism not isomorphism.

\(SU(2)\) describes rotations in a 2D complex space \(\mathbf{C} = \binom{z_1}{z_2}\) that leaves \(|z_1|^2 + |z_2|^2\) invariant.

Determinant +1 and thus 3 independent real parameters.

\(SO(3)\) Real orthogonal group describes rotations in 3D space letting \(x, y, z\) invariant: \(x^2 + y^2 + z^2 = \text{const.}\)

3 independent real parameters.

All these suggest there is a correspondence between \(SO(3)\) and \(SU(2)\).
\[ R = U : \text{unitary transformation} \]

\[ M' = U M U^* \quad \text{M: 2x2 matrix} \]

\[ M = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} x & -iy \\ ix & y \end{pmatrix} \]

Here we have excluded any projection on identity since we want \( M \) traceless.

\( M' \) should then be

\[ M' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3 \]

Determinant is invariant during unitary operations so \( (x^2 + y^2 + z^2) = (x'^2 + y'^2 + z'^2) \)

Operations of \( su(2) \) on \( M \) must produce rotations of the coordinates \( x, y, z \) appearing therein. This suggests \( su(2) \) and \( so(3) \) or homomorphic.

This is in fact the case.

We saw it for rotations along \( z \):

\[ U_z = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \]

\[ U_z \sigma_1 U_z^+ = x (\sigma_1 \cos(\theta) - \sigma_2 \sin(\theta)) \]

\[ U_z \sigma_2 U_z^+ = y (\sigma_1 \sin(\theta) + \sigma_2 \cos(\theta)) \]

\[ U_z \sigma_3 U_z^+ = z \sigma_3 \]
\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = -x \sin \theta + y \cos \theta \]
\[ z' = z \]

More generally, a sequence of 3 Euler rotations in \( \text{SU}(2) \) looks like
\[
e^{-\frac{i}{2} \sigma_3} e^{-i \beta \sigma_2} e^{-i \alpha \sigma_1}
\]

\[
\begin{bmatrix}
    -U(\alpha, \beta, \gamma) = U_3(\gamma) U_2(\beta) U_1(\alpha) \\
    \begin{pmatrix}
      \cos(\beta/2) & -i \sin(\beta/2) \\
      -i \sin(\beta/2) & \cos(\beta/2)
    \end{pmatrix}
\end{bmatrix}
\]

Defining \( x + y = \phi \), \(-x + iy = \psi\), \( \beta/2 = \theta \)

\[
\begin{pmatrix}
    \cos(\phi) & -i \sin(\phi) \\
    -i \sin(\phi) & \cos(\phi)
\end{pmatrix}
\begin{pmatrix}
    \cos(\psi) & i \sin(\psi) \\
    -i \sin(\psi) & \cos(\psi)
\end{pmatrix}
= \begin{pmatrix}
    \alpha & \beta \\
    -\beta & \alpha
\end{pmatrix}
\]

This is the form of a generic element in \( \text{SU}(2) \):
\( \det(u) = 1 \), \( u^* u = uu^* = 1 \)
There are 0 invariant operators, i.e., called Casimir operators, which commute with all
generators and are generalizations of $J^2$

Schur's Lemma

An operator $\mathcal{H}$ that commutes with all
group generators $\mathcal{H}_i$ of the Lie group $G$, and all group operators has as eigenvectors
all states of the multiplet and is degenerate
for all multiplet eigenvectors.

Moreover $\mathcal{H}$ commutes with all Casimir invariants

$[\mathcal{H}, C_i] = 0$

$SO(3)$ has $l = 1 \implies 1$ cosimir operator $L_x + L_y + L_z$
and $l = 1 \implies 1$ commuting generator $= L^2$

$\implies$ on Spherically symmetric Hamiltonian $\mathcal{H}$
will have the same energy (eigenvalue $\lambda + l$)
for all $2l+1$, i.e., states of a given $l$

Example of $SO(4)$ $l = 2$, $N = 6$

Let the independent variables be $x, y, z, t$

The generators can be chosen as

$M_1 = \frac{2}{t} \frac{\partial}{\partial y} - \frac{y}{2} \frac{\partial}{\partial t}$
$M_2 = \frac{2}{t} \frac{\partial}{\partial x} - \frac{x}{2} \frac{\partial}{\partial y}$
$M_3 = \frac{2}{t} \frac{\partial}{\partial z}$

$N_1 = \frac{2}{t} \frac{\partial}{\partial x} - \frac{x}{2} \frac{\partial}{\partial t}$
$N_2 = \frac{2}{t} \frac{\partial}{\partial y} - \frac{y}{2} \frac{\partial}{\partial t}$
$N_3 = \frac{2}{t} \frac{\partial}{\partial z}$
And the commutators are readily found to be

\[ [M_i, M_j] = E_{ij} M_k \quad [N_i, N_j] = 0 \]

\[ [M_i, N_j] = E_{ij} M_k \quad [N_i, N_j] = E_{ij} M_k \]

And they are usually replaced by

\[ S_i = M_i + N_i \quad \frac{1}{2} \]
\[ S_i = M_i - N_i \quad \frac{1}{2} \]

\[ [S_i, S_j] = E_{ij} S_k \quad [N_i, N_j] = E_{ij} N_k \]

\[ [N_i, N_j] = 0 \]

Note \((S_1, S_2, S_3)\) are closed under commutation

\[ (N_i, N_j, N_k) \]

These each form an \(SO(3)\) (subalgebra of \(SO(4)\))

and we say that the Lie group \(SO(4)\)

is isomorphic to

\(SO(3) \times SO(3)\)

Two Casimir operators for \(SO(4)\) are

\[ \lambda^2 + \bar{\lambda}^2 \]