Hamiltonians with extreme time dependence

We consider two extreme situations:

- **Sudden Approximation:** (Quench)
  
  Hamiltonian changes quickly, the system does not have time to adjust to the change.

  \[
  \text{State before change} = \text{state immediately after it}
  \]

  **Validity:** If change in Hamiltonian happens during a time \(T\), then the sudden change approximation is valid if

  \[
  \sqrt{T} \gg 2\pi \omega_b \quad \text{where} \quad E_{ab} = \hbar \omega_b
  \]

  Energy difference between Hamiltonian energy levels

- **Adiabatic Approximation:**

  Slow change in the parameters of the Hamiltonian.

  - If the system is in an eigenstate of the Hamiltonian at time \(t=0\), it will remain in the corresponding instantaneous eigenstate as the parameters slowly evolve.

  **Validity:** Parameters change at a time scale

  \[
  \sqrt{T} \ll 2\pi \omega_b
  \]

  In quantum mechanics, the adiabatic approximation can be cast in the form of a theorem:
If the Hamiltonian slowly changes from $\hat{H}^i \to \hat{H}^f$ and the particle was initially in the $n^{th}$ eigenstate of $\hat{H}^i$, it will be carried into the $n^{th}$ eigenstate of $\hat{H}^f$ in an adiabatic process.

\[ \text{NO CHANGE IN QUANTUM NUMBERS} \]

Note: Entropy IS conserved in an adiabatic process.
Energy IS NOT conserved
On the contrary
Energy IS conserved in a sudden process,
Entropy IS NOT

- In a time independent Hamiltonian

\[ \hat{H} |\psi_n> = E_n |\psi_n> \]

\[ (\exp \{-iHt}\] Then $|\psi_n(t)\rangle = e^{-iE_nt} |\psi_n> \] Evolution encoded in a phase

In a time dependent Hamiltonian, the eigenvalues and eigenstates are themselves time dependent

\[ \hat{H}(t) |\psi_n(t)\rangle = E_n(t) |\psi_n(t)\rangle \]

The instantaneous eigenstates form an orthogonal set

\[ \langle \psi_n(t) | \psi_m(t) \rangle = \delta_{nm} \]

and thus form a complete basis.

The time evolution of the time dependent Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \]

can then be written in terms of the instantaneous eigenstates as:

\[ i\partial \psi_n(t) \]
\[ |\psi(t)\rangle = \sum_n C_n(t) |\psi_n(t)\rangle e^{i\psi_n(t)} \]

with the phase \[ \psi_n(t) = -\frac{i}{\hbar} \int_0^t E_n(t') dt' \]

This generalizes the "standard" phase factor

Substituting this solution into the Schrödinger Eq yields:

\[ \sum_n \left( C_n(0) e^{i\psi_0} + C_n(t) e^{i\psi_n(t)} \right) = \sum_n C_n(t) e^{i\psi_n(t)} |\psi_n(t)\rangle \]

where time dependence of the various quantities has been assumed above.

Note that \( \psi_n = -\frac{E_n}{\hbar} t \) and \( H |\psi_n(t)\rangle = E_n |\psi_n(t)\rangle \)

\( \theta \) and \( \psi \) terms cancel

\[ \sum_n C_n(t) |\psi_n(t)\rangle e^{i\psi_0} = -\sum_n C_n(t) e^{i\psi_n(t)} |\psi_n(t)\rangle \]

\( x < \Psi_m(t) | \) above expression : inner product

\[ C_m e^{i\psi_0} = -\sum_n C_n(t) e^{i\psi_n(t)} < \Psi_m(t) | \Psi_n(t) > \]

or

\[ C_m(t) = -\sum_n C_n(t) e^{i(\psi_n(t) - \psi_m(t))} < \Psi_m(t) | \Psi_n(t) > \]

At the same time since \( H(t) |\psi_n(t)\rangle = E_n(t) |\psi_n(t)\rangle \)

\[ H(t) |\psi_n(t)\rangle + H(t) |\psi_m(t)\rangle = E_n(t) |\psi_n(t)\rangle + E_m(t) |\psi_m(t)\rangle \]

so

\[ < \Psi_m(t) | H(t) | \Psi_n(t) > = E_m \delta_{mn} - < \Psi_m(t) | H(t) | \Psi_n(t) > + E_n(t) | \Psi_n(t) > \]
\[ \langle \psi_m | H(t) | \psi_n(t) \rangle = \text{En} \delta_{mn} - \frac{\langle \psi_m | H(t) | \psi_n(t) \rangle + \text{En} \langle \psi_m | \psi_n(t) \rangle}{\text{En} | \psi_m(t) | \psi_n(t) \rangle} \]

So for \( \eta \neq m \)

\[ \langle \psi_m | H(t) | \psi_n(t) \rangle = (\text{En} - \text{Em}) | \psi_n(t) \rangle \]  \( \text{(**)} \)

\[ C_m(t) = C_m(0) e^{-i \gamma_m(t)} \]

So far this is exact. Now is when the adiabatic approximation is done. Assume \( \gamma \) is small and drop this term. Then

\[ C_m(t) = C_m(0) e^{-i \gamma_m(t)} \]

where

\[ \gamma_m(t) = i \int_0^t \langle \psi_m(t') | \frac{\partial}{\partial t} | \psi_m(t') \rangle dt' \]

In particular if at time \( t=0 \) the system starts in an eigenstate at \( \Psi(0) \) : \( C_n(0) = 1 \) \( C_m(0) = 0 \) Then

\[ |\Psi(t)\rangle = |\psi_n(t)\rangle e^{-i \gamma_m(t)} \]

This means that it remains an eigenstate up to a phase.

**Example:** Electron in a rotating magnetic field.

\[ B(t) = B_0 \left( \sin(wt) \hat{x} + \sin(wt) \hat{y} + \cos(2w) \hat{z} \right) \]

\[ \hat{A} = -e \mu_0 \hat{B} \]
\[ \hat{H} = -\vec{\mu} \cdot \vec{B} = \frac{e^2 B_0}{2m} \left( \cos \omega t \sigma_x + \sin \omega t \sigma_y + \cos \omega t \sigma_z \right) \]

\[ \hat{A} = \frac{\hbar \omega_s}{2} \begin{pmatrix} \cos \omega t & e^{i \omega t} \\ e^{-i \omega t} & -\cos \omega t \end{pmatrix} \]

The instantaneous eigenstates of \( \hat{A} \) can be obtained by a rotation

\[ X_\uparrow(t) = e^{\frac{i \sigma_y t}{2}} e^{\frac{i \sigma_z t}{2}} |\uparrow\rangle \]

\[ X_\downarrow(t) = e^{-\frac{i \sigma_y t}{2}} e^{-\frac{i \sigma_z t}{2}} |\downarrow\rangle \]

\[ X_\uparrow = \begin{pmatrix} \cos (\frac{\omega t}{2}) \\ e^{i \omega t} \sin (\frac{\omega t}{2}) \end{pmatrix} \quad X_\downarrow = \begin{pmatrix} e^{i \omega t} \\ -\cos (\frac{\omega t}{2}) \end{pmatrix} \]

The eigenvalues are \( E = \pm \frac{\hbar \omega_s}{2} \)

If at \( t=0 \) we start in \( X(0) = X_\uparrow(0) = \begin{pmatrix} \cos (\frac{\omega t}{2}) \\ \sin (\frac{\omega t}{2}) \end{pmatrix} \)

i.e. the spin is up along \( \vec{B}(0) \)

using the adiabatic approximation

\[ X(t) = \begin{pmatrix} \cos (\frac{\omega t}{2}) \\ e^{i \omega t} \sin (\frac{\omega t}{2}) \end{pmatrix} \quad i \theta(t) \quad e^{i \theta(t)} = e^{\frac{\hbar \omega_s}{2}} \int_0^t \frac{d\epsilon}{\hbar} = -\frac{\omega t}{2} \]

and

\[ Y(t) = i \int_0^t \begin{pmatrix} \cos (\omega t) & e^{i \omega t} \sin (\omega t) \\ i \omega e^{i \omega t} \sin (\omega t) \end{pmatrix} \begin{pmatrix} 0 \\ e^{i \omega t} \sin (\omega t) \end{pmatrix} \]

\[ -\omega t \sin (\omega t) = -\frac{\omega t}{2} (1 - \cos (\omega)) \]

validity? \( \omega t >> \omega \) for a complete cycle \( \omega t = 2\pi \)
the phase accumulated is \( \pi (1 - \cos \theta) \)

In general, \( \theta \) is called dynamical phase (generalization of the usual \( -\frac{\Omega t}{\hbar} \) factor when energy depends on time) and \( \gamma \) is called geometric phase or Berry phase.

**Berry Phase**

**Example: A classical pendulum**

Adiabatic transport of a pendulum over the surface of the earth. Pendulum will not longer be swinging in the same plane when it comes back. New plane makes an angle \( \Theta \) with the old one.

\( \Theta = \) solid angle subtended (at the center of the earth) by the path around which pendulum was carried. This path surrounds around \( \Theta/2\pi \) of the northern hemisphere so

\[
A = \frac{1}{2} \left( \frac{\Theta}{2\pi} \right) \pi R^2 = \Theta R^2 \quad (R: \text{earth radius})
\]

\[
\frac{\Theta}{R^2} = \Omega \quad \text{solid angle}
\]

Independent of the shape of the path

\[
\Theta = \int \sin \theta \, d\theta = 2\pi \left(1 - \cos \Theta \right) \bigg|_{\theta_0} = 2\pi \left(1 - \cos \Theta_0 \right)
\]

A system which does not return to its original state.
when transported around a close loop is called: Nonholonomic

We want to study the quantum mechanics of nonholonomic adiabatic processes.

Specific Question:

How does the final state differ from the initial state if the parameters of the Hamiltonian are carried adiabatically around a closed cycle? Answer: geometric phase

Let's try to work it out an alternative expression for the geometric phase

\[ Y_n = \int_0^t \langle \psi_n(t) | \frac{\partial}{\partial t} \psi_n(t) \rangle \, dt \]

Since \( \psi_n(t) \) depends on \( t \) because a parameter \( R \) of the Hamiltonian is changing with time. Thus

\[ \frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R} \cdot \frac{dR}{dt} \]

\[ Y_n = \int_0^t \langle \psi_n | \frac{\partial \psi_n}{\partial R} \rangle \, \frac{dR}{dt} \, dt = \int_{R_0}^{R_f} \langle \psi_n | \frac{d\psi_n}{dt} \rangle \, dR \]

where \( R_0 \) and \( R_f \) are the initial and final values of \( R(t) \).

In particular if \( R_f = R_i \) then \( Y_n(t) = 0 \) Trivial. However, if more than one parameter of the Hamiltonian is changing \((R_1(t), R_2(t), ..., R_w(t))\)

\[ \frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R_1} \cdot \frac{dR_1}{dt} + \frac{\partial \psi_n}{\partial R_2} \cdot \frac{dR_2}{dt} + ... + \frac{\partial \psi_n}{\partial R_w} \cdot \frac{dR_w}{dt} \]

\[ \mathbf{R} = (R_1, R_2, ..., R_w) \]

Then \( Y_n(t) = \int_{R_0}^{\mathbf{R}_f} \langle \psi_n | \frac{d\psi_n}{dt} \rangle \, d\mathbf{R} \)
If the Hamiltonian returns to the same form after time $T$

$$\chi = i \oint \langle \psi_0 | \overleftarrow{\nabla} \psi_0 \rangle \cdot d\vec{R}$$

line integral around a closed loop in parameter space

This is not necessarily zero

This equation was obtained by Michael Berry in 1984 and is called Berry's phase.

Note the Berry's phase is independent on how fast the path is traversed (as long as the adiabatic approximation is valid). It only depends on the path. On the contrary, the dynamic phase $2\pi \mathcal{E}(\mathrm{d}t)$ depends critically on elapsed time.

When the parameter space is 3D $\mathbf{R} = (R_1, R_2, R_3)$

Berry's phase resembles a magnetic flux, in terms of a vector potential ($\mathbf{A}$)

$$\Phi = \oint \overleftarrow{\mathbf{A}} \cdot d\mathbf{S} = \int \overline{\omega} \times \mathbf{A} \cdot d\mathbf{S} = \frac{1}{i} \oint \overleftarrow{\mathbf{A}} \cdot d\mathbf{S}$$

Stoke's theorem

In this case the Berry's phase can be thought as the flux associated to a magnetic field

$$\mathbf{B} = \overleftarrow{\nabla} \times \langle \psi_0 | \overleftarrow{\nabla} \psi_0 \rangle$$

So $\gamma_0(t) = \int \overleftarrow{\nabla} \times \langle \psi_0 | \overleftarrow{\nabla} \psi_0 \rangle \cdot d\mathbf{S}$

For the spin problem of a slowly rotating magnetic field we have

$$\gamma(t) = -(1 - \cos \frac{\omega t}{2}) \frac{\omega t}{2}$$

using $\gamma \rightarrow \theta$ $\chi = \left< \frac{\cos (\theta/2)}{\sin (\theta/2)} \right>$

in spherical coordinates one can show
\[ \langle x | \nabla x \rangle = \frac{i}{\hbar} \frac{\sin^2 \theta}{r \sin \theta} \hat{\phi} \]

\[ \vec{\nabla} \times \langle x | \nabla x \rangle = \frac{i}{2 \sin^2 \theta} \quad \gamma = -\frac{1}{2} \int \frac{1}{r^2} \mathbf{P} \cdot d\mathbf{a} \]

\[ d\mathbf{a} = v^2 \mathbf{r} \mathbf{d} \theta \phi \quad \text{so} \quad \gamma = -\frac{1}{2} \int v^2 \mathbf{r} \mathbf{d} \theta \phi = -\frac{1}{2} \pi \frac{1}{2} \pi \]

i.e. If a magnetic field is taken around a closed path the spin phase changes by half of the solid angle swept by the field vector.

In general for a spin S particle

\[ \gamma = -S \pi \]